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RESEARCH ARTICLE

ON RINGS IN WHICH ALL UNITS CAN BE PRESENTED IN THE FORM $\mathbf{1 + eR(1 - e)}$

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ABSTRACT

The aim of the present paper is to characterize associative rings R with unity in which $1 + eR(1 - e) = U(R)$ in terms of some important class of rings in the literature (for example, *NR*-rings, *UU*-rings, *UJ*-rings, *UR*-rings, exchange rings, 2-primal rings), where $e^2 = e \in R$ and $U(R)$ is the set of units of R.

Keywords: NR-rings, UU-rings, UJ-rings, UR-rings, Exchange rings

1. INTRODUCTION

For an associative ring R with unity, the Jacobson radical, the set of nilpotent elements and the set of units of R are denoted by $J(R)$, $N(R)$ and $U(R)$, respectively. We write $M_n(R)$, $T_n(R)$ and $R[t]$ for the $n \times n$ matrix ring, the $n \times n$ upper triangular matrix ring, and the polynomial ring over R, respectively.

A regular right self-injective ring R is purely infinite if it contains no nonzero directly finite central idempotents. It is well known that the group of units of a purely infinite regular right self-injective ring R is perfect and generated by transvections, i.e. units of the form $1 + x$ such that $x^2 = 0$ (equivalently, $x \in eR(1 - e)$ with $e^2 = e \in R$ in case R is a regular ring). In the case where R is the endomorphism ring of an infinite-dimensional vector space over a division ring has been obtained by Rosenberg [10].

In this study, some comparisons between NR-rings, UU-rings, UJ-rings, UR-rings, exchange rings, 2-primal rings, are investigated. We show that if R is a ring in which $1 + eR(1 - e) = U(R)$, then R is a UJ-ring (if every unit can be presented in a form $1 + x$, for some $x \in J(R)$ [5] (see also [6] and [11])) and R is an NR-ring (if the set of nilpotent elements of a ring is a subring [1]). In particular, we obtain that ring for which $1 + eR(1 - e) = U(R)$ satisfy Köthe's conjecture. A ring R is called 2-primal (respectively, NI-ring) if the set of nilpotent elements of the ring coincides with the prime radical (respectively, if the set of nilpotent elements of a ring is an ideal). We obtain that these notions coincide with over a ring R in which $1 + eR(1 - e) = U(R)$ whereas 2-primal rings are *NI*-ring, but the converse need not hold in general. Interestingly, if R is an exchange ring (if for each $a \in R$ there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$ [8]) in which $1 + eR(1 - e) = U(R)$ then, R is an NI-ring if and only if R is a 2-primal ring if and only if $J(R) = N(R) = eR(1 - e)$ is an ideal.

We also show that UR-rings, rings with a unique regular element without assuming commutativity [3] satisfy the presentation $1 + eR(1 - e) = U(R)$. Although whenever $n > 1$, the matrix ring $M_n(R)$ does not have the UR-property, we have that, over a ring R in which $1 + eR(1 - e) = U(R)$, eRe is a UR-ring if and only if $(1 - e)R(1 - e)$ is a UR-ring. On the other hand, it is shown that R is a UR-ring if and only if $1 + eR(1 - e) = U(R)$ and $e = eU(R)$.

Finally, the behavior of UJ-rings under various algebraic construction is investigated.

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2. THE RESULTS

If the set of nilpotent elements of a ring is a subring, then the ring is called an NR-ring (see [1]).

The ring R is said to be of bounded index if there exists a positive integer n such that $x^n = 0$ for all $x \in N(R)$, and R is of bounded index n if n is the least integer with this property.

A ring R is called UJ-ring if every unit can be presented in a form $1 + x$, for some $x \in J(R)$ ([5]).

Proposition 2.1 *Let R be a ring in which $1 + eR(1 - e) = U(R)$ for some $e^2 = e \in R$. Then:*

(1) $J(R) \subseteq eR(1 - e) \subseteq N(R)$.
(2) $eR(1 - e) = N(R)$.
(3) $N^2(R) = 0$ and R is of bounded index 2.
(4) $J(R) = N(R)$.
(5) R is a UJ-ring.
(6) $N(R)$ is additively closed.
(7) R is an NR-ring.

Proof. (1) Clearly, $eR(1 - e) \subseteq N(R)$. Let $x \in J(R)$. Then $1 + x \in U(R) = 1 + eR(1 - e)$ and so $x \in eR(1 - e)$, as desired.
(2) If $x \in N(R)$, then $1 + x \in U(R) = 1 + eR(1 - e)$ which implies $x \in eR(1 - e)$.
(3) This follows from (2).
(4) Since $N(R)$ is a nilpotent ideal, we get $N(R) \subseteq J(R)$.
(5) This follows from $1 + J(R) = 1 + N(R) = U(R)$.
(6) Clearly, if $x, y \in N(R)$ then $x + y \in N(R)$.
(7) This follows from (6).

Let us notice that $1 + eR(1 - e) \subseteq 1 + N(R)$ is always contained in $U(R)$. Recall that UU-rings, defined as rings with $U(R) = 1 + N(R)$ (i.e., rings with unipotent units) were studied in detail by Danchev and Lam in [7].

Remark 2.2 *Let R be a ring in which $1 + eR(1 - e) = U(R)$ for some $e^2 = e \in R$. It is clear that R is a UU-ring but the converse is not true. For example, $\mathbb{Z}_4$ is a UU-ring while has not the property $1 + eR(1 - e) = U(R)$.*

A ring R satisfies Köthe's conjecture if every nil left ideal of R is contained in a nil two-sided ideal. For a ring R and for two elements $x, y \in R$, we denote the operation $\circ$ by $x \circ y = x + y - xy$. Then $(R, \circ)$ is a monoid.

Lemma 2.3 *[12, Theorem 2.1] The following are equivalent for a ring R:*

(1) $N(R)$ is additively closed.
(2) $N(R)$ is multiplicatively closed and R satisfies Köthe's conjecture.
(3) $N(R)$ is closed under $\circ$.
(4) $N(R)$ is a subring of R.

Corollary 2.4 *Let R be a ring in which $1 + eR(1 - e) = U(R)$ for some $e^2 = e \in R$. Then R satisfies Köthe's conjecture.*

Proof. This follows from Proposition 2.1 and Lemma 2.3.

A ring R is called 2-primal if the set of nilpotent elements of the ring coincides with the prime radical $Nil_*(R)$ (i.e., $Nil_*(R) = N(R)$).

If the set of nilpotent elements of a ring is an ideal, then the ring is called an NI-ring (i.e., $Nil^*(R) = N(R)$).

Remark 2.5 *It is obvious that 2-primal rings are NI-rings, but the converse need not hold.*

Corollary 2.6 *Then the following statements are equivalent for a ring R in which $1 + eR(1 - e) = U(R)$ for some $e^2 = e \in R$:*
1. *R is an NI-ring,*
2. *R is a 2-primal ring.*

Proof. This follows from Proposition 2.1 and [4, Proposition 1.4].

A ring R is called exchange ([8]) if for each $a \in R$ there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$.

Corollary 2.7 *If R is an exchange ring in which $1 + eR(1 - e) = U(R)$ for some $e^2 = e \in R$ then:*
1. *R is an NI-ring.*
2. *$J(R) = N(R) = eR(1 - e)$ is an ideal.*
3. *R is a 2-primal ring.*
4. *$Nil_*(R) = Nil^*(R) = N(R) = J(R) = eR(1 - e)$.*

Proof. (1) This follows from Proposition 2.1.
(2) It is straightforward by the definition of the NI-ring.
(3) This follows from Corollary 2.6.
(4) It is straightforward by the definition of the 2-primal ring.

Cohn [2] introduced the term "0-ring" for commutative rings with 1, in which every element different from 1 is a zero-divisor (0-rings are also known as Cohn's rings, see for example [9]). Examples of 0-rings are Boolean rings.

Moreover, Henriksen in [3] introduced the concept of UR-rings, rings with a unique regular element without assuming commutativity, and generalized the concept of Cohn's rings.

Corollary 2.8 *Let R be a UR-ring. Then $1 + eR(1 - e) = U(R)$ for some $e^2 = e \in R$.*

Proof. Note that if $a = er(1 - e)$ is an arbitrary element in $eR(1 - e)$ then $a^2 = 0$. Hence $1 + a$ is a unit element and we conclude that $\{1\} \subseteq 1 + eR(1 - e) \subseteq U(R)$. First, suppose that R is a UR-ring. Then $U(R) = \{1\}$ and hence e $\{1\} \subseteq 1 + eR(1 - e) \subseteq U(R) = \{1\}$. Thus $1 + eR(1 - e) = U(R)$ (in particular, we have $eR(1 - e) = 0$).

Remark that the converse of Lemma 2.8 is true for the class of abelian rings (i.e., if every idempotent is central).

The following example shows the converse of Corollary 2.8 is not true in general. Also, it shows $1 + eR(1 - e) = U(R)$ does not imply $1 + (1 - e)Re = U(R)$.

Example 2.9 *Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $1 + eR(1 - e) = U(R)$ while is not a UR-ring.*

We have the following facts.

Proposition 2.10 *Let R be a ring in which $1 + eR(1-e) = U(R)$ for some $e^2 = e \in R$. Then:*

(1) $e = ue$ *for every* $u \in U(R)$
(2) $(1-e) = (1-e)u$ *for every* $u \in U(R)$
(3) $eU(R)e = e$
(4) $(1-e)U(R)(1-e) = (1-e)$.
(5) $(1-e)Re = 0$
(6) $eRe = Re$
(7) $(1-e)R(1-e) = (1-e)R$.
(8) $R = \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix}$
(9) $R = \begin{bmatrix} Re & eR(1-e) \\ 0 & (1-e)R \end{bmatrix}$

Proof. (1)-(4) It is clear. We only prove (5)-(9).
(5) Since the inclusion $(1-e)Re \subseteq N(R)$ always holds, we obtain $1 + (1-e)Re \subseteq U(R)$. By the assumption, $1 + (1-e)Re \subseteq 1 + eR(1-e)$ and so $(1-e)Re \subseteq eR(1-e)$. But, $(1-e)Re \cap eR(1-e) = \{0\}$. Therefore $(1-e)Re = 0$.
(6) By the Pierce decomposition, we have

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$$

Hence $Re = eRe \oplus 0 \oplus (1-e)Re \oplus 0$. By (5), $Re = eRe \oplus 0 \oplus 0 \oplus 0$. This means that $Re = eRe$.
(7) This is similar to (6).
(8) By the Pierce decomposition of $R = \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix}$ and (5), we have

$$R = \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix}.$$

(9) This is clear from (6), (7) and (8).

The following observations characterize UR-rings in which $1 + eR(1-e) = U(R)$.

Lemma 2.11 *Let R be a ring and $e \in R$ be an idempotent. Then the following are equivalent:*

(1) *For all* $x \in R$, $exe \in U(eRe)$
(2) *For all* $x \in R$, $exe + 1 - e$ *is unit*
(3) *For all* $x \in R$, $ex + 1 - e$ *is unit.*

Proof. Clear.

Let us observe that whenever $n > 1$, the matrix ring $\mathrm{M_n}(R)$ does not have the UR-property.

Theorem 2.12 *Let R be a ring in which $1 + eR(1-e) = U(R)$. Then:*

(1) eRe *is a* UR*-ring.*
(2) $(1-e)R(1-e)$ *is a* UR*-ring.*

Proof. (1) Let $exe \in U(eRe)$. Then $exe + 1 - e \in U(R)$. By Proposition 2.10 and Lemma 2.11, we have $e = (exe + 1 - e)e$ and so $e = exe$. Therefore e is the only unit in eRe. This means that eRe is a UR-ring.
(2) This is similar to (1).

The following example shows that the converse of Theorem 2.12 is not true in general.

Example 2.13 *Let* $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ *and* $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. *Then*

$$eRe = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$$

and

$$(1-e)R(1-e) = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is clear that eRe and $(1-e)R(1-e)$ are UR-rings but $1 + eR(1-e) \subset U(R)$.

Lemma 2.14 *Let R be a ring in which $1 + eR(1-e) = U(R)$. Then*

(1) $U(R) = \begin{bmatrix} e & eR(1-e) \\ 0 & 1-e \end{bmatrix}$.

(2) $eR(1-e) = eU(R) - e$.

Proof. (1) Let $u \in U(R)$. By Proposition 2.10 and Lemma 2.14, we conclude that

$$u = \begin{bmatrix} e & eR(1-e) \\ 0 & 1-e \end{bmatrix}.$$

(2) By the assumption, $1 + eR(1-e) = U(R)$, we have

$$\begin{aligned} e + eR(1-e) &= eU(R) \\ eR(1-e) &= eU(R)(1-e) \\ eR(1-e) &= eU(R)(1-e) \\ &= eU(R) - eU(R)e \end{aligned}$$

Now, by Proposition 2.10, we get $e = eUe$. Hence $eR(1-e) = eU(R) - e$.

Theorem 2.15 *The following statements are equivalent for a ring R:*

(1) R is a UR-ring.

(2) $1 + eR(1-e) = U(R)$ and $e = eU(R)$.

Proof. $(1) \Rightarrow (2)$ This is clear from Corollary 2.8.
$(2) \Rightarrow (1)$ By Lemma 2.14, we have $eR(1-e) = eU(R) - e = e - e = 0$. Hence $\{1\} = U(R)$ and we are done.

2. Some Algebraic Properties of Rings R Having the Representation $1 + eR(1-e) = U(R)$

For a ring R, we fix the following notation for convenience:

$$(P) \qquad 1 + eR(1-e) = U(R) \text{ for some } e^2 = e \in R.$$

Proposition 3.1 *Let R be a ring.*

(1) The ring $\prod_{i\in I} R_i$ has the representation (P) if and only if rings R_i have the representation (P), for all $i \in I$.

(2) (For any nil ideal $I \subseteq R$, if R has the representation (P), then R/I has the representation (P).

(3) Assume R has the representation (P). Then

  (a) $2 \in J(R)$

  (b) If R is a division ring, then $R = \mathbb{F}_2$. (i.e. the finite field of two elements).

  (c) If $J(R)$ is nil and $eR(1-e) \subseteq J(R)$ then $R/J(R)$ is reduced and hence abelian.

(4) Let $J(R)$ be a nil ideal. Then a (semi) local ring R has the representation (P) if and only if $R/J(R) \cong F_2 \times \cdots \times F_2$.

(5) If the polynomial ring $R[x]$ has the representation (P), then R has the representation (P).

Proof. (1) This is clear since

$$U\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} U(R_i) \text{and} \prod_{i\in I} R_i(1-e) = \prod_{i\in I} eR_i(1-e).$$

(2) It is enough to show that $\overline{1} + \overline{e}R/I\overline{(1-e)} \supseteq U(R/I)$. Let $\overline{u} \in U(R/I)$. Then there exists $\overline{v} \in U(R/I)$ such that $\overline{uv} = \overline{1}$, that is $uv - 1 \in I$. Hence $(uv-1)^n = 0$ for some $n > 0$. Now

$$\begin{aligned} 1_R &= (uv-1)^n + 1 \\ &= (uv - 1_R + 1_R)((-1)^{n-1}(uv-1_R)^{n-1} + \cdots + (-1)^0 1_R) \\ &= (uv)((-1)^{n-1}(uv-1_R)^{n-1} + \cdots + (-1)^0 1_R) \\ &= u[v((-1)^{n-1}(uv-1_R)^{n-1} + \cdots + (-1)^0 1_R)]. \end{aligned}$$

That is u is a unit in R. Since R has the representation (P) that is $u \in 1 + eR(1-e)$, clearly $u + I \in [1 + eR(1-e)] + I$ so there exists $r \in R$ such that

$$u + I = [1 + er(1-e)] + I = (1+I) + [(e+I)(r+I)(1-e+I)] \in \overline{1} + \overline{e}R/I\overline{(1-e)},$$

which completes the proof.

For the next conditions, we introduce an alternating proof of (2):

Alternating proof: If R is a division ring then it is well-known that there are no nilpotent elements other than 0 and no idempotent elements other than 0 and 1. Therefore a division ring R which has the representation (P) has only trivial units and so $R = \mathbb{F}_2$.

(3) (a) Write $-1 = 1 + er(1-e)$ so $-2 \in J(R)$ since $eR(1-e) = N(R) = J(R)$ by Lemma 2.1.

(b) If R is a division ring, then $e = 0$ or $e = 1$. Therefore, $U(R) = 1_R$ which gives the result.

(c) Let $a + J(R)$ be a nilpotent element of $R/J(R)$. We show that $a \in J(R)$. By the nilpotency of $a + J(R)$, we get $(a+1) + J(R) \in U\big(R/J(R)\big)$. Since $J(R)$ is nil, $R/J(R)$ has the representation (P). So $(a+1) + J(R) = [1 + J(R)] + [er(1-e) + J(R)]$ for some $r \in R$. Then $a + J(R) \in eR(1-e) \subseteq N(R) \subseteq J(R)$ which proves $a \in J(R)$.

(4) Since $R/J(R)$ is semisimple by the definition and reduced by (3c), we obtain that $R/J(R)$ is a finite direct product of division rings. Hence (3b) completes the proof.

(5) This is clear.

Corollary 3.2 *Let $T_n(R)$ be the $n \times n$ upper triangular matrices over a ring R, where $n \geq 1$ is a fixed integer. If $T_n(R)$ has the representation (P) then R has the representation (P).*

Proof. Let $I = [\mathrm{a_{ij}}] \in T_n$ with all $a_{ii} = 0$. It is a nil ideal in T_n, with $T_n/I \cong R^n$. Therefore, the result follows from Proposition 3.1 (1) and (2).

As we mentioned before, rings with $1 + eR(1-e) = U(R)$ are easy to describe using Pierce decomposition. Namely, it is clear that the imposed condition implies $(1-e)Re = 0$. In particular $eR(1-e)$ is an ideal (equal to the Jacobson radical of R) and $R/eR(1-e)$ has trivial unit (i.e., the only unit is 1). This yields that following Corollary 3.2.

Corollary 3.3 *A ring R has the representation (P) if and only if there are rings S and T with trivial units (in particular they are reduced) and an (S,T)-bimodule M such that R is the triangular ring* $\begin{bmatrix} Se & M \\ 0 & T \end{bmatrix}$.

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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