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Lorentz para-Kenmotsu manifoldlarının generic altmanifoldları

Generic submanifolds of Lorentzian para-Kenmotsu manifolds

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Özet. Bu makalede, Lorentz metrikli para-kontakt manifoldların özel bir sınıfı olan Lorentz para-Kenmotsu manifoldların altmanifoldları çalışılmıştır. Öncelikle, bir Lorentz para-Kenmotsu manifoldun bir almanifoldunun bazı özellikleri elde edilmiştir. Daha sonra, holomorfik ve sadece reel dağılımların integrallenebilirlik koşulları incelenmiştir. Son olarak, paralel kanonik yapılar için bazı sonuçlar verilmiştir.

Anahtar Kelimeler: generic altmanifold, Lorentz para-Kenmotsu, paralel dağılım.

Abstract. In this paper, we study on submanifolds of a Lorentzian para-Kenmotsu manifold which is a special kind of para-contact manifolds with Lorentzian metric. Firstly, we obtain some basic properties for a submanifold of a Lorentzian para-Kenmotsu manifold. Then, we examine integrability conditions of holomorphic distribution and purely real distribution. Finally, we obtain some results for parallel canonical structures.

Keywords: generic submanifold, Lorentzian para-Kenmotsu, parallel distribution.

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1. Introduction

An almost complex structure J is an endomorphism on a Hermitian manifold that $J = -I$. We recall the manifold an almost complex manifold and also if J is integrable manifold becomes complex. On the other hand, similar to complex structure a para-complex structure is defined on a Hermitian manifold M with the endomorphism $J : \Gamma(TM) \longrightarrow \Gamma(TM)$, such that $J^2 = I$. Para-complex structures have many different geometric properties from classical complex structures. A canonical example is the product of two manifolds $M_+ \times M_-$ of the same dimension. Also there are many applications of these manifolds in physics [19].

Contact manifolds have been studied in the tensorial viewpoint since 1960s. In this way, some geometric properties of almost complex and complex manifolds have guided researchers in the study of contact manifolds. For example, Sasakian manifolds, which are an important example of contact manifolds, are considered as one-dimensional analogues of Kähler manifolds. Similarily, para-Kähler structures have been studied in para-complex geometry and para-Sasakian structure have been studied in para-contact geometry. An almost para-contact structure was defined by Kaneyuki Willams [6]. Many authors worked on almost para-contact metric manifolds and their subclasses [1, 12, 19, 14, 16].

In 1969, Takahashi [15] studied on Sasakian manifolds with pseudo-Riemannian metric. In 1990, K. L. Duggal [5] analyzed the paper of Takahashi and they considered space time manifolds with contact structure. Lorentzian Kenmotsu manifolds have been defined by Roşça [9]. Sari and Turgut

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Vanlı [10, 11] worked on Lorentzian Kenmotsu manifolds. In [20], Tirpathi and De presented a survey on tLorentzian para-contact manifolds. Also, some authors investigated LP-Kenmotsu manifolds in [2, 7, 8].

In the presented paper, we study on generic submanifolds of Lorentzian para-Kenmotsu manifolds. Firstly, we obtain some basic properties for a submanifold of a LP-Kenmotsu manifold. Then, we examine integrability conditions of holomorphic distribution and purely real distribution. Finally, we obtain some results for parallel canonical structures.

2. Preliminaries and basic results

In this section, we give some fundamental facts of LP-Kenmotsu manifolds and the submanifold theory.

Definition 1. *Let $\overline{M}$ be an $n-$dimensional differentiable manifold with Lorentzian metric $\bar{g}$. If we have a para-contact structure (φ, ξ, η) on $\overline{M}$ as the following :*

$$\varphi^2 = I + \eta \otimes \xi, \qquad \eta(\xi) = -1$$

$$\bar{(}\varphi X, \varphi Y) = \bar{g}(X, Y) + \eta(X)\eta(Y),$$

where φ is a tensor field of type $(1,1)$, η is a $1-$form, ξ is a vector field, then $\overline{M}$ is called a Lorentzian almost para-contact metric manifold [20].

From the definition, it is clear that $g(\varphi X, Y) = g(X, \varphi Y)$. Similar to any contact or para-contact structure the fundamental $2-$form Φ is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

for any $X, Y \in \Gamma(T\bar{M})$. Moreover, an almost para-contact metric manifold is *normal* if

$$N = [\varphi, \varphi] + 2d\eta \otimes \xi = 0$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ.

Definition 2. *A Lorentzian almost para-contact metric manifold $\overline{M}$ is said to be a Lorentzian almost para-Kenmotsu manifold if 1-form η is closed and $d\Phi = -2\eta \wedge \Phi$.*

If $\overline{M}$ is also normal, then $\overline{M}$ is called a Lorentzian para-Kenmotsu manifold. Except for the definitions, also we can classify para-contact manifolds by covariant derivation of φ. By the following theorem, a Lorentzian para-contact metric manifold is characterized as LP-Kenmotsu manifold.

Theorem 1. *A Lorentzian almost para-contact metric manifold $\overline{M}$ is a LP-Kenmotsu manifold if and only if*

$$\left(\overline{\nabla}_X \varphi\right) Y = -\bar{g}(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{1}$$

for all $X, Y \in \Gamma(T\overline{M})$ [7].

Corollary 1. *Let $\left(\overline{M}, \varphi, \xi, \eta, g\right)$ be a Lorentzian para-Kenmotsu manifold. Then we have*

$$\overline{\nabla}_X \xi = \varphi^2 X \tag{2}$$

for all $X \in \Gamma(T\overline{M})$.

Gauss and Weingarten formulas on an $m-$dimensional submanifold of a LP-Kenmotsu manifold with induced metric g are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{3}$$

$$\overline{\nabla}_X V = \nabla^{\perp}_X V - A_V X \tag{4}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental from of $\overline{M}$ and A_V is the Weingarten endomorphism associated with V.

The second fundamental form h and the shape operator A have the following relation

$$g(h(X, Y), V) = g(A_V X, Y). \tag{5}$$

A submanifold M of $\overline{M}$ is said to be totally geodesic if $h(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$.

For $X \in \Gamma(TM)$ we put

$$\varphi X = PX + NX \tag{6}$$

where PX and NX denote tangential and normal components of φX, respectively. Similarly, for $V \in \Gamma(T^{\perp}M)$, we write

$$\varphi V = pV + nV \tag{7}$$

where pV and nV denote tangential and normal components of φV, respectively.

For $X, Y \in \Gamma(TM)$, we have

$$g(\varphi X, Y) = g(X, \varphi Y).$$

From (6), we can state

$$g(PX + NX, Y) - g(X, PY + NY) = 0.$$

Thus we obtain $g(X, PY) = g(PX, Y)$. On the other hand, for a vector field $V \in \Gamma(T^{\perp}M)$ we have

$$g(\varphi X, V) = g(X, \varphi V).$$

From (7), we get $g(X, pV) = g(NX, V)$.

Finally, for $K \in \Gamma(T^{\perp}M)$ we can state

$$g(pK + nK, V) - g(K, pV + nV) = 0.$$

From (7) we obtain $g(K, nV) = g(nK, V)$. We can summarize all these results by the following lemma.

Lemma 1. *For all $X, Y \in \Gamma(TM)$, $V, K \in \Gamma(T^{\perp}M)$ we have*

$$g(X, PY) = g(PX, Y),$$
$$g(X, pV) = g(NX, V),$$
$$g(K, nV) = g(nK, V).$$

Suppose that $\xi \in \Gamma(TM)$. Then we have

$$0 = \varphi\xi = P\xi + N\xi = 0.$$

Since, $T\overline{M} = TM \oplus T^{\perp}M$, it is obvious that $P\xi = N\xi = 0$. On the other hand we have

$$0 = \eta(\varphi X) = g(\xi, \varphi X) = g(\xi, PX) + g(\xi, NX) = \eta(PX) + \eta(NX) = 0$$

and thus we get $\eta \circ P = \eta \circ N = 0$. After following similar steps we have

$$\varphi^2 X = \varphi(PX + NX) = P^2X + NPX + pNX + nNX.$$

Since $P^2X + pNX \in \Gamma(TM)$ and $NPX + nNX \in \Gamma(T^{\perp}M)$, we get $P^2 + pN = I + \eta \otimes \xi$ and $NP + nN = 0$.

Finally we have

$$\varphi^2 P = \varphi(pV + nV) = PpV + NpV + pnV + n^2V$$

Since $n^2V + NpV \in \Gamma(T^{\perp}M)$ and $PpV + pnV \in \Gamma(TM)$, we get $n^2 + Np = I$ and $pn + Pp = 0$. We can summarize all these results as following.

Lemma 2. *On a submanifold of a LP-Kenmotsu manifold we have*

$$P\xi = 0 = N\xi$$
$$\eta \circ P = 0 = \eta \circ N$$
$$P^2 + pN = I + \eta \otimes \xi$$
$$NP + nN = 0$$
$$n^2 + Np = I$$
$$pn + Pp = 0.$$

By using formula (1), we obtain covariant derivation of T and N via the induced connection ∇ and $\nabla^{\perp}$. For $X, Y \in \Gamma(TM)$ from (3), (4), (6) and (7) we get

$$(\overline{\nabla}_X\varphi)Y = \nabla_X PY + h(X, PY) - A_{NY}X + \nabla^{\perp}_X NY - P\nabla_X Y - N\nabla_X Y - ph(X, Y) - nh(X, Y).$$

From (1) and with consider tangential and normal parts we obtain

$$-g(PX, Y)\xi - \eta(Y)PX = (\nabla_X P)Y - A_{NY}X - ph(X, Y)$$

and

$$-\eta(Y)NX = (\nabla_X N)Y - h(X, PY) - A_{NY}X.$$

Similarly, we can obtain covariant derivation of p and n via the induced connection ∇ and $\nabla^{\perp}$. For $V \in \Gamma(T^{\perp}M)$ and from (3), (4), (6) and (7) we have

$$(\overline{\nabla}_X\varphi)V = \nabla_X pV + h(X, pV) + A_{nV}X + \nabla^{\perp}_X nV + PA_V X + NA_V X - p\nabla^{\perp}_X V - n\nabla^{\perp}_X V.$$

From (1) and with consider tangential and normal parts we obtain

$$-g(NX, V)\xi = (\nabla_X p)V + PA_V X - A_{nV}X$$

and

$$0 = (\nabla_X n)V + h(X, pV) + NA_V X.$$

We can summarize all these results as following.

Lemma 3. *On a submanifold of a LP-Kenmotsu manifold, we have*

$$(\nabla_X P)Y = A_{NY}X + ph(X,Y) - g(PX,Y)\xi - \eta(Y)PX \tag{8}$$

$$(\nabla_X N)Y = nh(X,Y) + h(X,PY) + \eta(Y)NX \tag{9}$$

$$(\nabla_X p)V = A_{nV}X + PA_V X - g(NX,V)\xi \tag{10}$$

$$(\nabla_X n)V = -h(X,pV) - NA_V X. \tag{11}$$

3. Main results

In this section, we give the definition of a generic submanifold of a Lorentzian para-Kenmotsu manifold and we obtain some results.

A generic submanifold of a Kähler manifold was defined and studied by Chen in [3]. In [21], the authors studied on generic submanifolds of a Sasakian manifold with a special vector field and they give the definition of a generic submanifold with a different way. In this paper, we follow Chen's definition for para-contact case and use this definition to construct our results for LP-Kenmotsu manifolds.

Let define the maximal invariant subspace under φ by

$$\mathcal{H}_x = T_xM \cap \varphi T_xM \qquad x \in M.$$

Definition 3. *If $\mathcal{H}_x$ has constant dimension along M, then M is called a generic submanifold of $\overline{M}$. $\mathcal{H}_x$ is said to be $\varphi-$holomorphic distribution.*

A generic submanifold M in a LP-Kenmotsu manifold $\overline{M}$ is called a semi invariant submanifold if the orthogonal complementary distribution $\mathcal{H}^{\perp}$ of $\mathcal{H}$ in TM is totally real, i.e. $\varphi\mathcal{H}^{\perp}_x \subseteq T^{\perp}_xM$, $T^{\perp}_xM$ the normal space of M at x.

Consider the orthogonal complementary distribution $\mathcal{H}^{\perp}_x$ in TM. If we have

- $\mathcal{H}_x \perp \mathcal{H}^{\perp}_x$,
- $P\mathcal{H}^{\perp}_x \subseteq \mathcal{H}^{\perp}_x$,
- $\mathcal{H}_x \cap \varphi\mathcal{H}^{\perp}_x = \{0\}$,

then $\mathcal{H}^{\perp}_x$ is called purely real $\varphi-$distribution.

Let take a vector field V in $\Gamma(T^{\perp}_xM)$. As we know $\varphi V = pV + nV$ which shows φV has tangential and normal components. Thus φV could be in TM if $nV = 0$ or in $T^{\perp}_xM$ if $p = 0$. So the the geometric properties of φV gives some important information. Suppose that also we have $\varphi V \in \Gamma(\varphi T^{\perp}_xM)$. Thus $V \in \Gamma(T^{\perp}_xM \cap \varphi T^{\perp}_xM)$. We recall these vector fields as $\varphi-$holomorphic normal vector fields. The space of these vector fields are stated by μ_x and defines a differentiable vector subbundle of $T^{\perp}M$. μ is said to be $\varphi-$holomorphic normal vector subbundle. Finally, we can say that a vector field V has components in tangential part of purely real $\varphi-$distribution and $\varphi-$holomorphic normal vector subbundle. Thus we can write

$$T^{\perp}M = N\mathcal{H}^{\perp} \oplus (T^{\perp}_xM \cap \varphi T^{\perp}_xM)$$

On the other hand p project $T^{\perp}_xM$ to $\mathcal{H}^{\perp}$ and $P\mathcal{H}^{\perp}$ become orthogonal to $\varphi-$holomorphic normal vector subbundle μ. In the other words, we have

$$T^{\perp}M = N\mathcal{H}^{\perp} \oplus \mu, \ \ p(T^{\perp}M) = \mathcal{H}^{\perp}, \ \ g(N\mathcal{H}^{\perp}, \mu) = 0.$$

By above constructions we define three distributions of a generic submanifold of a M in a LP-Kenmotsu manifold $\overline{M}$: $\varphi-$holomorphic distribution $\mathcal{H}$, purely real $\varphi-$distribution $\mathcal{H}^{\perp}$ and $\varphi-$holomorphic normal distribution μ

In the following results, by using the basic properties of generic submanifolds, we obtain integrability conditions.

Theorem 2. *The $\varphi-$holomorphic distribution $\mathcal{H}$ is always integrable.*

Proof. Firstly, for $Y \in \Gamma(\mathcal{H})$ we know that,

$$g(Y,\xi) = 0.$$

Then we conclude that,

$$g(\nabla_X Y,\xi) = -g(\nabla_X \xi, Y). \tag{12}$$

On the other hand, for all $X, Y \in \Gamma(\mathcal{H})$ we have

$$g([X,Y],\xi) = g(\nabla_X Y,\xi) - g(\nabla_Y X,\xi).$$

By virtue of (2) and (12), we get desired result. □

Theorem 3. *The holomorphic distribution $\mathcal{H}^\perp \oplus \{\xi\}$ of a generic submanifold M is integrable if and only if $(A_{nZ}W - A_{nW}Z + \nabla_Z pW - \nabla_W pZ) \in \mathcal{H}^\perp$ for any vector $Z, W \in \Gamma(\mathcal{H}^\perp)$.*

Proof. For all $Z, W \in \Gamma(\mathcal{H}^\perp)$ and $X \in \Gamma(\mathcal{H})$ we get

$$g([Z,W],\varphi X) = g(\nabla_Z W,\varphi X) - g(\nabla_W Z,\varphi X).$$

Then, using (3), we have

$$g([Z,W],\varphi X) = g(\overline{\nabla}_Z W,\varphi X) - g(\overline{\nabla}_W Z,\varphi X).$$

On the other hand, from (1), we obtain

$$g([Z,W],\varphi X) = g(\overline{\nabla}_Z \varphi W, X) - g(\overline{\nabla}_W \varphi Z, X).$$

Then, using (3), (4) and (7), we get

$$g([Z,W],\varphi X) = g(\nabla_Z pW, X) - g(A_{nW}Z, X) - g(\nabla_W pZ, X) + g(A_{nZ}W, X)$$

which completes the proof. □

Theorem 4. *The holomorphic distribution $\mathcal{H} \oplus \{\xi\}$ and its leaves are totally geodesic in M if and only if $g(h(\mathcal{H},\mathcal{H}), n\mathcal{H}^\perp) = 0$.*

Proof. For all $X, Y \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{H}^\perp)$, from (5) we get

$$g(h(X,Y), nV) = -g(A_{nV}X, Y) = g(\overline{\nabla}_X nV, Y).$$

On the other hand, using (7) we have

$$g(h(X,Y), nV) = g(\overline{\nabla}_X \varphi V, Y) - g(\overline{\nabla}_X pV, Y) = g(\overline{\nabla}_X V, \varphi Y)$$

since $\nabla_{\mathcal{H}}\mathcal{H}^\perp \subset \mathcal{H}^\perp$ we obtain

$$g(h(X,Y), nV) \quad = \quad g(\nabla_X V, Y) = 0$$

which completes the proof. □

Proposition 1. *Let M be a generic submanifold of a LP-Kenmotsu manifold $\overline{M}$. Then we have the following results;*

(1) *P is parallel if and only if $A_{NX}Y = A_{NY}X$ for any vector $X, Y \in \Gamma(TM)$.*
(2) *N is parallel if and only if $A_{nV}X = A_V PX$ for all $X \in \Gamma(TM)$, $V \in \Gamma(T^\perp M)$.*
(3) *n is parallel if and only if $A_V PU = A_U pV$ for all $U \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.*

Proof. For any vector $X, Y \in \Gamma(TM)$, using (5) and (8), we get (1). On the other hand, for all $X \in \Gamma(TM)$, $V \in \Gamma(T^\perp M)$and from (9), we get

$$g((\nabla_Y N)X, V) = g(h(Y,X), nV) - g(h(Y,PX), V).$$

Then, by using (5) we have

$$g((\nabla_Y N)X, V) = g(A_{nV}X, Y) - g(A_V PX, Y)$$

which proves (2). Finally, for all $U, X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, using (11) we have

$$g((\nabla_X n)V, U) = g(NA_V X, U) - g(h(X, tV), U).$$

After some calculations, we get

$$g((\nabla_X n)V, U) = g(X, A_V TU) - g(A_U tV, X)$$

which gives (3). □

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