ELASTIC CURVES IN THE GALILEAN PLANE

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ABSTRACT

An elastic curve or elastica introduced by Jacques Bernoulli in 1692 is the solution of a variational problem which minimizes the integral of the total squared curvature for curves of a fixed length satisfying given first order boundary conditions. Many works related to elastica problem, which plays a large role in many area such as engineering, computer science, biology, chemistry, ship building, bridges ,DNA etc in our life have been done by many researchers in Euclidean and non-Euclidean spaces.

In this work, we consider the classical variational problem in the Galilean plane. We derive Euler-Lagrange equation as a second order differential equation. Then, we obtain the curvature of the elastic curves parameterized by arc length in such a plane. Next, we give an example which represents the position vector of an elastic curve in explicit form in the Galilean plane.

Keywords: Elastic curve, Elastica, Galilean plane, Euler-Lagrange equation.

1. Introduction

Differential geometry of curves appears in many area such as engineering, computer science, biology, chemistry, ship building etc. One of the most fundamental problems on the curve theory is how to characterize special curves in different types of planes and spaces.

The elastic curve problem has been considered using different approaches since the middle of the 18^{th} century. It was first posed by Galileo in 1639 as a solution of the problem which asked about weight required to break a beam set into a wall. Many researchers followed Galileo's results in coming decade. However, the complete solution of the elastica problem was obtained by Euler in 1744 by developing the variational method. In 1740, J. Bernoulli posed a geometric model for the elastic curve problem for which the bending energy functional $E = \int \kappa^2 \, ds$ minimizes and solved the problem to characterize the family of curves known as elastic curves or elastica by using calculus of variations. Elastic curves classified in Euclidean plane by Euler in 1743 were studied by Radon who derived and integrated the Euler-Lagrange equations in 1928 [6, 11].

In the 19^{th} century, A. Cayley and F. Klein discovered the hyperbolic and elliptic geometry by considering Euclidean and non-Euclidean geometries as mathematical structures living inside projective-metric spaces and they outlined their idea with respect to the real projective plane. These hyperbolic and elliptic geometries established by A. Cayley and F. Klein are known as Cayley-Klein geometries (Table 1.1) [3].

Table 1.1: Cayley-Klein Geometries in the Plane

		Measure of length between two points		
		Elliptic ($\boldsymbol{\kappa_1 > 0}$)	**Parabolic ($\boldsymbol{\kappa_1 = 0}$)**	**Hyperbolic ($\boldsymbol{\kappa_1 < 0}$)**
Measure of angles between two lines	**Elliptic ($\boldsymbol{\kappa_2 > 0}$)**	Elliptic Geometry	Euclidean Geometry	Hyperbolic Geometry
	Parabolic ($\boldsymbol{\kappa_2 = 0}$)	co-Euclidean Geometry (Anti-Newton Hooke)	Galilean Geometry	co-Minkowskian Geometry (Newton-Hooke)
	Hyperbolic ($\boldsymbol{\kappa_2 < 0}$)	co-Hyperbolic Geometry (Anti-De-Sitter)	Minkowskian Geometry	doubly-Hyperbolic Geometry (De-Sitter)

In this work we consider the problem of elastic curves in Galilean plane which is a Cayley-Klein plane.

Although the notion of elastica is quite old, similar problems have been still studied by many authors in Euclidean and non-Euclidean spaces by using classical and modern approaches [1,2,3,4, etc.]. J. Langer and D. A. Singer classified all closed elastica in the Euclidean space [5]. Then, D.A Singer obtained and solved Euler-Lagrange equations of the elastic curves in three-dimensional Euclidean space [7]. In recent years, the characterizations of some curves in Galilean spaces are obtained [8,9,10]. In [8], the authors derived Euler-Lagrange and solved the elastica problem for the 3-dimensional Galilean space.

The local theory of curves states that a curve lies in a plane if and only if its torsion vanishes. The classification of the elastic curves in the Galilean plane can easily be concluded from equations given in [8] for $\tau = 0$. The results given in the present work are deduced for the Galilean plane from the project course of the students enrolled in the TEBIP High Performers Program .

In this paper, we derive Euler-Lagrange equations for a non-isotropic curve in the Galilean plane and then we obtain the curvature of the elastic curves by solving Euler-Lagrange equation. Next, we give an example for an elastic curve in such a plane.

2. Preliminaries

Consider the Euclidean plane $\mathbb{R}^2$ equipped the bilinear form

$$< \boldsymbol{x}, \boldsymbol{y} > = \mathcal{E}_1 x_1 y_1 + \mathcal{E}_2 x_2 y_2 \,, \quad (\mathcal{E}_1, \mathcal{E}_2 = 0,\ \pm 1)$$

where $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$.

When $\mathcal{E}_1 = \mathcal{E}_2 = 1$ we have Euclidean plane denoted by E^2,
when $\mathcal{E}_1 = 1$ and $\mathcal{E}_2 = -1$ we obtain Lorentz-Minkowski plane denoted by L^2 and
when $\mathcal{E}_1 = 1$ and $\mathcal{E}_2 = 0$ or $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 1$ obtain Galilean plane denoted by G^2 [3],[7],[9].

A vector whose the first component is zero is an isotropic vector and a vector whose the first component is nonzero is called a non-isotropic vector.
The inner or scalar product of the vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ is defined by

$$< \boldsymbol{x}, \boldsymbol{y} >= \begin{cases} x_1 y_1, & if\ x_1 \neq 0\ or\ y_1 \neq 0 \\ x_2 y_2, & if\ x_1 = 0\ or\ y_1 = 0. \end{cases} \tag{2.1}$$

Hence a Galilean plane is a Cayley-Klein plane equipped with the scalar product given by (2.1).

The vectoral product of the vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ is given by

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{cases} \begin{vmatrix} \boldsymbol{0} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}, & if\ x_1 \neq 0\ or\ y_1 \neq 0 \\ \\ \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}, & if\ x_1 = 0\ or\ y_1 = 0 \end{cases} \tag{2.2}$$

The metric of the vectors $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ is defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} |x_2 - x_1|, & if\ x_1 \neq x_2 \\ |y_2 - y_1|, & if\ x_1 = x_2 \end{cases} \tag{2.3}$$

Let $\gamma : I = [a_1, a_2] \subset \mathbb{R} \rightarrow G_2$ be a smooth curve parameterized by the arc length parameter "s", where the arc length is a Galilean invariant.

The Frenet vector fields along $\gamma(s)$ in G_2 are defined by

$$\boldsymbol{T}(s) = \frac{d\gamma}{ds} \quad \text{and} \quad \boldsymbol{N}(s) = \frac{1}{\kappa(s)} \frac{d\boldsymbol{T}}{ds}, \tag{2.4}$$

where $\kappa(s)$, $\boldsymbol{T}(s)$ and $\boldsymbol{N}(s)$ are the curvature, tangent and principal normal vector fields of the curve, respectively.

A curve whose tangent vector is an isotropic is called an isotropic curve, otherwise it is called a non-isotropic curve.

In Galilean plane, every non-isotropic curve $\gamma(s)$ with unit speed can be parameterized as

$$\gamma(s) = \big(s, x(s)\big), \quad (\|\gamma'(s)\| = 1) \tag{2.5}$$

which implies that

$$\frac{d\gamma}{ds} = \boldsymbol{T}(s) = \big(1, x'(s)\big), \tag{2.6}$$

$$\frac{d^2\gamma}{ds^2} = \frac{d\boldsymbol{T}}{ds} = \big(0, x''(s)\big) \tag{2.7}$$

and

$$\frac{d\boldsymbol{N}}{ds} = \frac{d}{ds}\left(\frac{\boldsymbol{T'}}{\|\boldsymbol{T'}\|}\right) = \frac{d}{ds}\left[\frac{\big(0, x''(s)\big)}{x''}\right] = \frac{d}{ds}[(0,1)] = (0,0).$$

Hence the orthogonal Frenet frame of the curve $\gamma(s)$ is obtained as

$$\begin{aligned} \boldsymbol{T}'(s) &= \kappa \boldsymbol{N}, \\ \boldsymbol{N}'(s) &= \boldsymbol{0}, \end{aligned} \tag{2.8}$$

where

$$\kappa(s) = \frac{\|\gamma' \wedge \gamma''\|}{[(\gamma')^2]^{3/2}} = \frac{x''}{1} = x''.$$

3. Euler-Lagrange Equations in the Galilean Plane

The solution of the elastica problem gives the smooth curves, which minimize total squared curvature given by

$$\int \kappa(s)^2 \, ds = \int_{a_1}^{a_2} \kappa(t)^2 \, v(t) \, dt \tag{3.1}$$

among curves of the same length under the first order boundary conditions.

By the classical Lagrange multiplier method, there is a multiplier $\Lambda \in \mathbb{R}$ such that any critical point γ is also a critical point of the one-parameter family of functionals

$$\int_{\gamma} \kappa(s)^2 \, ds + \Lambda \int_{\gamma} ds \tag{3.2}$$

with fixed length and the same boundary condition.

Assume that $\gamma : I = [a_1, a_2] \subset \mathbb{R} \to G_2$ is an extremum of (3.2).

If

$$\Omega = \{\gamma \mid \gamma(a_i) = \gamma_i, \; \gamma'(a_i) = \gamma_i'\}$$

is the space of the curves and

$$\Omega_u = \{\gamma \in \mathbb{R} \mid \|\gamma'\| \equiv 1\}$$

is the subspace of limit speed curves then (3.2) can be written as the functional

$$\mathcal{F}^{\Lambda}(\gamma) = \int [\, \|\gamma'\|^2 + \Lambda(s)(\|\gamma'\|^2 - 1)\,]\, ds, \tag{3.3}$$

where $\Lambda(s)$ is a pointwise Lagrange multiplier.

Then, if $\boldsymbol{W}$ is a vector field along γ then an infinitesimal variation of the curve satisfies the equation

$$\frac{\partial \mathcal{F}^{\Lambda}}{\partial \varepsilon}(\boldsymbol{W}) = \frac{\partial}{\partial \varepsilon}\mathcal{F}(\gamma + \varepsilon \boldsymbol{W})_{\varepsilon=0} \tag{3.4}$$

From (3.3) and (3.4) it can be easily seen that

$$\begin{aligned}
0 &= \frac{\partial}{\partial \varepsilon}\int_{a_1}^{a_2} [\langle \gamma'' + \varepsilon \boldsymbol{W}'', \gamma'' + \varepsilon \boldsymbol{W}''\rangle + \Lambda(s)\langle \gamma' + \varepsilon \boldsymbol{W}', \gamma' + \varepsilon \boldsymbol{W}'\rangle - \Lambda(s)]\, ds \,\Big|_{\varepsilon = 0} \\
&= \frac{\partial}{\partial \varepsilon}\int_{a_1}^{a_2} [\langle \gamma'', \gamma''\rangle + 2\varepsilon\langle \gamma'', \boldsymbol{W}''\rangle + \varepsilon^2\langle \boldsymbol{W}'', \boldsymbol{W}''\rangle \\
&\qquad + \Lambda(s)(\langle \gamma', \gamma'\rangle + 2\varepsilon\langle \gamma', \boldsymbol{W}'\rangle + \varepsilon^2\langle \boldsymbol{W}', \boldsymbol{W}'\rangle) - \Lambda(s)]\, ds \,\Big|_{\varepsilon = 0} \\
&= \int_{a_1}^{a_2} [2\langle \gamma'', \boldsymbol{W}''\rangle + 2\varepsilon\langle \boldsymbol{W}'', \boldsymbol{W}''\rangle - \Lambda(s)(\, 2\langle \gamma', \boldsymbol{W}'\rangle + 2\varepsilon\langle \boldsymbol{W}', \boldsymbol{W}'\rangle)\,]\, ds \,\Big|_{\varepsilon = 0} \\
&= 2\int_{a_1}^{a_2} [\langle \gamma'', \boldsymbol{W}''\rangle - \Lambda(s)\langle \gamma', \boldsymbol{W}'\rangle]\, ds \,\Big|_{\varepsilon = 0} \,.
\end{aligned} \tag{3.5}$$

Integration by parts gives

$$\begin{aligned}
0 &= \langle \gamma'', \boldsymbol{W}'\rangle\Big|_{a_1}^{a_2} - \int_{a_1}^{a_2} \langle \gamma''', \boldsymbol{W}'\rangle\, ds - \langle \Lambda\gamma', \boldsymbol{W}\rangle\Big|_{a_1}^{a_2} + \int_{a_1}^{a_2} \langle (\Lambda\gamma')', \boldsymbol{W}\rangle\, ds \\
&= \langle \gamma'', \boldsymbol{W}'\rangle\Big|_{a_1}^{a_2} - \langle \gamma''', \boldsymbol{W}\rangle\Big|_{a_1}^{a_2} + \int_{a_1}^{a_2} \langle \gamma^{(4)}, \boldsymbol{W}\rangle\, ds - \langle \Lambda\gamma', \boldsymbol{W}\rangle\Big|_{a_1}^{a_2} + \int_{a_1}^{a_2} \langle (\Lambda\gamma')', \boldsymbol{W}\rangle\, ds \\
&= \int_{a_1}^{a_2} \langle \gamma^{(4)} + (\Lambda\gamma')', \boldsymbol{W}\rangle\, ds + [\langle \gamma'', \boldsymbol{W}'\rangle - \langle \gamma''' + \Lambda\gamma', \boldsymbol{W}\rangle]\Big|_{a_1}^{a_2} ,
\end{aligned} \tag{3.6}$$

or putting

$$J = \gamma''' + \Lambda\gamma'$$

and

$$E(\gamma) = \gamma^{(4)} + (\Lambda\gamma')'$$

(3.6) reduces to

$$0 = \int_{a_1}^{a_2} \langle E(\gamma), \boldsymbol{W} \rangle ds + [\langle \gamma'', \boldsymbol{W}' \rangle - \langle J, \boldsymbol{W} \rangle] \Big|_{a_1}^{a_2}. \quad (3.7)$$

Since (3.7) is hold for any $\boldsymbol{W}$, which is any vector field along the critical γ the equation

$$E(\gamma) = \gamma^{(4)} + (\Lambda\gamma')' = 0 \quad (3.8)$$

must be satisfied for some $\Lambda(s)$.

Integration of (3.8) with respect to "s" gives

$$J = \gamma''' + \Lambda\gamma' = const. \quad (3.9)$$

On the other hand (2.5), (2.6), (2.7) and Frenet formulas in G_2 imply that

$$\begin{aligned} \gamma^{(4)} - (\Lambda\gamma')' &= (\kappa\boldsymbol{N})'' - (\boldsymbol{T}\Lambda)' \\ &= (-\Lambda')\boldsymbol{T} + (\kappa'' - \kappa\Lambda)\boldsymbol{N}. \end{aligned} \quad (3.10)$$

Substitution (3.10) into (3.8) gives

$$-\Lambda' = 0 \quad (3.11)$$

and

$$\kappa'' - \kappa\Lambda = 0. \quad (3.12)$$

From (3.11) we obtain

$$\Lambda = C^2 = const. \quad (3.13)$$

Substituting (3.13) into (3.12) we get

$$\kappa'' - \kappa C^2 = 0 \quad (3.14)$$

which represents the Euler-Lagrange equation obtained in [8] for $\tau = 0$. So, we can conclude that

Lemma 3.1 The necessary condition for a curve $\gamma : [a_1, a_2] \to G_2$ with fixed length to be an elastic curve is that Euler-Lagrange equation

$$\kappa'' - \kappa C^2 = 0$$

is hold.

Multiplication both sides of (3.14) by $2\kappa'$ leads to

$$\frac{d}{ds}[(\kappa')^2 - (C\kappa)^2] = 0, \tag{3.15}$$

which implies that

$$(\kappa\,')^2 - \kappa^2 C^2 = D = const. \tag{3.16}$$

Setting $\kappa' = u$, (3.15) reduces to

$$u^2 - \kappa^2 C^2 = D$$

and differentiating both sides with respect to "s" we get

$$2u(u' - C^2\kappa) = 0. \tag{3.17}$$

If $u = 0$ then $\kappa = constant$ and from (3.14), we obtain $\kappa = 0$, which states that the curve reduces to a straight line in the Galilean plane.

If $u \neq 0$ then we have the first order homogeneous differential equation

$$u' - C^2\kappa = 0 \tag{3.18}$$

Differentiating (3.18) and using $u = \kappa'$ we obtain the differential equation

$$u'' - C^2 u = 0. \tag{3.19}$$

The general solution of the homogeneous differential equation (3.19) is

$$u = C_1 e^{Cs} + C_2 e^{-Cs},$$

where C_1 and C_2 are constants.
Substituting $u = \kappa'$ we get

$$\kappa' = C_1 e^{Cs} + C_2 e^{-Cs}, \tag{3.20}$$

which leads to

$$\kappa(s) = \frac{C_1}{C} e^{Cs} - \frac{C_2}{C} e^{-Cs} \tag{3.21}$$

Taking into account all above we end up with parametrization of $\gamma(s) = \big(s, x(s)\big)$ as

$$\begin{aligned} \gamma(s) &= (s, x(s)) \\ &= \left(s, \tfrac{C_1}{C^3} e^{Cs} - \tfrac{C_2}{C^3} e^{-Cs}\right), \end{aligned}$$

where $c = \Lambda$ is the Lagrange multiplier.

Theorem 3.2. An elastic curve given by $\gamma : \mathrm{I} = [a_1, a_2] \to G_2$ in the Galilean plane is classified as

$$\gamma(s) = \left(s, \frac{C_1}{C^3} e^{Cs} - \frac{C_2}{C^3} e^{-C}\right),$$

where $C_1, C_{2,}\ C \in \mathbb{R}$ such that $C_1 \neq C_{2,}$ and C^2 represents Lagrange multiplier.

Example 3.3. Let consider the curve $\gamma(s) = (s, \cosh s)$ defined in the Galilean plane.

The curvature of $\gamma(s)$ is $\kappa(s) = \cosh s$ and it satisfies the necessary condition for a curve to be an elastic curve in G_2 given by $\kappa'' - \kappa C^2 = 0$, where $C_1 = \frac{1}{2}$, $C_2 = -\frac{1}{2}$ and the Lagrange multiplier $C = 1$. Then the curve $\gamma(s) = (s, \cosh s)$ is an elastic curve in the Galilean plane.

4. Conclusion

The elastica problem has a number of analogies with physical and biological systems including the mathematical models used for shipbuilding, bridge building and similar applications. The classification problem of elastic curves and its generalizations in real space forms have been studied by using different approaches by many researchers. In this work, the elastica problem of has been considered for the Galilean plane associated with Galileo's principle of relativity.

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