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The Quasi Parallel Curve of a Space Curve

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Abstract

A parallel or offset curve is defined as a curve whose points are a fixed distance from a given curve. These curves are not parallel transport. Often, it has a more complex mathematical structure than the first curve. Offset curves are important in numerically controlled machining, for example, where a biaxial machine defines the shape of the cut made with a round cutting tool. In this study, the quasi parallel curve is defined with the help of the quasi frame of a given curve. According to all selections of the projection vector, the quasi frame equations of this curve are expressed in terms of the quasi elements of the given curve. The curvatures of the quasi parallel curve were obtained depending on the quasi curvatures of the main curve. The study was supported by examples. The examples confirm that the quasi parallel curve is not parallel transport.

Keywords: Space curves, parallel curves, Frenet frame, q-frame

1. Introduction

Curves are the building blocks of differential geometry. The study of special curves has an important place in the theory of curves. A parallel curve or offset is defined as a curve whose points are at a constant normal distance from a given curve. Some of the offset curves that have been subject to many studies are involute evolute offsets, Bertrand offsets, Mannheim offsets curves. The authors studied Mannheim partner curves [1, 2]. The offset surface of a surface is at a constant distance from the surface. The offset surface appears as the expansion or contraction of the main surface. The authors are define Bertrand offset ruled surfaces and investigate the developable ruled surface conditions of these surfaces, [3]. In [4], They using networks of bicubic patches for approximating the offsets to general piecewise parametric surfaces. In [5,6], the authors are define the Mannheim offsets and the involute-evolute offsets of ruled surfaces. Offset surfaces are also seen in other subjects, [7,8]. Parallel curves are important, for example, in numerically controlled machining, where a two-axis machine defines the shape of the cut made with a circular cutting tool. The authors investigated motion of parallel curves and surfaces in Euclidean 3-space, [9]. The Frenet frame defined on the curve gives the properties of the curve. Alternative frame definitions are available in the literature for cases where the Frenet frame cannot be defined. Rotation minimizing frame, which is among the most important of these, was defined by bishop, [10]. In [11], they defined a new frame for a space curve called as the q-frame. This frame can be defined even along a line. They They obtain some conditions for the given curve to be the inextensible flows of curves using the q-frame, [12]. The authors have researched the translation surfaces using q-frame, [13]. In [14] by using directional quasi fields , they obtain new optical conditions of quasi magnetic Lorentz flux. The authors have found relations between the motion of curves according to q-frame and the motion of their spherical image,[15]. In this paper, we define parallel curve using the q-frame of a curve and obtain its q-frame and q-curvature according to q-frame of given curve. Also, the study was supported by examples.

2. Materials and Methods

Let $\alpha : I \to IR^3$, $s \to \alpha(s)$ be a unit speed curve. The orthonormal frame $\{T(s), N(s), B(s)\}$ is called the Frenet frame of the curve α, where $T(s)$, $N(s)$, $B(s)$ are the unit tangent, the principal normal and the binormal vector fields of α, respectively, and they are defined by

$$\begin{aligned} T(s) &= \alpha'(s), \\ N(s) &= \alpha''(s)/\|\alpha''(s)\|, \\ B(s) &= T(s)\times N(s). \end{aligned} \qquad \textbf{(2.1)}$$

The curvature $\kappa(s)$ and torsion $\tau(s)$ of α are given by $\kappa(s) = \|\alpha''(s)\|, \tau(s) = \langle N'(s), B(s)\rangle$. Then the famous Frenet formula is,

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix}\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}. \qquad \textbf{(2.2)}$$

The q-frame $\{T, N_q, B_q, k\}$ of the curve $\alpha(s)$ is given by

$$T = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \quad N_q = \frac{T\times k}{\|T\times k\|}, \quad B_q = T\times N_q, \qquad \textbf{(2.3)}$$

where k is the projection vector.

If T and k are parallel, then the projection vector can be k = (1, 0, 0) or k = (0, 1, 0) or k = (0, 0, 1), [11].
The relationship matrix between the Frenet frame $\{T, N, B\}$ and the q-frame $\{T, N_q, B_q, k\}$ of the curve $\alpha(s)$ is given by

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}, \qquad \textbf{(2.4)}$$

where the angle θ between the principal normal N and quasi-normal N_q vectors. The first-order angular variation of the q-frame may be expressed as

$$\begin{bmatrix} T' \\ N'_q \\ B'_q \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix}\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}, \qquad \textbf{(2.5)}$$

where $k_1 = \kappa\cos\theta$, $k_2 = -\kappa\sin\theta$, $k_3 = d\theta + \tau$ are the q-curvatures of the curve $\alpha(s)$, [11]

2. Results and Discussion

Definition: The quasi parallel curve or q-parallel curve $\bar{\alpha}(\bar{s})$ of a unit speed curve $\alpha(s)$ is defined by as

$$\bar{\alpha}(\bar{s}) = \alpha(s) + dB_q(s) \qquad \textbf{(3.1)}$$

where d is a nonzero real constant, $s = s(\bar{s})$ is the arc length of $\alpha(s)$ and $B_q(s)$ is q-binormal vector fields of the curve $\alpha(s)$, and

$$\Omega = ds/d\bar{s} = 1/\sqrt{(1-dk_2)^2 + (dk_3)^2}.$$

If the derivative is taken from Eq. (3.1) according to the arc length s and usinq Eq. (2.5), then the tangent vector $\bar{T}$ of q-parallel curve $\bar{\alpha}(\bar{s})$ is

$$\bar{T} = \Omega(1-dk_2)T - d\Omega k_3 N_q. \qquad \textbf{(3.2)}$$

The q-normal vector $\bar{N}_q$ and q-binormal vector $\bar{B}_q$ of q-parallel curve $\bar{\alpha}(\bar{s})$ are

$$\bar{N}_q = \frac{\bar{T}\times k}{\|\bar{T}\times k\|}, \quad \bar{B}_q = \bar{T}\times\bar{N}_q, \qquad \textbf{(3.3)}$$

where k is the projection vector.

Usinq Eqs. (3.1) and (3.3), we can write as follows corollories.

Corollary 1. If choosen the projection vector k = (1, 0, 0), then the relationship between the q- frame $\{\bar{T}, \bar{N}_q, \bar{B}_q\}$ of q-parallel cuve $\bar{\alpha}(\bar{s})$ and the q-frame $\{T, N_q, B_q\}$ of the curve $\alpha(s)$ is given by

$$\begin{aligned} \bar{T} &= \bar{k_1}T - \bar{k_2}N_q \\ \bar{N}_q &= \bar{k_2}B_q \\ \bar{B}_q &= -\bar{k_2}^2 T - \bar{k_1}\bar{k_2}N_q \end{aligned}, \qquad \textbf{(3.4)}$$

where $\bar{k_1} = \Omega(1-dk_2)$, $\bar{k_2} = d\Omega k_3$.

Corollary 2. If choosen the projection vector k = (0, 1, 0), then the relationship between the q- frame

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$\left\{\overline{T}, \overline{N}_q, \overline{B}_q\right\}$ of q-parallel cuve $\overline{\alpha}\left(\overline{s}\right)$ and the q-frame $\left\{T, N_q, B_q\right\}$ of the curve $\alpha(s)$ is given by

$$\begin{aligned}\overline{T} &= \overline{k_1}\,T - \overline{k_2}\,N_q\\ \overline{N}_q &= \overline{k_1}\,B_q \qquad ,\\ \overline{B}_q &= -\overline{k_1}\,\overline{k_2}\,T - \overline{k_1}^{\,2}\,N_q\end{aligned} \tag{3.5}$$

where $\overline{k_1} = \Omega(1-dk_2)$, $\overline{k_2} = d\Omega k_3$.

Corollary 3. If the projection vector k = (0, 0, 1) , then the relationship between the q- frame $\left\{\overline{T}, \overline{N}_q, \overline{B}_q\right\}$ of q-parallel cuve $\overline{\alpha}\left(\overline{s}\right)$ and the q-frame $\left\{T, N_q, B_q\right\}$ of the curve $\alpha(s)$ is given by

$$\begin{aligned}\overline{T} &= \overline{k_1}\,T - \overline{k_2}\,N_q\\ \overline{N}_q &= -\overline{k_2}\,T - \overline{k_1}\,N_q \qquad ,\\ \overline{B}_q &= -\left(\overline{k_1}^{\,2} + \overline{k_2}^{\,2}\right)B_q\end{aligned} \tag{3.6}$$

where $\overline{k_1} = \Omega(1-dk_2)$, $\overline{k_2} = d\Omega k_3$.

Example 1: The unit speed helix curve $\alpha(s) = \left(\frac{3}{5}\sin s, \frac{3}{5}\cos s, \frac{4}{5}s\right)$ (Figure 1., blue), has the following q-frame

$$\begin{aligned}T &= \left(\frac{3}{5}\cos s, -\frac{3}{5}\sin s, \frac{4}{5}\right)\\ N_q &= \left(-\sin s, -\cos s, 0\right) \qquad ,\\ B_q &= \left(\frac{4}{5}\cos s, -\frac{4}{5}\sin s, -\frac{3}{5}\right)\end{aligned}$$

where the projection vector is k = (0, 0, 1) .
The q-parallel curve $\overline{\alpha}\left(\overline{s}\right)$ of the curve $\alpha(s)$ is

$$\overline{\alpha}\left(\overline{s}\right) = \begin{pmatrix}\frac{32}{5}\cos s + \frac{3}{5}\sin s,\\ \frac{3}{5}\cos s - \frac{32}{5}\sin s,\ \frac{4}{5}s - \frac{24}{5}\end{pmatrix}$$

where d=8, (Figure 1., red).

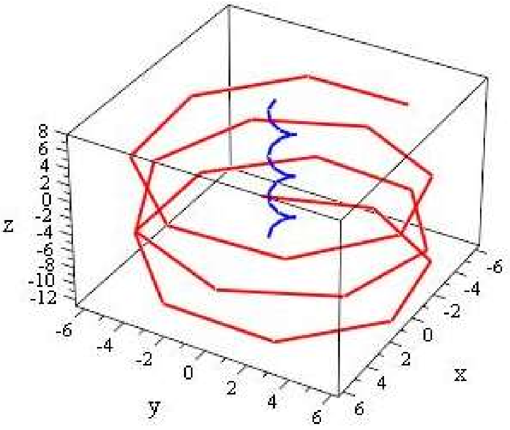

Figure 1. The helix curve and its q-parallel curve.

Example 2 : The curve $\alpha(s) = \left(s^3, s, s^2\right)$ (Figure 2., blue), has the following q-frame

$$\begin{aligned}T &= \frac{1}{\sqrt{9s^4+4s^2+1}}\left(3s^2, 1, 2s\right)\\ N_q &= \frac{1}{\sqrt{9s^4+1}}\left(1, -3s^2, 0\right) \qquad ,\\ B_q &= \frac{1}{\sqrt{\left(9s^4+4s^2+1\right)\left(9s^4+1\right)}}\left(6s^3, 2s, -9s^4-1\right)\end{aligned}$$

where the projection vector is k = (0, 0, 1) .
The q-parallel curve $\overline{\alpha}\left(\overline{s}\right)$ of the curve $\alpha(s)$ is

$$\overline{\alpha}\left(\overline{s}\right) = \begin{pmatrix} s^3 + \dfrac{48}{\sqrt{\left(9s^4+1\right)\left(9s^4+4s^2+1\right)}},\\ s + \dfrac{16s}{\sqrt{\left(9s^4+1\right)\left(9s^4+4s^2+1\right)}},\\ s^2 - \dfrac{8\left(9s^4+1\right)}{\sqrt{\left(9s^4+1\right)\left(9s^4+4s^2+1\right)}}\end{pmatrix}$$

where d=8, (Figure 2., red).

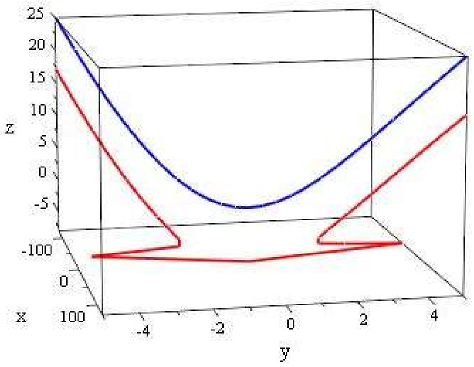

Figure 2. The curve $\alpha(s)$ and its q-parallel curve.

Example 3: The curve

$$\alpha(s)=\left(\frac{4}{5}\cos s,1-\sin s,-\frac{3}{5}\cos s\right)$$

(Figure 3., blue), has the following q-frame

$$T=\left(-\frac{4}{5}\sin s,\ -\cos s,\ \frac{3}{5}\sin s\right)$$

$$N_q=\left(-\frac{3}{5},0,-\frac{4}{5}\right),$$

$$B_q=\left(-\frac{4}{5}\cos s,-\frac{7}{25}\sin s,-\frac{3}{5}\cos s\right)$$

where the projection vector is $k=(0, 1, 0)$.
The q-parallel curve $\bar{\alpha}\left(\bar{s}\right)$ of the curve $\alpha(s)$ is

$$\bar{\alpha}\left(\bar{s}\right)=\left(-\frac{36}{5}\cos s,1\text{-}\frac{19}{5}\sin s,-\frac{33}{5}\cos s\right)$$

where d=10, (Figure 3, red).

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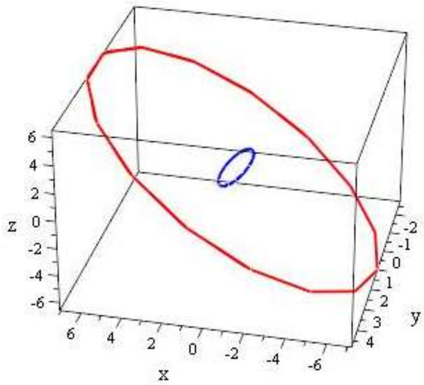

Figure 3. The curve $\alpha(s)$ and its q-parallel curve.

3. Conclusion

The parallel or offset curve points is a curve with a fixed distance from a given curve which are not parallel transport. In this study, we defined the q-parallel curve of a given curve. The equations of the q- frame of the q-parallel curve were obtained according to q-frame of given curve. Also, the curvatures of q-parallel curve are obtain. It was supported by the examples given that the q-parallel curve is not a parallel transport of the main curve.

Author's Contributions

Fatma Güler: Drafted and wrote the manuscript.

Ethics

There are no ethical issues after the publication of this manuscript.

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