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ON *-BOUNDEDNESS AND *-LOCAL BOUNDEDNESS OF NON-NEWTONIAN SUPERPOSITION OPERATORS IN $c_{0,\alpha}$ AND c_α TO $\ell_{1,\beta}$

FATMANUR ERDOĞAN AND BIRSEN SAĞIR

0000-0001-5638-5294 and 0000-0001-5954-2005

Abstract. Many investigations have been made about of non-Newtonian calculus and superposition operators until today. Non-Newtonian superposition operator was defined by Sağır and Erdoğan in [9]. In this study, we have defined *- boundedness and *-locally boundedness of operator. We have proved that the non-Newtonian superposition operator ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition (NA_2'). Then we have shown that the necessary and sufficient conditions for the *-boundedness of ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$. Finally, the similar results have been also obtained for ${}_N P_f : c_\alpha \to \ell_{1,\beta}$.

1. Introduction and Preliminaries

Non-Newtonian calculus was firstly introduced and worked by Michael Grossman and Robert Katz between years 1967 and 1970. They published the book about fundamentals of non-Newtonian calculus and which includes some special calculus such as geometric, harmonic, quadratic. Çakmak and Başar [5] obtained some results on sequence spaces with respect to non-Newtonian calculus. Duyar and Erdogan [7] worked on non-Newtonian real number series. Also, Güngör [11] studied on some geometric properties of $\ell_p(N)$.

Many studies are done until today on superposition operator which is one of the non-linear operators. Dedagich and Zabreiko [2] studied on the superposition operators in the space ℓ_p. After, some properties of superposition operator, such as boundedness, continuity, were studied by Tainchai [3], Sama-ae [4], Sağır and Güngör [6] and many others. Non-Newtonian superposition operator was defined and characterized in some non-Newtonian sequence spaces by Sağır and Erdoğan in [9]. In this article, we define *- boundedness and *-locally boundedness of operator. We prove that the non-Newtonian superposition operator ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition (NA_2'). Then we show that

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the necessary and sufficient conditions for the *-boundedness of ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$. Also the similar results are obtained for ${}_N P_f : c_\alpha \to \ell_{1,\beta}$.

A *generator* is defined as an injective function with domain $\mathbb{R}$ and the range of generator is a subset of $\mathbb{R}$. Let take any α generator with range $A = \mathbb{R}_\alpha$. Let define $\alpha-$addition, $\alpha-$subtraction, $\alpha-$multiplication, $\alpha-$division and $\alpha-$order as follows;

$\alpha-$addition	$x\dot{+}y = \alpha\left(\alpha^{-1}(x) + \alpha^{-1}(y)\right)$
$\alpha-$subtraction	$x\dot{-}y = \alpha\left(\alpha^{-1}(x) - \alpha^{-1}(y)\right)$
$\alpha-$multiplication	$x\dot{\times}y = \alpha\left(\alpha^{-1}(x) \times \alpha^{-1}(y)\right)$
$\alpha-$division	$x\dot{/}y = \alpha\left(\alpha^{-1}(x) / \alpha^{-1}(y)\right) \ (y \neq \dot{0})$
$\alpha-$order	$x\dot{<}y \ \left(x\dot{\leq}y\right) \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y) \ \left(\alpha^{-1}(x) \leq \alpha^{-1}(y)\right)$

for $x, y \in \mathbb{R}_\alpha$ [1].

$(\mathbb{R}_\alpha, \dot{+}, \dot{\times}, \dot{\leq})$ is totally ordered field [5].

The numbers $x\dot{>}\dot{0}$ are $\alpha-$positive numbers and the numbers $x\dot{<}\dot{0}$ are $\alpha-$negative numbers in $\mathbb{R}_\alpha$. $\alpha-$integers are obtained by successive $\alpha-$addition of $\dot{1}$ to $\dot{0}$ and successive $\alpha-$subtraction of $\dot{1}$ from $\dot{0}$. For each integer n, we set $\dot{n} = \alpha(n)$.

$\alpha-$absolute value of a number $x \in \mathbb{R}_\alpha$ is defined by

$$|x|_\alpha = \alpha\left(\left|\alpha^{-1}(x)\right|\right) = \begin{cases} x & if \quad x\dot{>}\dot{0} \\ \dot{0} & if \quad x = \dot{0} \\ \dot{0}\dot{-}x & if \quad x\dot{<}\dot{0} \end{cases}.$$

For $x \in \mathbb{R}_\alpha$, $\sqrt[p]{x}^{\alpha} = \alpha\left(\sqrt[p]{\alpha^{-1}(x)}\right)$ and $x^{p_\alpha} = \alpha\left\{\left[\alpha^{-1}(x)\right]^p\right\}$.

Grossman and Katz described the *-calculus with the help of two arbitrary selected generators. In this paper, we study according to *-calculus. Let take any generators α and β and let * ("star") is shown the ordered pair of arithmetics ($\alpha-$arithmetic, $\beta-$arithmetic). The following notations will be used.

	$\alpha-$arithmetic	$\beta-$ arithmetic
Realm	$A\,(=\mathbb{R}_\alpha)$	$B\,(=\mathbb{R}_\beta)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Ordering	$\dot{<}$	$\ddot{<}$

In the $*-$calculus, $\alpha-$arithmetic is used on arguments and $\beta-$arithmetic is used on values.

The isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic is the unique function $\imath$(iota) that possesses the following three properties.

1. $\imath$ is one-to-one.
2. $\imath$ is on A and onto B.
3. For any numbers u and v in A,

$$\begin{aligned} \iota\left(u\dot{+}v\right) &= \iota(u)\,\ddot{+}\,\iota(v), \ \iota\left(u\dot{-}v\right) = \iota(u)\,\ddot{-}\,\iota(v), \\ \iota\left(u\dot{\times}v\right) &= \iota(u)\,\ddot{\times}\,\iota(v), \ \iota\left(u\dot{/}v\right) = \iota(u)\,\ddot{/}\,\iota(v), v \neq \dot{0} \\ u &\dot{<} v \Longleftrightarrow \iota(u)\,\ddot{<}\,\iota(v). \end{aligned}$$

It turns out that $\iota(x) = \beta\left\{\alpha^{-1}(x)\right\}$ for every number x in A and that $\iota(\dot{n}) = \ddot{n}$ for every integer n [1].

In non-Newtonian metric space, the definitions of $\alpha-$accumulation point of a set, $\alpha-$convergence of a sequence and $\alpha-$bounded sequence have been given in the studies which are numbered[5, 10]. The definitions of *-limit and *-continuity of the function $f : X \subset \mathbb{R}_\alpha \to \mathbb{R}_\beta$ have been introduced by Sağır and Erdogan[10]. Duyar and Erdogan introduced $\alpha-$series and its $\alpha-$convergence[7].

Let X be a vector space over the field $\mathbb{R}_\alpha$ and $\|.\|_{X,\alpha}$ be a function from X to $\mathbb{R}_\alpha^+ \cup \{\dot{0}\}$ satisfying the following non-Newtonian norm axioms. For $x, y \in X$ and $\lambda \in \mathbb{R}_\alpha$,

(NN1) $\|x\|_{X,\alpha} = \dot{0} \Leftrightarrow x = \dot{0}$,

(NN2) $\|\lambda \dot{\times} x\|_{X,\alpha} = |\lambda|_\alpha \dot{\times} \|x\|_{X,\alpha}$,

(NN3) $\|x \dot{+} y\|_{X,\alpha} \dot{\leq} \|x\|_{X,\alpha} \dot{+} \|y\|_{X,\alpha}$.

Then $\left(X, \|.\|_{X,\alpha}\right)$ is said to be a non-Newtonian normed space.

The non-Newtonian sequence spaces S_α, $\ell_{\infty,\alpha}$, c_α, $c_{0,\alpha}$ and $\ell_{p,\alpha}$ over the non-Newtonian real field $\mathbb{R}_\alpha$ are defined as following:

$S_\alpha = \{x = (x_k) : \forall k \in \mathbb{N}, x_k \in \mathbb{R}_\alpha\}$

$$\ell_{\infty,\alpha} = \left\{ x = (x_k) \in S_\alpha : {}^\alpha \sup_{k\in\mathbb{N}} |x_k|_\alpha \dot{<} \dot{+}\infty \right\},$$

$$c_\alpha = \left\{ x = (x_k) \in S_\alpha : \exists l \in \mathbb{R}_\alpha \ni {}^\alpha \lim_{k\to\infty} \left|x_k \dot{-} l\right|_\alpha = \dot{0} \right\},$$

$$c_{0,\alpha} = \left\{ x = (x_k) \in S_\alpha : {}^\alpha \lim_{k\to\infty} |x_k|_\alpha = \dot{0} \right\},$$

$$\ell_{p,\alpha} = \left\{ x = (x_k) \in S_\alpha : {}_\alpha \sum_{k=1}^{\infty} |x_k|_\alpha^{p_\alpha} \dot{<} \dot{+}\infty \right\} \ (1 \leq p < \infty).$$

The sequence spaces $\ell_{\infty,\alpha}$, c_α, $c_{0,\alpha}$ are non-Newtonian normed spaces with the non-Newtonian norm $\|.\|_{\ell_{\infty,\alpha}}$ which is defined as $\|x\|_{\ell_{\infty,\alpha}} = {}^\alpha \sup_{k\in\mathbb{N}} |x_k|_\alpha$ and the sequence space $\ell_{p,\alpha}$ is a non-Newtonian normed space with the non-Newtonian norm $\|.\|_{\ell_{p,\alpha}}$ which is defined as $\|x\|_{\ell_{p,\alpha}} = \left({}_\alpha \sum_{k=1}^{\infty} |x_k|_\alpha^{p_\alpha} \right)^{\left(\frac{1}{p}\right)_\alpha}$ [5]. The $\alpha-$sequence $e_n^{(k)}$ is defined as $e_n^{(k)} = \begin{cases} \dot{1}, & k = n \\ \dot{0}, & k \neq n \end{cases}$.

Let S_N be space of non-Newtonian real number sequences, X_α be a sequence space on $\mathbb{R}_\alpha$ and Y_β be a sequence space on $\mathbb{R}_\beta$. A non-Newtonian superposition operator ${}_N P_f$ on X_α is a mapping from X_α into S_N defined by ${}_N P_f(x) = (f(k, x_k))_{k=1}^{\infty}$ where $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ satisfies condition (NA_1) as follows;

(NA_1) $f(k, \dot{0}) = \ddot{0}$ for all $k \in \mathbb{N}$.

If ${}_N P_f(x) \in Y_\beta$ for all $x = (x_k) \in X_\alpha$, we say that ${}_N P_f$ acts from X_α into Y_β and write ${}_N P_f : X_\alpha \to Y_\beta$ [9].

Also, we shall assume the following conditions:

(NA_2) $f(k, .)$ is *-continuous for all $k \in \mathbb{N}$.

(NA_2') $f(k, .)$ is $\beta-$bounded on every $\alpha-$bounded subset of $\mathbb{R}_\alpha$ for all $k \in \mathbb{N}$.

Sağır and Erdoğan [9] have characterized the non-Newtonian superposition operators ${}_N P_f$ on $c_{0,\alpha}$ and c_α as the following.

Theorem 1.1. *Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ satisfies the condition (NA_2'). Then ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ if and only if there exist a $\alpha-$number $\mu \dot{>} \dot{0}$ and a $\beta-$sequence $(c_k) \in \ell_{1,\beta}$ such that $|f(k,t)|_\beta \ddot{\leq} c_k$ when $|t|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$.*

Theorem 1.2. *Let $f:\mathbb{N}\times\mathbb{R}_\alpha\rightarrow\mathbb{R}_\beta$ satisfies the condition (NA_2'). Then ${}_NP_f: c_\alpha\rightarrow\ell_{1,\beta}$ if and only if there exist a $\alpha-$number $\mu\dot{>}\dot{0}$ and a $\beta-$sequence $(c_k)\in\ell_{1,\beta}$ such that $|f(k,t)|_\beta\ddot{\leq}c_k$ when $\left|t\dot{-}z\right|_\alpha\dot{\leq}\mu$ for all $z\in\mathbb{R}_\alpha$ and for all $k\in\mathbb{N}$.*

2. Main Results

Definition 2.1. Let (X_α,d_α) and $\left(Y_\beta,d_\beta'\right)$ be non-Newtonian sequence spaces. An operator $F:X_\alpha\rightarrow Y_\beta$ is *-bounded if $F(A)$ is $\beta-$bounded for every $\alpha-$bounded subset A of X_α .

Definition 2.2. Let (X_α,d_α) and $\left(Y_\beta,d_\beta'\right)$ be non-Newtonian sequence spaces. An operator $F:X_\alpha\rightarrow Y_\beta$ is *-locally bounded at $x_0\in X_\alpha$ if there exist $\alpha-$number $\mu\dot{>}\dot{0}$ and $\beta-$number $\eta\ddot{>}\ddot{0}$ such that $F(x)\in B_{d_\beta'}[F(x_0),\eta]$ for $x\in B_{d_\alpha}[x_0,\mu]$. F is *-locally bounded if it is *-locally bounded for every $x\in X_\alpha$.

Theorem 2.3. *Let (X_α,d_α) and $\left(Y_\beta,d_\beta'\right)$ be non-Newtonian metric sequence spaces. An operator $F:X_\alpha\rightarrow Y_\beta$ is *-locally bounded if F is *-bounded.*

Proof. Let $x\in X_\alpha$ with $x\in B_{d_\alpha}[x_0,\mu]$ for $x_0\in X_\alpha$ and $\mu\dot{>}\dot{0}$. Since F is *-bounded, $F(B_{d_\alpha}[x_0,\mu])$ is $\beta-$bounded set. Then there exists a $\beta-$number $\eta\ddot{>}\ddot{0}$ such that $d_\beta'(F(x),F(x_0))\ddot{\leq}\eta$. So we obtain that $F(x)\in B_{d_\beta'}[F(x_0),\eta]$. Thus F is *-locally bounded at $x_0\in X_\alpha$. □

Corollary 2.4. Let X_α be an $\alpha-$sequence space. $F:X_\alpha\rightarrow\ell_{1,\beta}$ is *-locally bounded if F is *-bounded.

Theorem 2.5. *If the function $f:\mathbb{N}\times\mathbb{R}_\alpha\rightarrow\mathbb{R}_\beta$ is *-locally bounded, it is satisfies the condition (NA_2').*

Proof. Let A be an $\alpha-$bounded subset of $\mathbb{R}_\alpha$. Then there exists $\dot{[}a,b\dot{]}\subset\mathbb{R}_\alpha$ such that $A\subset\dot{[}a,b\dot{]}$. Let $c\in\dot{[}a,b\dot{]}$. Since f is *-locally bounded, there exists $\delta_c\dot{>}\dot{0}$ and $\gamma_c\ddot{>}\ddot{0}$ such that

$$\left|f(x)\ddot{-}f(c)\right|_\beta\ddot{\leq}\gamma_c \text{ with } \left|x\dot{-}c\right|_\alpha\dot{\leq}\delta_c\ .$$

Then it is written that $f(x)\in B_\beta[f(c),\gamma_c]$ for $x\in B_\alpha[c,\delta_c]$. Since

$$\left||f(x)|_\beta\ddot{-}|f(c)|_\beta\right|_\beta\ddot{\leq}\left|f(x)\ddot{-}f(c)\right|_\beta\ddot{\leq}\gamma_c\ ,$$

we get

$$|f(x)|_\beta\ddot{\leq}\gamma_c\ddot{+}|f(c)|_\beta$$

when $x\in B_\alpha[c,\delta_c]$. Every $\alpha-$closed interval $\dot{[}a,b\dot{]}$ on $\mathbb{R}_\alpha$ is $\alpha-$compact by *-Heine Borel Theorem in [9]. Then there exist $c_1,c_2,...,c_n\in\dot{[}a,b\dot{]}$ such that $\dot{[}a,b\dot{]}\subset\bigcup_{k=1}^{n}B_\alpha[c_k,\delta_{c_k}]$, since $\dot{[}a,b\dot{]}\subset\bigcup_{c\in\dot{[}a,b\dot{]}}B_\alpha[c,\delta_c]$. So we have $|f(x)|_\beta\ddot{\leq}\iota(c_k)\ddot{+}|f(c_k)|_\beta$ for each $x\in B_\alpha[c_k,\delta_{c_k}]$ where $1\leq k\leq n$. If $M={}^\beta\max\left\{\iota(c_k)\ddot{+}|f(c_k)|_\beta:1\leq k\leq n\right\}$, then $|f(x)|_\beta\ddot{\leq}M$ for $x\in\bigcup_{k=1}^{n}B_\alpha[c_k,\delta_{c_k}]$. Since $A\subset\dot{[}a,b\dot{]}\subset\bigcup_{k=1}^{n}B_\alpha[c_k,\delta_{c_k}]$, we get $|f(x)|_\beta\ddot{\leq}M$ for $x\in A$. □

Theorem 2.6. *Let* $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$. *Then the non-Newtonian superposition operator* ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ *is *-locally bounded if and only if* f *satisfies the condition* (NA_2').

Proof. Assume that f satisfies the condition (NA_2'). Let $z = (z_k) \in c_{0,\alpha}$. Since ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ and f satisfies (NA_2'), by Theorem 1.1, there exist $\mu \dot{>} \dot{0}$ and $(c_k) \in \ell_{1,\beta}$ such that

$$|f(k,t)|_\beta \ddot{\leq} c_k \text{ whenever } |t|_\alpha \dot{\leq} \mu \tag{2.1}$$

for all $k \in \mathbb{N}$. Let $\varphi = \frac{\mu}{\dot{2}}\alpha$ and $x \in c_{0,\alpha}$ such that $\left\|x \dot{-} z\right\|_{c_{0,\alpha}} \dot{\leq} \varphi$. Since ${}^{\alpha}\lim_{k\to\infty} |z_k|_\alpha = \dot{0}$, there exists a positive integer r such that $|z_k|_\alpha \dot{\leq} \varphi$ for all $k \geq r$. Then

$$\|z_\lambda\|_{c_{0,\alpha}} = {}^{\alpha}\sup_{k\geq r} |z_k|_\alpha \dot{\leq} \varphi \tag{2.2}$$

for $\lambda \in \{r, r+1, ...\}$. Since $\left\|x \dot{-} z\right\|_{c_{0,\alpha}} \dot{\leq} \varphi$, we get that

$$ {}^{\alpha}\sup_{k} \left|x_k \dot{-} z_k\right|_\alpha \dot{\leq} \varphi \tag{2.3}$$

By (2.2) and (2.3), it is written that

$$\begin{aligned} |x_k|_\alpha &\dot{\leq} {}^{\alpha}\sup_{n\geq r} |x_n|_\alpha \\ &= {}^{\alpha}\sup_{n\geq r} \left|x_n \dot{-} z_n \dot{+} z_n\right|_\alpha \\ &\dot{\leq} {}^{\alpha}\sup_{n\geq r} \left|x_n \dot{-} z_n\right|_\alpha \dot{+} {}^{\alpha}\sup_{n\geq r} |z_n|_\alpha \\ &\dot{\leq} \varphi \dot{+} \varphi \\ &= \mu \end{aligned}$$

for all $k \geq r$. From (2.1), we have $|f(k,x_k)|_\beta \ddot{\leq} c_k$ for all $k \geq r$. Then

$$ {}_{\beta}\sum_{k=r}^{\infty} |f(k,x_k)|_\beta \ddot{\leq} {}_{\beta}\sum_{k=r}^{\infty} c_k = {}_{\beta}\sum_{k=r}^{\infty} |c_k|_\beta \ddot{\leq} {}_{\beta}\sum_{k=1}^{\infty} |c_k|_\beta = \|(c_k)\|_{\ell_{1,\beta}} . \tag{2.4}$$

Let $m_k = {}^{\beta}\sup_{\left|t \dot{-} z_k\right|_\alpha \dot{\leq} \varphi} |f(k,t)|_\beta$ for all $k \in \mathbb{N}$. Since f satisfies the condition (NA_2'), it is seen that $m_k \ddot{<} \ddot{+}\infty$ for all $k \in \mathbb{N}$. So we get $\left|x_k \dot{-} z_k\right|_\alpha \dot{\leq} \varphi$ for all $k \in \mathbb{N}$ by (2.3). Then we have

$$|f(k,x_k)|_\beta \ddot{\leq} m_k \tag{2.5}$$

for all $k \in \mathbb{N}$. Using the relations (2.4) and (2.5), it is obtained that

$$\begin{aligned} \|{}_N P_f(x)\|_{\ell_{1,\beta}} &= {}_{\beta}\sum_{k=1}^{\infty} |f(k,x_k)|_\beta \\ &= {}_{\beta}\sum_{k=1}^{r-1} |f(k,x_k)|_\beta \ddot{+} {}_{\beta}\sum_{k=r}^{\infty} |f(k,x_k)|_\beta \\ &\ddot{\leq} {}_{\beta}\sum_{k=1}^{r-1} m_k \ddot{+} \|(c_k)\|_{\ell_{1,\beta}} . \end{aligned}$$

Then we have

$$\begin{aligned}\left\|{}_NP_f(x)\ddot{-}{}_NP_f(z)\right\|_{\ell_{1,\beta}} &\ddot{\leq} \left\|{}_NP_f(x)\right\|_{\ell_{1,\beta}}\ddot{+}\left\|{}_NP_f(z)\right\|_{\ell_{1,\beta}} \\ &\ddot{\leq} {}_\beta\sum_{k=1}^{r-1} m_k\ddot{+}\left\|(c_k)\right\|_{\ell_{1,\beta}}\ddot{+}\left\|{}_NP_f(z)\right\|_{\ell_{1,\beta}}\ .\end{aligned}$$

Therefore we get that

$$\left\|{}_NP_f(x)\ddot{-}{}_NP_f(z)\right\|_{\ell_{1,\beta}}\ddot{\leq}\gamma \text{ when } \gamma=\left\|{}_NP_f(z)\right\|_{\ell_{1,\beta}}\ddot{+}{}_\beta\sum_{k=1}^{r-1} m_k\ddot{+}\left\|(c_k)\right\|_{\ell_{1,\beta}}.$$

Hence, the non-Newtonian operator ${}_NP_f$ *-locally bounded at z.

Conversely assume that ${}_NP_f : c_{0,\alpha}\to\ell_{1,\beta}$ is *-locally bounded. Let $k\in\mathbb{N}$ and $b\in\mathbb{R}_\alpha$. Let $y=(y_n)$ be defined as $y_n=\begin{cases} b\ , & n=k \\ \dot{0}\ , & n\neq k\end{cases}$. Then $(y_n)\in c_{0,\alpha}$. By assumption, there exist $\mu\dot{>}\dot{0}$ and $\eta\ddot{>}\ddot{0}$ such that

$$\left\|{}_NP_f(x)\ddot{-}{}_NP_f(y)\right\|_{\ell_{1,\beta}}\ddot{\leq}\eta \text{ whenever } \left\|x\dot{-}y\right\|_{c_{0,\alpha}}\dot{\leq}\mu. \tag{2.6}$$

Let $a\in\mathbb{R}_\alpha$ with $\left|a\dot{-}b\right|_\alpha\dot{\leq}\mu$ and let $x=(x_n)$ with $x_n=\begin{cases} a\ , & n=k \\ \dot{0}\ , & n\neq k\end{cases}$. Then $x\in c_{0,\alpha}$. Since

$$\left\|x\dot{-}y\right\|_{c_{0,\alpha}}={}^\alpha\sup_n\left|x_n\dot{-}y_n\right|_\alpha=\left|a\dot{-}b\right|_\alpha\dot{\leq}\mu,$$

we get $\left\|{}_NP_f(x)\ddot{-}{}_NP_f(y)\right\|_{\ell_{1,\beta}}\ddot{\leq}\eta$ by (2.6). Then we have

$$\begin{aligned}\left|f(k,a)\ddot{-}f(k,b)\right|_\beta &\ddot{\leq} {}_\beta\sum_{n=1}^{\infty}\left|f(n,x_n)\ddot{-}f(n,y_n)\right|_\beta \\ &= \left\|{}_NP_f(x)\ddot{-}{}_NP_f(y)\right\|_{\ell_{1,\beta}} \\ &\ddot{\leq} \eta\end{aligned}$$

Hence $f(k,.)$ is *-locally bounded at b. Since $b\in\mathbb{R}_\alpha$ is arbitrary, $f(k,.)$ is *-locally bounded. Thus $f(k,.)$ satisfies the condition (NA_2'). □

Corollary 2.7. Let $f:\mathbb{N}\times\mathbb{R}_\alpha\to\mathbb{R}_\beta$ satisfies the condition (NA_2). The non-Newtonian superposition operator ${}_NP_f$ is *-locally bounded if ${}_NP_f : c_{0,\alpha}\to\ell_{1,\beta}$.

Corollary 2.8. Let $f:\mathbb{N}\times\mathbb{R}_\alpha\to\mathbb{R}_\beta$. If ${}_NP_f : c_{0,\alpha}\to\ell_{1,\beta}$ is *-bounded, f satisfies the condition (NA_2').

Proposition 2.9. Assume that $f:\mathbb{N}\times\mathbb{R}_\alpha\to\mathbb{R}_\beta$ satisfies the condition (NA_2'). If for each $\mu\dot{>}\dot{0}$ there exists a $\beta-$number $\eta(\mu)\ddot{>}\ddot{0}$ such that

$${}_\beta\sum_{k=1}^{\infty}\left|f(k,x_k)\right|_\beta\ddot{\leq}\eta(\mu)\ \text{ whenever }\ \left|x_k\right|_\alpha\dot{\leq}\mu$$

for all $k\in\mathbb{N}$, then there exists a $c(\mu)=(c_k(\mu))\in\ell_{1,\beta}$ with $c_k(\mu)\ddot{\geq}\ddot{0}$ and $\left\|c(\mu)\right\|_{\ell_{1,\beta}}\ddot{\leq}\eta(\mu)$ for all $k\in\mathbb{N}$ such that

$$\left|f(k,t)\right|_\beta\ddot{\leq}c_k(\mu)\ \text{ whenever }\ \left|t\right|_\alpha\dot{\leq}\mu\ .$$

Proof. Let $\mu\dot{>}\dot{0}$. We define

$$A(\mu)=\left\{t\in\mathbb{R}_\alpha:|t|_\alpha\dot{\leq}\mu\right\}\text{ and }c_k(\mu)={}^{\beta}\sup\left\{|f(k,t)|_\beta:t\in A(\mu)\right\}$$

for all $k\in\mathbb{N}$. Then $|f(k,t)|_\beta\ddot{\leq}c_k(\mu)$ where $|t|_\alpha\dot{\leq}\mu$. Since f satisfies the condition (NA_2'), it is obtained that $\ddot{0}\ddot{\leq}c_k(\mu)\ddot{<}\ddot{+}\infty$ for all $k\in\mathbb{N}$. For each $\varepsilon\ddot{>}\ddot{0}$. there exists an α-sequence $x=(x_k)$ when $|x_k|_\alpha\dot{\leq}\mu$ such that

$$c_k(\mu)\ddot{\leq}|f(k,x_k)|_\beta\ddot{+}\frac{\varepsilon}{\ddot{2}^{k_\beta}}\beta \tag{2.7}$$

for all $k\in\mathbb{N}$. By (2.7), we have

$$_\beta\sum_{k=1}^{\infty}c_k(\mu)={}_\beta\sum_{k=1}^{\infty}|c_k(\mu)|_\beta\ddot{\leq}{}_\beta\sum_{k=1}^{\infty}|f(k,x_k)|_\beta\ddot{+}{}_\beta\sum_{k=1}^{\infty}\frac{\varepsilon}{\ddot{2}^{k_\beta}}\beta\ddot{\leq}\eta(\mu)\ddot{+}\varepsilon\ .$$

Thus, $\|c_k(\mu)\|_{\ell_{1,\beta}}\ddot{\leq}\eta(\mu)\ddot{+}\varepsilon$. Since ε is arbitrary, it is written that $\|c(\mu)\|_{\ell_{1,\beta}}\ddot{\leq}\eta(\mu)$ with $c(\mu)=(c_k(\mu))$. So there exists a β-sequence $c(\mu)=(c_k(\mu))\in\ell_{1,\beta}$ with $c_k(\mu)\ddot{\geq}\ddot{0}$ and $\|c(\mu)\|_{\ell_{1,\beta}}\ddot{\leq}\eta(\mu)$ such that $|f(k,t)|_\beta\ddot{\leq}c_k(\mu)$ whenever $|t|_\alpha\dot{\leq}\mu$ for each $k\in\mathbb{N}$. □

Theorem 2.10. *Let $f:\mathbb{N}\times\mathbb{R}_\alpha\to\mathbb{R}_\beta$. The non-Newtonian superposition operator $_NP_f:c_{0,\alpha}\to\ell_{1,\beta}$ is *-bounded if and only if for all $\mu\dot{>}\dot{0}$ there exists a β-sequence $c(\mu)=(c_k(\mu))\in\ell_{1,\beta}$ such that*

$$|f(k,t)|_\beta\ddot{\leq}c_k(\mu)\ \textit{ whenever }\ |t|_\alpha\dot{\leq}\mu$$

for each $k\in\mathbb{N}$.

Proof. Let $x\in c_{0,\alpha}$ and $\mu\dot{>}\dot{0}$ with $\|x\|_{c_{0,\alpha}}\ddot{\leq}\mu$. Then $|x_k|_\alpha\dot{\leq}\mu$ for all $k\in\mathbb{N}$. By the hypothesis, there exists a β-sequence $c(\mu)=(c_k(\mu))\in\ell_{1,\beta}$ such that $|f(k,t)|_\beta\ddot{\leq}c_k(\mu)$ for each $k\in\mathbb{N}$. Then

$$\|_NP_f(x)\|_{\ell_{1,\beta}}={}_\beta\sum_{k=1}^{\infty}|f(k,x_k)|_\beta\ddot{\leq}{}_\beta\sum_{k=1}^{\infty}c_k(\mu)={}_\beta\sum_{k=1}^{\infty}|c_k(\mu)|_\beta=\|c(\mu)\|_{\ell_{1,\beta}}\ .$$

Thus, $_NP_f:c_{0,\alpha}\to\ell_{1,\beta}$ is *-bounded.

Conversely, assume that $_NP_f:c_{0,\alpha}\to\ell_{1,\beta}$ is *-bounded. Let $\mu\dot{>}\dot{0}$. Then for each $x\in c_{0,\alpha}$ with $\|x\|_{c_{0,\alpha}}\ddot{\leq}\mu$, it is obtained that

$$\|_NP_f(x)\|_{\ell_{1,\beta}}={}_\beta\sum_{k=1}^{\infty}|f(k,x_k)|_\beta\ddot{\leq}\eta(\mu)\ddot{<}\ddot{+}\infty$$

for a β-positive integer $\eta(\mu)$. By Corollary 2.8, f satisfies the condition (NA_2'). In view of Proposition 2.9, there exists a β-sequence $c(\mu)=(c_k(\mu))\in\ell_{1,\beta}$ with $\|c(\mu)\|_{\ell_{1,\beta}}\ddot{\leq}\eta(\mu)$ such that $|f(k,t)|_\beta\ddot{\leq}c_k(\mu)$ whenever $|t|_\alpha\dot{\leq}\mu$ for each $k\in\mathbb{N}$. □

Example 2.11. Let function $f:\mathbb{N}\times\mathbb{R}_\alpha\to\mathbb{R}_\beta$ be defined by $f(k,t)=\dfrac{|\iota(t)|_\beta}{\ddot{5}^{k_\beta}}\beta$ for all $k\in\mathbb{N}$ and $t\in\mathbb{R}_\alpha$. Since there exist $\gamma=\dot{1}$ and $(c_k)=\left(\dfrac{\ddot{1}}{\ddot{5}^{k_\beta}}\beta\right)\in\ell_{1,\beta}$ such that $|f(k,t)|_\beta\ddot{\leq}c_k$ whenever $|t|_\alpha\dot{\leq}\dot{1}$ for each $k\in\mathbb{N}$, the non-Newtonian superposition operator ${}_N P_f$ acts from $c_{0,\alpha}$ to $\ell_{1,\beta}$. Let $\mu \dot{>} \dot{0}$ and $t \in \mathbb{R}_\alpha$ with $|t|_\alpha \dot{\leq} \mu$. Then, for all $k \in \mathbb{N}$

$$|f(k,t)|_\beta = \frac{|\iota(t)|_\beta}{\ddot{5}^{k_\beta}} \beta \ddot{\leq} \frac{\iota(\mu)}{\ddot{5}^{k_\beta}} \beta \text{ and } {}_\beta\sum_{k=1}^{\infty} \frac{\iota(\mu)}{\ddot{5}^{k_\beta}} \beta = \left(\frac{\iota(\mu)}{\ddot{4}^{k_\beta}} \beta\right) .$$

Hence we obtaine that $|f(k,t)|_\beta \ddot{\leq} c_k(\mu)$ whenever $(c_k(\mu)) = \left(\frac{\iota(\mu)}{\ddot{5}^{k_\beta}} \beta\right) \in \ell_{1,\beta}$ for all $k \in \mathbb{N}$. Then, ${}_N P_f : c_{0,\alpha} \to \ell_{1,\beta}$ is *-bounded by Theorem 2.10.

Theorem 2.12. *Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$. The non-Newtonian superposition operator ${}_N P_f : c_\alpha \to \ell_{1,\beta}$ is *-locally bounded if and only if f satisfies the condition (NA_2').*

Proof. Assume that f satisfies the condition (NA_2'). Let $z = (z_k) \in c_\alpha$. By Theorem 1.2 there exist $\mu \dot{>} \dot{0}$ and $(c_k) \in \ell_{1,\beta}$ such that

$$|f(k,t)|_\beta \ddot{\leq} c_k \text{ whenever } \left|t \dot{-} a\right|_\alpha \dot{\leq} \mu \tag{2.8}$$

for each $a \in \mathbb{R}_\alpha$ and for all $k \in \mathbb{N}$. Let $\eta \dot{>} \dot{0}$ and $x \in c_\alpha$ with $\left\|x \dot{-} z\right\|_{c,\alpha} \dot{\leq} \eta$. Since $x \in c_\alpha$, there exists $a \in \mathbb{R}_\alpha$ such that

$${}^\alpha\lim_{k\to\infty} \left|x_k \dot{-} a\right|_\alpha = \dot{0} . \tag{2.9}$$

From (2.8), there exist a $\rho \dot{>} \dot{0}$ and a $(c_k) \in \ell_{1,\beta}$ such that

$$|f(k,t)|_\beta \ddot{\leq} c_k \text{ whenever } \left|t \dot{-} a\right|_\alpha \dot{\leq} \rho \tag{2.10}$$

for all $k \in \mathbb{N}$. By (2.9), there exists $i \in \mathbb{N}$

$$\left|x_k \dot{-} a\right|_\alpha \dot{\leq} \rho \tag{2.11}$$

for all $k \geq i$. By (2.10) and (2.11), we obtaine that $|f(k,x_k)|_\beta \ddot{\leq} c_k$ for all $k \geq i$. Then

$${}_\beta\sum_{k=i}^{\infty} |f(k,x_k)|_\beta \ddot{\leq} {}_\beta\sum_{k=i}^{\infty} c_k = {}_\beta\sum_{k=i}^{\infty} |c_k|_\beta \ddot{\leq} {}_\beta\sum_{k=1}^{\infty} |c_k|_\beta = \|c_k\|_{\ell_{1,\beta}} \tag{2.12}$$

Let $m_k = {}^\beta\sup_{\left|t \dot{-} z_k\right|_\alpha \dot{\leq} \eta} |f(k,t)|_\beta$ for each $k \in \mathbb{N}$. Since f satisfies the condition (NA_2'), $m_k \ddot{<} \ddot{+}\infty$ for all $k \in \mathbb{N}$. Since $\left\|x \dot{-} z\right\|_{c,\alpha} \dot{\leq} \eta$, we have that $\left|x_k \dot{-} z_k\right|_\alpha \dot{\leq} \eta$ for all $k \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$

$$|f(k,x_k)|_\beta \ddot{\leq} m_k \tag{2.13}$$

By (2.12) and (2.13),

$$\begin{aligned} \|{}_N P_f(x)\|_{\ell_{1,\beta}} &= {}_\beta\sum_{k=1}^{\infty} |f(k,x_k)|_\beta \\ &= {}_\beta\sum_{k=1}^{i-1} |f(k,x_k)|_\beta \ddot{+} {}_\beta\sum_{k=i}^{\infty} |f(k,x_k)|_\beta \\ &\ddot{\leq} {}_\beta\sum_{k=1}^{i-1} m_k \ddot{+} \|(c_k)\|_{\ell_{1,\beta}} . \end{aligned}$$

Then

$$\left\|{}_{N}P_{f}\left(x\right)\ddot{-}\ {}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}} \ddot{\leq} \left\|{}_{N}P_{f}\left(x\right)\right\|_{\ell_{1,\beta}} \ddot{+} \left\|{}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}} \ddot{\leq} {}_{\beta}\sum_{k=1}^{i-1} m_{k} \ddot{+} \left\|(c_{k})\right\|_{\ell_{1,\beta}} \ddot{+} \left\|{}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}} .$$

Therefore we have that

$$\left\|{}_{N}P_{f}\left(x\right)\ddot{-}\ {}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}} \ddot{\leq}\gamma \text{ when } \gamma = \left\|{}_{N}P_{f}\left(z\right)\right\|_{\ell_{1,\beta}} \ddot{+}\ {}_{\beta}\sum_{k=1}^{i-1} m_{k} \ddot{+} \left\|(c_{k})\right\|_{\ell_{1,\beta}} .$$

Hence ${}_{N}P_{f}$ *-locally bounded at z.

Conversely, assume that ${}_{N}P_{f} : c_{\alpha} \to \ell_{1,\beta}$ is *-locally bounded. Let $k \in \mathbb{N}$ and $b \in \mathbb{R}_{\alpha}$. Let $y = (y_n)$ be as follows

$$y_n = \begin{cases} b\ , & n = k \\ \dot{0}\ , & n \neq k \end{cases}$$

for all $k \in \mathbb{N}$ and $b \in \mathbb{R}_{\alpha}$. Then $y \in c_{\alpha}$. By the hypothesis, there exist $\mu \dot{>} \dot{0}$ and $\varphi \ddot{>} \ddot{0}$ such that

$$\left\|{}_{N}P_{f}\left(x\right)\ddot{-}\ {}_{N}P_{f}\left(y\right)\right\|_{\ell_{1,\beta}} \ddot{\leq}\varphi \text{ whenever } \left\|x\dot{-}y\right\|_{c,\alpha} \dot{\leq}\mu\ . \tag{2.14}$$

Let $a \in \mathbb{R}_{\alpha}$ with $\left|a\dot{-}b\right|_{\alpha} \dot{\leq}\mu$ and $x = (x_n)$ with $x_n = \begin{cases} a\ , & n = k \\ \dot{0}\ , & n \neq k \end{cases}$. Then $x \in c_{\alpha}$. Since

$$\left\|x\dot{-}y\right\|_{c,\alpha} = {}^{\alpha}\sup_{n}\left|x_n\dot{-}y_n\right|_{\alpha} = \left|a\dot{-}b\right|_{\alpha} \dot{\leq}\mu,$$

by virtue of (2.14), it is written that $\left\|{}_{N}P_{f}\left(x\right)\ddot{-}\ {}_{N}P_{f}\left(y\right)\right\|_{\ell_{1,\beta}} \ddot{\leq}\varphi$. Then we have

$$\left|f\left(k,a\right)\ddot{-}f\left(k,b\right)\right|_{\beta} \ddot{\leq} {}_{\beta}\sum_{n=1}^{\infty}\left|f\left(n,x_n\right)\ddot{-}f\left(n,y_n\right)\right|_{\beta} = \left\|{}_{N}P_{f}\left(x\right)\ddot{-}\ {}_{N}P_{f}\left(y\right)\right\|_{\ell_{1,\beta}} \ddot{\leq} \varphi$$

Therefore $f\left(k,.\right)$ is *-locally bounded at b. Since $b \in \mathbb{R}_{\alpha}$ is arbitrary, $f\left(k,.\right)$ is *-locally bounded. Hence $f\left(k,.\right)$ satisfies the condition (NA_2'). □

Corollary 2.13. Let the function $f : \mathbb{N} \times \mathbb{R}_{\alpha} \to \mathbb{R}_{\beta}$ satisfy the condition (NA_2). Then ${}_{N}P_{f} : c_{\alpha} \to \ell_{1,\beta}$ is *-locally bounded.

Corollary 2.14. Let $f : \mathbb{N} \times \mathbb{R}_{\alpha} \to \mathbb{R}_{\beta}$. If ${}_{N}P_{f} : c_{\alpha} \to \ell_{1,\beta}$ is *-bounded, f satisfies the condition (NA_2').

Theorem 2.15. *Let $f : \mathbb{N} \times \mathbb{R}_{\alpha} \to \mathbb{R}_{\beta}$. ${}_{N}P_{f} : c_{\alpha} \to \ell_{1,\beta}$ is *-bounded if and only if for every $\mu \dot{>} \dot{0}$ there exists a sequence $c\left(\mu\right) = \left(c_k\left(\mu\right)\right) \in \ell_{1,\beta}$ such that*

$$\left|f\left(k,t\right)\right|_{\beta} \ddot{\leq} c_k\left(\mu\right) \text{ whenever } \left|t\right|_{\alpha} \dot{\leq}\mu$$

for all $k \in \mathbb{N}$.

Proof. Let $\mu \dot{>} \dot{0}$ and $x \in c_\alpha$ with $\|x\|_{c,\alpha} \dot{\leq} \mu$. Then $|x_k|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$. By the hypothesis, for each $k \in \mathbb{N}$ there exists a sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ such that $|f(k, x_k)|_\beta \ddot{\leq} c_k(\mu)$. Then it is written that

$$\|{}_N P_f(x)\|_{\ell_{1,\beta}} = {}_\beta\sum_{k=1}^{\infty} |f(k, x_k)|_\beta \ddot{\leq} {}_\beta\sum_{k=1}^{\infty} c_k(\mu) = {}_\beta\sum_{k=1}^{\infty} |c_k(\mu)|_\beta = \|c(\mu)\|_{\ell_{1,\beta}} .$$

Thus ${}_N P_f : c_\alpha \to \ell_{1,\beta}$ is *-bounded.

Conversely, assume that ${}_N P_f : c_\alpha \to \ell_{1,\beta}$ is *-bounded. Let $\mu \dot{>} \dot{0}$. There exists a positive $\beta-$number $\eta(\mu)$ such that

$$\|{}_N P_f(x)\|_{\ell_{1,\beta}} = {}_\beta\sum_{k=1}^{\infty} |f(k, x_k)|_\beta \ddot{\leq} \eta(\mu)$$

for each $x \in c_\alpha$ with $\|x\|_{c,\alpha} \dot{\leq} \mu$. From Corollary 2.13, f satisfies the condition (NA_2'). By Proposition 2.9, there exists a β-sequence $c(\mu) = (c_k(\mu)) \in \ell_{1,\beta}$ with $\|c(\mu)\|_{\ell_{1,\beta}} \ddot{\leq} \eta(\mu)$ such that $|f(k,t)|_\beta \ddot{\leq} c_k(\mu)$ whenever $|t|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$. □

Example 2.16. Let $f : \mathbb{N} \times \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be as follows

$$f(k,t) = \frac{(\iota(t))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta$$

for all $k \in \mathbb{N}$. Let $\mu \dot{>} \dot{0}$ and $t \in \mathbb{R}_\alpha$ with $|t|_\alpha \dot{\leq} \mu$. Then

$$|f(k,t)|_\beta = \frac{(\iota(t))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta \ddot{\leq} \frac{(\iota(\mu))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta$$

for each $k \in \mathbb{N}$. Since

$${}_\beta\sum_{k=1}^{\infty} \frac{(\iota(\mu))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta = (\iota(\mu))^{2_\beta} \ddot{\times} {}_\beta\sum_{k=1}^{\infty} \frac{\ddot{1}}{\ddot{5}^{k_\beta}} \beta = (\iota(\mu))^{2_\beta} \ddot{\times} \frac{\ddot{1}}{\ddot{5}} \beta \ddot{\times} \frac{\ddot{1}}{\ddot{1} \ddot{-} \frac{\ddot{1}}{\ddot{5}} \beta} \beta = \frac{(\iota(\mu))^{2_\beta}}{\ddot{4}} \beta \ddot{<} \ddot{+}\infty,$$

we have $|f(k,t)|_\beta \ddot{\leq} c_k$ when $c_k = \dfrac{(\iota(\mu))^{2_\beta}}{\ddot{5}^{k_\beta}} \beta$ for each $k \in \mathbb{N}$. Hence ${}_N P_f : c_\alpha \to \ell_{1,\beta}$ is *-bounded by Theorem 2.15.

3. Conclusion

In this paper, the well-known boundedness and locally boundedness in classical calculus were extended to non-Newtonian calculus. Also their properties on some non-Newtonian sequence spaces were investigated.

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(F.Erdoğan) Ondokuz Mayis University, Institute of Graduate Education, Department of Mathematics, 55200, Samsun, TURKEY
Email address, F.Erdoğan: `fatmanurkilic89@hotmail.com`

(B.Sağır) Ondokuz Mayis University, Faculty of Sciences and Arts, Department of Mathematics, 55200, Samsun, TURKEY
Email address, B.Sağır: `bduyar@omu.edu.tr`