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Strong and weak convergences in 2-probabilistic normed spaces

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Abstract

In this paper, we have introduced the notions of strong and weak convergences in 2-probabilistic normed spaces (2-PN spaces) and established some of its properties. Later, we have defined the strong and weak boundedness of a linear map between two 2-PN spaces and proved a necessary and sufficient condition for the linear map between two 2-PN spaces to be strongly and weakly bounded.

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1. Introduction and Preliminaries

Probabilistic normed spaces (PN-spaces) are vector spaces V over a real field in which the norm of any vector in V is a distribution function instead of a real number. The theory of PN spaces was initiated by Serstnev in 1963. Karl Menger considered the distribution function instead of nonnegative real numbers as values of the metric. This lead to the situation, when we do not know exactly the distance between two points, but we are aware about only the probabilities of available values of this distance. The theory of PN spaces is s a generalization of deterministic results of normed linear spaces and also the study of

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random operator theory. Recently many authors had studied the I-convergence in probabilistic n-normed space [18], statistically convergent multiple sequences [19], best approximation [17] and statistically lacunary convergence of generalized difference sequences [6]. Alizera et. al. [15] established the probabilistic norms on the homeomorphisms of a group. The concept of 2-Probabilistic normed spaces has been introduced by Fatemeh Lael and Kourosh Nourouzi ([4]). In 2005, Ioan Golet [5] had generalized the notion of 2-Probabilistic normed spaces from Random 2-normed spaces, which was established in 1988 by him. Recently, P.K. Harikrishnan at.el. [7][9] had studied about accretive operators, $\mathcal{D}-$ compactness, convex sets and convex series properties in 2- probabilistic normed spaces. N. Eghbali had discussed the Frechet differentiation between Menger probabilistic normed spaces in [3] and this idea motivated us to study the similar notions in 2-probabilistic normed spaces. In this paper, we have proved some new examples for 2-PN spaces and discussed about various properties of strong and weak convergences; and strong , weak boundedness of linear mappings in 2-PN spaces.

Let X be a real linear space of dimension greater than 1. We recall the definition of a 2-norm on $X \times X$:

Definition 1.1. *([16][10]) Let X be a real linear space of dimension greater than 1 and $\|\cdot,\cdot\|$ be a real valued function on $X \times X$ satisfying the following properties, for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$*

1. $\|x, y\| = 0$ *if and only if x and y are linearly dependent;*
2. $\|x, y\| = \|y, x\|$;
3. $\|\alpha x, y\| = |\alpha|\|y, x\|$;
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$;

then the function $\|\cdot,\cdot\|$ is called a 2-norm on X. The pair $(X,\|\cdot,\cdot\|)$ is called a linear 2- normed space.

Remark 1.2. *One can find from the definition that 2-norm is non-negative. That is, For every $x, y \in X$, ,* $0 = \|x + y, x + y\| \leq 2\|x, y\|$ *implies* $\|x, y\| \geq 0$.

Remark 1.3. *In any real linear 2-normed spaces $(X,\|\cdot,\cdot\|)$, it is true that $\|x, y + \alpha x\| = \|x, y\|$ for every $x, y \in X$ and $\alpha \in \mathbb{R}$. That is, $\|x, y + \alpha x\| = \|y + \alpha x, x\|$ implies $\|x, y + \alpha x\| \leq \|y, x\| + |\alpha|\|x, x\| = \|x, y\|$. And, $\|x, y\| = \|x, y + \alpha x - \alpha x\| = \|y + \alpha x - \alpha x, x\|$ implies $\|x, y\| \leq \|y + \alpha x, x\| + |\alpha|\|x, x\|$ implies* $\|x, y\| \leq \|y + \alpha x, x\|$.

Example 1.4. *[16] A standard example of a 2-normed space is $\mathbb{R}^2$ equipped with the 2-norm $\|x, y\|$ = area of the parallelogram determined by the vector x and y as the adjacent sides.*

$\mathbb{R}^3$ is a 2-normed space equipped with 2-norm $\|x, y\|$ is the length of the cross product of the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$.

More examples for 2-norm and related results can be found in Freese and Cho [16].

Definition 1.5. *([1]) A distribution function (= d.f.) is a function $F : \overline{\mathbb{R}} \to [0, 1]$ that is non decreasing and left-continuous on $\mathbb{R}$; moreover, $F(-\infty) = 0$ and $F(+\infty) = 1$. Here $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The set of all the d.f.'s will be denoted by Δ and the subset of those d.f.'s called distance d.f.'s, such that $F(0) = 0$, by Δ^+. We shall also consider $\mathcal{D}$ and $\mathcal{D}^+$, the subsets of Δ and Δ^+, respectively, formed by the proper d.f.'s, i.e., by those d.f.'s $F \in \Delta$ that satisfy the conditions*

$$\lim_{x\to-\infty} F(x) = 0 \quad \textit{and} \quad \lim_{x\to+\infty} F(x) = 1$$

respectively.

For every $a \in \mathbb{R}, \varepsilon_a$ is the d.f. defined by

$$\varepsilon_a(t) := \begin{cases} 0, & t \leq a, \\ 1, & t > a. \end{cases}$$

The set Δ, as well as its subsets, are POSET with respect to the usual pointwise partial order . ε_0 is the maximal element in Δ^+ with respect to this partial order.

A triangle function is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, non-decreasing in each place and has ε_0 as identity, this is, for all F, G and H in Δ^+:

(TF1) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$,

(TF2) $\tau(F, G) = \tau(G, F)$,

(TF3) $F \leq G \Longrightarrow \tau(F, H) \leq \tau(G, H)$,

(TF4) $\tau(F, \varepsilon_0) = \tau(\varepsilon_0, F) = F$.

Typical continuous triangle functions [14] are

$$\tau_T(F, G)(x) = \sup_{s+t=x} \{T(F(s), G(t))\}$$

and

$$\tau_{T^*}(F, G) = \inf_{s+t=x} \{T^*(F(s), G(t)\}.$$

Here T is a continuous t-norm, i.e. a continuous binary operation on $[0, 1]$ that is commutative, associative, non-decreasing in each variable and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0, 1]$ which is related to the continuous t-norm T through $T^*(x, y) = 1 - T(1 - x, 1 - y)$. Let us recall among the triangular function one has the function defined via $T(x, y) = \min(x, y) = M(x, y)$ and $T^*(x, y) = \max(x, y)$ or $T(x, y) = \Pi(x, y) = xy$ and $T^*(x, y) = \Pi^*(x, y) = x + y - xy$.

Definition 1.6. *([2], [14]) A probabilistic normed space is a quadruple (V, N, τ, τ^*), where V is a real linear space, τ and τ^* are continuous triangle functions and the mapping $N : V \to \Delta^+$ satisfies, for all p and q in V, the conditions*

1. *$N_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);*
2. *$\forall p \in V \quad N_{-p} = N_p$;*
3. *$N_{p+q} \geq \tau(N_p, N_q)$;*
4. *$\forall\, \alpha \in [0, 1] \quad N_p \leq \tau^*\left(N_{\alpha\, p}, N_{(1-\alpha)\, p}\right)$.*

If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t–norm T and its t–conorm T^* then $(V, N, \tau_T, \tau_{T^*})$ is denoted by (V, N, T) and is said to be a Menger PN space.

Definition 1.7. *[7][8] A Menger's 2- Probabilistic Normed Space (briefly a Menger's 2- PN space), is a triplet $(X, N, *)$, where X is a real vector space of $dim(X) > 1$, $*$ is a binary operation, a t-norm, and the mapping $N : X \times X \to \triangle^+$ (for each $(x, y) \in X \times X$ the distribution function $N(x, y)$ is denoted by $N_{x,y}$ and $N_{x,y}(n)$ is the value of $N_{x,y}$ at $n \in \mathbb{R}$) satisfying the axioms:*

1. *$N_{x,y}(0) = 0$ for all $x, y \in X$;*
2. *$N_{x,y}(n) = 1$ for all $n > 0$ if and only if x, y are linearly dependent;*

3. $N_{x,y}(n) = N_{y,x}(t)$ *for all* $x, y \in X$;

4. $N_{\alpha x,y}(n) = N_{x,y}\left(\dfrac{n}{|\alpha|}\right)$ *for all* $\alpha \in \mathbb{R}\{0\}$ *and for all* $x, y \in X$;

5. $N_{x+y,z}(m+n) \geq N_{x,z}(m) * N_{y,z}(n)$ *for all* $x, y, z \in X$ *and* $m, n \in \mathbb{R}$.

Example 1.8. *[7] Let* $(X, \|.,.\|)$ *be a 2-normed space with t-norm* $x * y = \min(x, y)$. *Every 2-norm induces a 2-PN norm on* X *as follows:*

$$N_{x,y}(n)) = \begin{cases} \frac{1}{\|x,y\|}, & \text{if } n > 0 \\ 0, & \text{if } n \leq 0. \end{cases}$$

This 2-probabilistic norm is called the standard 2-PN norm.

More examples on 2-probabilistic norm and the related results can be found in Harikrishnan et. al.[7].

Definition 1.9. *[7] Let* $(X, N, *)$ *be a 2-PN space, and* $\{x_n\}$ *be a sequence of* X. *Then the sequence* $\{x_n\}$ *is said to be convergent to* x, *if* $\lim_{n\to\infty} N_{x_n - x, z}(t) = 1$ *for all* $z \in X$ *and* $t > 0$.

Definition 1.10. *[7] Let* $((X, N, *)$ *be a 2-PN space then a sequence* $\{x_n\} \in X$ *is said to be a Cauchy sequence, if* $\lim_{n,m\to\infty} N_{x_m - x_n, z}(t) = 1$ *for all* $z \in X, t > 0$ *and* $m > n$.

Definition 1.11. *[7] A 2-PN space* $((X, N, *)$ *is said to be complete if every Cauchy sequence in* X *is convergent to a point of* X. *A complete 2-PN space is called 2-Probabilistic Banach space.*

Definition 1.12. *[7] Let* $(X, N, *)$ *be a 2-PN space,* E *be a subset of* X *then the closure of* E *is* $\overline{E} = \{x \in X;$ *there is a sequence* $\{x_n\}$ *of* E *such that* $x_n \to x\}$. *We say,* E *is sequentially closed if* $E = \overline{E}$.

2. Strong and weak convergence in 2-PN spaces

In this section, we begin with two theorems which gives a new example for 2-PN space, induced from a 2-normed space.

Theorem 2.1. *Let* $(X, \|.,.\|)$ *be a 2-normed space with t-norm* $x * y = \min(x, y)$. *Define*

$$N_{x,y}(n) = \begin{cases} \dfrac{n - \|x,y\|}{n + \|x,y\|}, & \text{if } n > \|x,y\| \\ 0, & \text{if } n \leq \|x,y\| \end{cases}$$

where $x, y \in X$ *and* $n \in \mathbb{R}$ *then* $(X, N, *)$ *is a 2-PN space.*

Proof. (i) Since $\|x,y\| \geq 0$, we have $N_{x,y}(0) = 0$ for every $x, y \in X$.

(ii) $N_{x,y}(1) = 1 \iff 1 - \|x,y\| = 1 + \|x,y\| \iff \|x,y\| = 0$
$\iff x, y$ are linearly dependent.

(iii) Since $\|x,y\| = \|y,x\|$ we have $N_{x,y}(n) = N_{y,x}(n)$ for all $x, y \in X$ and $n \in \mathbb{R}$.

(iv)

$$\begin{aligned} N_{\alpha x,y}(n) = \frac{n - \|\alpha x, y\|}{n + \|\alpha x, y\|} &= \frac{n - |\alpha|\|x,y\|}{n + |\alpha|\|x,y\|} \\ &= \frac{\frac{n}{|\alpha|} - \|x,y\|}{\frac{n}{|\alpha|} + \|x,y\|} = N_{x,y}\left(\frac{n}{|\alpha|}\right). \end{aligned}$$

(v) We have,

$$N_{x+y,z}(m+n) = \begin{cases} \dfrac{(m+n) - \|x+y,z\|}{(s+t) + \|x+y,z\|}, & \text{if } m+n > \|x+y,z\| \\ 0, & \text{if } m+n \leq \|x+y,z\| \end{cases}$$

$$N_{x,z}(m) = \begin{cases} \dfrac{m - \|x,z\|}{m + \|x,z\|}, & \text{if } m > \|x,z\| \\ 0, & \text{if } m \leq \|x,z\| \end{cases}$$

and

$$N_{y,z}(n) = \begin{cases} \dfrac{n - \|y,z\|}{t + \|y,z\|}, & \text{if } n > \|y,z\| \\ 0, & \text{if } n \leq \|y,z\| \end{cases}$$

Let $M = \max\{\|x,z\|, \|y,z\|\}$ then $\min\{N_{x,z}(m), N_{y,z}(n)\} = \dfrac{m-M}{n+M}$, and

$$\begin{aligned} N_{x+y,z}(m+n) \geq \frac{m - \|x+y,z\|}{n + \|x+y,z\|} &\geq N_{x+y,z}(m) \\ &= \frac{m - \|x+y,z\|}{m + \|x+y,z\|} \\ &\geq \frac{m - \|x,z\| - \|y,z\|}{m + \|x,z\| + \|y,z\|} \\ &\geq \frac{m - \|x,z\| - \|y,z\|}{n + \|x,z\| + \|y,z\|} \\ &\geq \frac{m-M}{n+M} \\ &= \min\{N_{x,z}(m), N_{y,z}(n)\} \\ &= N_{x,z}(m) * N_{y,z}(n). \end{aligned}$$

Hence, $(X, N, *)$ is a 2-PN space. □

Theorem 2.2. *Let $(X, \|.,.\|)$ be a 2-normed space with t-norm $x * y = \Pi(x,y) = xy$. Define*

$$N_{x,y}(n) = \begin{cases} \dfrac{n}{n + \|x,y\|}, & \textit{if } n > 0 \\ 0, & \textit{if } n \leq 0 \end{cases}$$

where $x, y \in X$ and $t \in \mathbb{R}$ then (X, N, Π) is a 2-PN space.

Proof. (i) $N_{x,y}(0) = 0$ for every $x, y \in X$.

(ii) $N_{x,y}(1) = 1 \iff \dfrac{n}{n + \|x,y\|} = 1 \iff \|x,y\| = 0$
$\iff x, y$ are linearly dependent.

(iii) $N_{x,y}(n) = N_{y,x}(n)$ for all $x, y \in X$ and $n \in \mathbb{R}$.

(iv) We have

$$N_{\alpha x,y}(n) = \frac{n}{n + \|\alpha x, y\|} = \frac{n}{n + |\alpha|\|x,y\|} = \frac{\frac{n}{|\alpha|}}{\frac{n}{|\alpha|} + \|x,y\|} = N_{x,y}\left(\frac{n}{|\alpha|}\right).$$

(v) We have,

$$N_{x+y,z}(m+n) = \frac{(m+n)}{(m+n)+\|x+y,z\|} \geq \frac{m}{m+\|x,z\|}.\frac{n}{n+\|y,z\|}$$
$$= \frac{mn}{mn+m\|y,z\|+n\|x,z\|+\|x,z\|\|y,z\|}.$$

Now it is sufficient to check,

$$(m+n)(mn+m\|y,z\|+n\|x,z\|+\|x,z\|\|y,z\|) \geq mn(m+n+\|x,z\|+\|y,z\|)$$

equivalent to $m^2\|y,z\|+n^2\|x,z\|+(m+n)\|x,z\|\|\|y,z\| \geq 0$
implies $\frac{m^2}{\|x,z\|}+m+\frac{n^2}{\|y,z\|}+n \geq 0$
and $u^2+\sqrt{\|x,z\|}$ with $\frac{m}{\sqrt{\|x,z\|}} = u$; $v^2+\sqrt{\|y,z\|}$ with $\frac{n}{\sqrt{\|y,z\|}} = v$.
Finally, (X, N, Π) is a 2-PN space. □

Theorem 2.3. *Let $(X, \|.,.\|)$ be a 2-normed space with t-norm $x * y = \min(x, y)$. Define*

$$N_{x,y}(n) = \begin{cases} \frac{n}{n+\|x,y\|}, & \text{if } n > 0 \\ 0, & \text{if } n \leq 0 \end{cases}$$

where $x, y \in X$ and $n \in \mathbb{R}$ then $(X, N, *)$ is a 2-PN space.

Proof. It is sufficient to verify that $N_{x+y,z}(m+n) \geq \min\{N_{x,z}(m), N_{y,z}(n)\}$.
We have,

$$N_{x+y,z}(m+n) = \begin{cases} \frac{(m+n)}{(m+n)+\|x+y,z\|}, & \text{if } m+n > 0 \\ 0, & \text{if } m+n \leq 0 \end{cases}$$
$$\geq \frac{m}{m+\|x+y,z\|} \quad \text{or} \quad \frac{n}{n+\|x+y,z\|}$$

Therefore, $N_{x+y,z}(m+n) \geq \min\{N_{x,z}(m), N_{y,z}(n)\}$.
Now, choose $f(x) = \frac{x}{x+k}$ then $f(x) = \frac{k}{(x+k)^2}$ implies f is increasing.

$$N_{x,z}(m) = \begin{cases} \frac{m}{m+\|x,z\|}, & \text{if } m > 0 \\ 0, & \text{if } m \leq 0 \end{cases}$$

and

$$N_{y,z}(n) = \begin{cases} \frac{n}{n+\|y,z\|}, & \text{if } n > 0 \\ 0, & \text{if } n \leq 0 \end{cases}$$

We can assume that $\min\{\frac{m}{m+\|x+y,z\|}, \frac{n}{n+\|x+y,z\|}\} = \frac{m}{m+\|x+y,z\|}$ when $m \leq n$.
Then we need, $\frac{m}{m+\|x+y,z\|} \geq \frac{m}{m+\|x,z\|} \iff \|x+y,z\| \leq \|x,z\|$.
Thus, $\frac{m}{m+\|x+y,z\|} \geq \frac{n}{n+\|y,z\|} \iff 1 \leq \frac{t}{m} < \frac{\|y,z\|}{\|x+y,z\|}$ equivalent to $\|x+y,z\| \leq \|y,z\|$.
Finally, $N_{x+y,z}(m+n) \geq \min\{N_{x,z}(m), N_{y,z}(n)\}$. Hence, $(X, N, *)$ is a 2-PN space. □

Definition 2.4. *Let $(X,N,*)$ be a 2-PN space and $U \subset X$, U is said to be open if for each $x \in U$ there exists some $n>0$ and some $\alpha \in (0,1)$ such that $B(x,\alpha,n) \subseteq U$ where $B(x,\alpha,n) = \{y; N_{x-y,z}(n) > 1-\alpha \ \ \forall z \in X\}$.*

Theorem 2.5. *Let $(X,N,*)$ be a 2-PN space with the condition*

$$N_{x,y}(n) > 0 \ \ \textit{for all} \ \ n>0 \ \ \textit{implies} \ \ x \textit{ and } y \textit{ are dependent.} \tag{1}$$

Let $\|x,y\|_\alpha = \inf\{n>0 \ : \ N_{x,y}(n) \geq \alpha\}$, for all $\alpha \in (0,1)$. Then $\{\|\cdot,\cdot\|_\alpha \ : \ \alpha \in (0,1)\}$ is an ascending family of 2-norms on X. These 2-norms are called $\alpha-2-$norms on X corresponding to a 2-probabilistic norm.

Proof. (i) Let $x,y \in X$ and $\alpha \in (0,1)$ then

$$\begin{aligned} \|x,y\|_\alpha = 0 \ \text{ implies } \ & \inf\{n>0 \ : \ N_{x,y}(n) \geq \alpha\} = 0 \ \text{ for each } \alpha \in (0,1) \\ \text{implies } \ & N_{x,y}(n) > 0 \ \ \forall \ \ n>0 \\ \text{implies } \ & \text{x,y are dependent.} \end{aligned}$$

(ii) For each $\alpha \in (0,1)$ and $x,y,z \in X$,

$$\begin{aligned} \|x,y\|_\alpha &= \inf\{n>0 \ : \ N_{x,y}(n) \geq \alpha\} \\ &= \inf\{n>0 \ : \ N_{y,x}(n) \geq \alpha\} \\ &= \|y,x\|_\alpha. \end{aligned}$$

(iii) For each $\alpha \in (0,1)$ and $x,y,z \in X, k \in \mathbb{R}$,

$$\begin{aligned} \|kx,y\|_\alpha &= \inf\{n>0 \ : \ N_{kx,y}(n) \geq \alpha\} \\ &= \inf\left\{n>0 \ : \ N_{x,y}\left(\frac{n}{k}\right) \geq \alpha\right\} \\ &= k\|x,y\|_\alpha. \end{aligned}$$

(iv) For each $\alpha \in (0,1)$ and $x,y,z \in X$,

$$\begin{aligned} \|x,y\|_\alpha + \|y,z\|_\alpha &= \inf\{m>0 \ : \ N_{x,y}(m) \geq \alpha\} + \inf\{n>0 \ : \ N_{x,y}(n) \geq \alpha\} \\ &= \inf\{m+n \ : \ N_{x,y}(m) \geq \alpha, N_{x,y}(n) \geq \alpha\} \\ &= \inf\{m+n \ : \ N_{x,y}(m) * N_{x,y}(n) \geq \alpha\} \\ &\geq \inf\{m+n \ : \ N_{x+y,z}(m+n) \geq \alpha\} \\ &= \|x+y,z\|_\alpha. \end{aligned}$$

Now, let $0 < \alpha_1 < \alpha_2 < 1$ then $\|x,y\|_{\alpha_1} = \inf\{m>0 \ : \ N_{x,y}(m) \geq \alpha_1\}$ and $\|x,y\|_{\alpha_2} = \inf\{m > 0 \ : \ N_{x,y}(m) \geq \alpha_2\}$. Since $\alpha_1 < \alpha_2, \{m>0 \ : \ N_{x,y}(m) \geq \alpha_2\} \subseteq \{m>0 \ : \ N_{x,y}(m) \geq \alpha_1\}$ then $\|x,y\|_{\alpha_2} \geq \|x,y\|_{\alpha_1}$.

Hence $\{\|\cdot,\cdot\|_\alpha \ : \ \alpha \in (0,1)\}$ is an ascending family of 2-norms on X. □

Definition 2.6. *Let $\{x_n\}$ be a sequence in a Menger 2-PN space $(X,N,*)$. Then*

1. *$\{x_n\}$ is said to be weakly convergent to $x \in X$, denoted by $x_n \xrightarrow{w} x$ iff, for every $\alpha \in (0,1)$ and $\varepsilon > 0$ there exists some $k = k(\alpha,\varepsilon)$ such that $n \geq k$ implies $N_{x_n-x,z}(\varepsilon) \geq 1-\alpha$ for every $z \in X$.*

2. *$\{x_n\}$ is said to be strongly convergent to $x \in X$, denoted by $x_n \xrightarrow{s} x$ iff, for every $\alpha \in (0,1)$ there exists some $k = k(\alpha)$ such that $n \geq k$ implies $N_{x_n-x,z}(t) \geq 1-\alpha$ for every $t>0$ and $z \in X$.*

Theorem 2.7. *Let $\{x_n\}$ be a sequence in a Menger 2-PN space $(X,N,*)$ satisfying the condition (2.1). Then*

1. $x_n \xrightarrow{w} x$ *if and only if, for each* $\alpha \in (0,1)$, $\lim_{n\to\infty} \|x_n - x, z\|_\alpha = 0$ *for every* $z \in X$.

2. $x_n \xrightarrow{s} x$ *if and only if, for each* $\alpha \in (0,1)$, *and for every* $z \in X$, $\lim_{n\to\infty} \|x_n - x, z\|_\alpha = 0$ *uniformly in* $\alpha \in (0,1)$.

Proof. (1) Suppose $x_n \xrightarrow{\text{w}} x$. Choose $\alpha \in (0,1)$ and $t > 0$, then there exists $k \in \mathbb{N}$ such that $N_{x_n - x, z}(\varepsilon) \geq 1-\alpha$ for all $n \geq k \implies \|x_n - x, z\|_{1-\alpha} \to 0$ for every $z \in X$.

Conversely, Let $\|x_n - x, z\|_\alpha \to 0$, for every $\alpha \in (0,1)$ and $t > 0$. There exists $k \in \mathbb{N}$ such that $\inf\{r > 0 : N_{x_n - x, z}(r) \geq 1-\alpha\} < t$, for all $n \geq k$ and $z \in X$. It implies that $N_{x_n-x,z}(t) \geq 1-\alpha$, for all $n \geq k \implies x_n \xrightarrow{\text{w}} x$.

(2) We can prove (2) using the similar assertions applied in (1), □

Theorem 2.8. *Let* $\{x_n\}$ *be a sequence in a Menger 2-PN space* $(X, N, *)$. *If* $\{x_n\}$ *is strongly convergent, then it is weakly convergent to the same limit. But the converse need not be true.*

Proof. This is immediate from the definition (2.3). But the next example shows that the converse of this result need not be true. □

Example 2.9. *Let* $(X, \|.,\|)$ *be a linear 2-normed space and define* N *on* X *by*

$$N_{x,y}(t) = \begin{cases} \dfrac{t - \|x,y\|}{t + \|x,y\|}, & \text{if } t > \|x,y\| \\ 0, & \text{if } t \leq \|x,y\| \end{cases}$$

Define $x * y = \Pi(x,y)$. *Then* (X, N, Π) *is a Menger 2-PN space. Since* N *satisfies the condition (2.1), we can find the* $\alpha{-}2$ *norm of* N. *Thus,* $N_{x,y}(t) \geq \alpha \iff \dfrac{t - \|x,y\|}{t + \|x,y\|} \geq \alpha \iff \left(\dfrac{1+\alpha}{1-\alpha}\right)\|x,y\| \leq t.$

This shows that $\|x,y\|_\alpha = \inf\{t > 0; N_{x,y}(t) \geq \alpha\} \leq \left(\dfrac{1+\alpha}{1-\alpha}\right)\|x,y\|$. *Furthermore,* $N_{x,y}\left(\left(\dfrac{1+\alpha}{1-\alpha}\right)\|x,y\|\right) =$

$$\frac{\left(\dfrac{1+\alpha}{1-\alpha}\right)\|x,y\| + \|x,y\|}{\left(\dfrac{1+\alpha}{1-\alpha}\right)\|x,y\| - \|x,y\|} = \alpha$$

implies $\left(\dfrac{1+\alpha}{1-\alpha}\right)\|x,y\| \in \{t > 0; N_{x,y}(t) \geq \alpha\}$.

This means, $\|x,y\|_\alpha = \left(\dfrac{1+\alpha}{1-\alpha}\right)\|x,y\|$.

For $z \in X$, *let* $y \in S_X = \{x \in X; \|x,z\| = 1\}$ *be fixed. Define a sequence* $\{x_n\} = \{\frac{y}{n}\}$. *For each* $\alpha \in (0,1)$ *and* $z \in X$,

$$\|x_n - 0, z\|_\alpha = \left(\frac{1+\alpha}{1-\alpha}\right)\frac{\|y,z\|}{n} \to 0, \quad \text{as } n \to \infty,$$

this convergence is uniform in α.

We have, for given

$$\varepsilon > 0, \|x,z\|_\alpha = \left(\frac{1+\alpha}{1-\alpha}\right)\frac{\|y,z\|}{n} < \varepsilon \iff \frac{1-\alpha}{(1-\alpha)\varepsilon} < n,$$

it is clear that we cannot find such n *as* $\dfrac{1-\alpha}{(1-\alpha)\varepsilon} \to \infty$ *as* $\alpha \to \infty$.

Definition 2.10. *Let* $(X, N, *)$ *and* $\left(Y, N', *\right)$ *be two Menger 2-PN spaces and* $f : X \to Y$ *be a mapping then*

1. *f is said to be weakly continuous at $x_0 \in X$ if for given $\varepsilon > 0$ and $\alpha \in (0,1)$, there exists $\delta = \delta(\varepsilon, \alpha) > 0$ such that for all $x, z \in X$*

$$N_{x-x_0,z}(\delta) \geq \alpha \ \ \textit{implies} \ \ N'_{f(x)-f(x_0),f(z)}(\varepsilon) \geq \alpha.$$

2. *f is said to be strongly continuous at $x_0 \in X$ if for given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, z \in X$*

$$N'_{f(x)-f(x_0),f(z)}(\varepsilon) \geq N_{x-x_0,z}(\delta).$$

Definition 2.11. *Let $(X, N, *)$ and $\left(Y, N', *\right)$ be two Menger 2-PN spaces and $f : X \to Y$ be a linear mapping then*

1. *f is said to be weakly bounded on X if for every $\alpha \in (0,1)$, there exists some $m_\alpha > 0$ such that, for all $x, y \in X$*

$$N_{x,y}\left(\frac{t}{m_\alpha}\right) \geq \alpha \implies N'_{f(x),f(y)}(t) \geq \alpha \ \ \textit{for every} \ \ t > 0.$$

2. *f is said to be strongly bounded on X if for every $\alpha \in (0,1)$, there exists some $M > 0$ such that, for all $x, y \in X$*

$$N'_{f(x),f(y)}(t) \geq N_{x,y}\left(\frac{t}{M}\right) \ \ \textit{for every} \ \ t > 0.$$

Theorem 2.12. *Let $(X, N, *)$ and $\left(Y, N', *\right)$ be two Menger 2-PN spaces and $f : X \to Y$ be a linear mapping. Then, f is strongly (weakly) continuous if and only if it is strongly (weakly) bounded.*

Proof. Suppose that f is strongly bounded. Then there exists $M > 0$ such that, for all $x, y \in X$

$$N'_{f(x),f(y)}(t) \geq N_{x,y}\left(\frac{t}{M}\right) \ \ \text{for every} \ \ t > 0$$

$$\implies N'_{f(x)-f(0),f(y)}(t) \geq N_{x-0,y}\left(\frac{t}{M}\right) \ \ \text{for every} \ \ t > 0.$$

Let $\varepsilon > 0$ be given. Choose $\delta = \dfrac{\varepsilon}{M}$ then,

$$\implies N'_{f(x)-f(0),f(y)}(\varepsilon) \geq N_{x-0,y}(\delta) \ \ \text{for every} \ \ t > 0$$

implies f is strongly continuous at 0, hence f is continuous on X.
Conversly, assume that f is strongly continuous on X. Then f is strongly continuous at $x = 0$.
For $\varepsilon = 1$, there exists $\delta > 0$ such that

$$N'_{f(x)-f(0),f(y)}(1) \geq N_{x-0,y}(\delta) \ \ \text{for every} \ \ x \in X.$$

Suppose that $x \neq 0$ and $t > 0$. Take $w = \dfrac{x}{t}$ then

$$\begin{aligned} N'_{f(x)-f(0),f(y)}(t) &= N'_{tf(w)-f(0),f(y)}(t) \\ &= N'_{f(w)-f(0),f(y)}(1) \\ &\geq N_{w-0,y}(\delta) \\ &= N_{x-0,y}(t\delta) \\ &= N_{x-0,y}\left(\frac{t}{M}\right) \end{aligned}$$

where $M = \dfrac{1}{\delta}$.

If $x \neq 0$ and $t \leq 0$ then for every $y \in X$,

$$N'_{f(x),f(y)}(t) = 0 = N_{x,y}\left(\frac{t}{M}\right).$$

If $x = 0$ and $t \in \mathbb{R}$ then $f(0) = 0$ and for every $y \in X$,

$$N'_{0,y}(t) = N_{0,y}\left(\frac{t}{M}\right) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

Hence, f is strongly bounded. □

Theorem 2.13. *Let $(X, N, *)$ be a Menger 2-PN space, satisfying (2.1) and the condition*

$$\text{For } x \neq 0 \text{ and } y \in X, N_{x,y}(.) \text{ is continuous on } \mathbb{R} \text{ and strictly increasing on} \tag{2}$$

$$\{t : 0 < N_{x,y}(t) < 1\}.$$

$$\text{For } x \neq 0, \alpha \in (0,1), t' > 0 \text{and for every } y \in X \tag{3}$$

we have $\|x, y\|_\alpha = t' \iff N_{x,y}(t') = \alpha$.

Also suppose that $\{\|\cdot,\cdot\|_\alpha : \alpha \in (0,1)\}$ *be the family of corresponding* $\alpha - 2 - norms$ *of* N *on* X *defined by* $\|x, y\|_\alpha = \inf\{t > 0 \ : \ N_{x,y}(t) \geq \alpha\}$, *for all* $\alpha \in (0,1)$ *. then for any increasing (or decreasing) sequence* $\{\alpha_n\}$ *in* $(0,1), \alpha_n \to \alpha$ *implies* $\|x, y\|_{\alpha_n} \to \|x, y\|_\alpha$ *for every* $x, y \in X$.

Proof. For $x = 0$ and $y \in X$, it is obvious that $\|x, y\|_{\alpha_n} \to \|x, y\|_\alpha$.

Suppose $x \neq 0$. Then by equation (2.3), we have $\alpha \in (0,1), t' > 0$ and for every $y \in X$ we have $\|x, y\|_\alpha = t'$ if and only if $N_{x,y}(t') = \alpha$.

Let $\{\alpha_n\}$ be an increasing sequence in $(0,1)$ such that $\alpha_n \to \alpha$ in $(0,1)$.

Let $\|x\|\|_{\alpha_n} = t_n$ and $\|x\|_\alpha = t$ then $N_{x,y}(t_n) = \alpha_n$ and $N_{x,y}(t) = \alpha$.

We know that $\{t_n\}$ is an increasing sequence of real numbers and bounded above by t. So, $\{t_n\}$ is convergent to some $t \in \mathbb{R}$.

Since, $N_{x,y}(t)$ is sequentially continuous, one can say that $\{t_n\} \to t \iff N_{x,y}(t_n) \to N_{x,y}(t)$. Hence $\lim\limits_{n\to\infty} \|x, y\|_{\alpha_n} = \|x\|_\alpha$.

Similarly, if $\{\alpha_n\}$ be an decreasing sequence in $(0,1)$ then we can prove the theorem. □

Theorem 2.14. *Let $(X, N, *)$ and $\left(Y, N', *\right)$ be two Menger 2-PN spaces satisfying (2.1), (2.2) and (2.3). Suppose that $f : X \to Y$ be a linear mapping then,*

1. *f is weakly bounded if and only if f is bounded with respect to the $\alpha - 2-$ norms of N and N', for each $\alpha \in (0,1)$.*

2. *f is strongly bounded if and only if f is uniformly bounded with respect to the $\alpha - 2-$ norms of N and N'.*

Proof. 1) Assume that f is weakly bounded then for every $\alpha \in (0,1)$ there exists $m_\alpha > 0$ such that for all $x, y \in X$ and $t \in \mathbb{R}$ we have $N_{x,y}\left(\dfrac{t}{m_\alpha}\right) \geq \alpha$ and then $N'_{f(x).f(y)}(t) \geq \alpha$.

Hence $\sup\{\beta \in (0,1) : \|m_\beta x, y\|^1_\alpha \leq t\} \geq \alpha$

$\implies \sup\{\beta \in (0,1) : \|f(x), f(y)\|^2_\beta \leq t\} \geq \alpha$ where $\|\cdot,\cdot\|^1_\alpha$ and $\|\cdot,\cdot\|^2_\alpha$ are the $\alpha-$ 2 - norms of N and N' respectively.

Now we prove that $\sup\{\beta \in (0,1) : \|m_\alpha x, y\|^1_\beta \leq t\} \geq \alpha \iff \|m_\alpha x, y\|^1_\alpha \leq t$.

The relation is obvious when $x = 0$. Suppose $x \neq 0$.

Now, if

$$\sup\{\beta \in (0,1) : \|m_\alpha x, y\|_\beta^1 \leq t\} > \alpha \quad \text{then} \quad \|m_\alpha x, y\|_\alpha^1 \leq t. \tag{4}$$

If $\sup\{\beta \in (0,1) : \|m_\alpha x, y\|_\beta^1 \leq t\} = \alpha$, then there exists an increasing sequence $\{\alpha_n\}$ such that $\alpha_n \to \alpha$ and $\|m_\alpha x, y\|_{\alpha_n}^1 \leq t$. Then by the above theorem, we have

$$\|m_\alpha x, y\|_\alpha^1 \leq t. \tag{5}$$

Thus from (2.4) and (2.5) we get,

$$\sup\{\beta \in (0,1) : \|m_\alpha x, y\|_\beta^1 \leq t\} \geq \alpha \implies \|m_\alpha x, y\|_\alpha^1 \leq t.$$

Next we suppose that

$$\|m_\alpha x, y\|_\alpha^1 \leq t \tag{6}$$

If $\|m_\alpha x, y\|_\alpha^1 < t$ then $N_{m_\alpha x, y}(t) \geq \alpha$. So

$$\sup\{\beta \in (0,1) : \|m_\alpha x, y\|_\beta^1 \leq t\} \geq \alpha \tag{7}$$

If $\|m_\alpha x, y\|_\alpha^1 = t$, then there exists a decreasing sequence $\{s_n\} \in \mathbb{R}$ such that $s_n \to t$ and $N_{m_\alpha x, y}(s_n) \geq \alpha$ implies $N_{m_\alpha x, y}\left(\lim_{n\to\infty} s_n\right) \geq \alpha$ so we get $N_{m_\alpha x, y}(t) \geq \alpha$.

Hence,

$$\sup\{\beta \in (0,1) : \|m_\alpha x, y\|_\beta^1 \leq t\} \geq \alpha. \tag{8}$$

It follows that,

$$\|m_\alpha x, y\|_\alpha^1 \leq t \quad \text{implies} \quad \sup\{\beta \in (0,1) : \|m_\alpha x, y\|_\beta^1 \leq t\} \geq \alpha. \tag{9}$$

Hence,

$$\sup\{\beta \in (0,1) : \|m_\alpha x, y\|_\beta^1 \leq t\} \geq \alpha \iff \|m_\alpha x, y\|_\alpha^1 \leq t. \tag{10}$$

In a similar manner, we can show that,

$$\sup\{\beta \in (0,1) : \|f(x), f(y)\|_\beta^2 \leq t\} \geq \alpha \iff \|f(x), f(y)\|_\alpha^2 \leq t. \tag{11}$$

Therefore, from (2.10) and (2.11) we have $N_{m_\alpha x, y}(t) \geq \alpha \implies N'_{f(x),f(y)}(t) \geq \alpha$

then $\|m_\alpha x, y\|_\alpha^1 \leq t \implies \|f(x), f(y)\|_\alpha^2 \leq t$.

This implies $\|f(x), f(y)\|_\alpha^2 \leq m_\alpha \|x, y\|_\alpha^1$.

Conversely suppose that for every $\alpha \in (0,1)$, there exists $m_\alpha > 0$ such that $\|f(x), f(y)\|_\alpha^2 \leq m_\alpha \|x, y\|_\alpha^1$. Then for $x \neq 0, \inf\{s : N_{m_\alpha x, y}(s) \geq \alpha\} \leq t$ implies $\inf\{s : N'_{f(x),f(y)}(s) \geq \alpha\} \leq t$.

In a similar way as above we can prove that $\inf\{s : N_{m_\alpha x, y}(s) \geq \alpha\} \leq t$ iff $N_{m_\alpha x, y}(t) \geq \alpha$ and $\inf\{s : N'_{f(x),f(y)}(s) \geq \alpha\} \leq t$ iff $N'_{f(x),f(y)}(t) \geq \alpha$. Thus we have $N_{x,y}\left(\frac{t}{m_\alpha}\right) \geq \alpha \implies N'_{f(x),f(y)}(t) \geq \alpha$.

If $x \neq 0, t \leq 0$ and $x = 0, t > 0$ then the above relation is obvious. Hence the theorem follows.

2) Let $\|\cdot,\cdot\|_\alpha^1$ and $\|\cdot,\cdot\|_\alpha^2$ be the $\alpha-$ 2 - norms of N and N' respectively.

First we suppose that f is strongly bounded. Then there exists $M > 0$ such that for all $x \in X$ and $s \in \mathbb{R}$ we have $N'_{f(x),f(y)}(s) \geq N_{x,y}\left(\frac{s}{M}\right)$.

Therefore, $N'_{f(x),f(y)}(s) \geq N_{Mx,y}(s)$.

Now $\|Mx, y\|_\alpha^1 < t \implies \inf\{s : N_{Mx,y}(s) \geq \alpha\} < t$ implies there exists $s_0 < t$ such that $N_{Mx,y}(s_0) \geq \alpha$ implies there exists $s_0 < t$ such that $N'_{f(x),f(y)}(s_0) \geq \alpha$ implies $\|f(x), f(y)\|_\alpha^2 \leq s_0 < t$.

Hence $\|f(x), f(y)\|_\alpha^2 \leq \|Mx, y\|_\alpha^1 = M\|x, y\|_\alpha^1$. This implies that T is uniformly bounded with respect to $\alpha-$ 2- norms.

Conversely suppose that there exists $M > 0$ such that $\|f(x), f(y)\|_\alpha^2 \leq M\|x, y\|_\alpha^1$ holds for all $\alpha \in (0, 1)$ and $x, y \in x$.

Now $r < N_{Mx,y}(s)$ implies $r < \sup\{\alpha \in (0,1) : \|Mx, y\|_\alpha^1 \leq s\}$
implies there exists $\alpha_0 \in (0, 1)$ such that $r < \alpha_0$ and $\|Mx, y\|_{\alpha_0}^1 \leq s$

$$N'_{f(x),f(y)}(s) \geq N_{Mx,y}(s) = N_{x,y}\left(\frac{s}{M}\right).$$

Therefore, f is strongly bounded.

□

3. Conclusion

We have established new examples for 2-probabilistic normed spaces and introduced the notions of strong and weak convergences in 2-probabilistic normed spaces with several properties. Subsequently, we have defined the strong and weak boundedness of a linear map between two 2-PN spaces and established a necessary and sufficient condition for the linear map between two 2-PN spaces to be strongly and weakly bounded.

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