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On Fuzzy Differential Equations with Finite Delay via ψ-type Riemann-Liouville Fractional Derivative

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Abstract
In the article, the existence of a solution for a class of boundary value problem for a fuzzy differential equation with finite delay is discussed. By applying the contraction mapping principle, we gain an existence of a solution

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1. Introduction

The idea of this paper is to look into the existence of fuzzy solution for three-point boundary value problem for ψ-type fractional differential equation:

$$\begin{cases} \mathscr{D}^{\alpha;\psi}x(t) = f\left(t, x_t, \mathscr{D}^{\beta;\psi}x(t)\right), & t \in J := [0,1],\ 1 < \alpha < 2, \\ x(t) = \phi(t), & t \in [-r,0], \\ x(\zeta) = x(1), \end{cases} \tag{1.1}$$

where $\mathscr{D}^{\alpha;\psi}$, $\mathscr{D}^{\beta;\psi}$ are ψ-type Riemann-Liouville (R-L) fractional derivatives, $\alpha-\beta \geq 1$, $\varepsilon \in [0,1)$, $f: J \times C_0 \times E^n \to E^n$ is a fuzzy function, $\phi \in C_0$, $\phi(0) = \widehat{0}$, and $C_0 = C([-r,0], E^n)$. For any function x defined on $[-r,1]$ and any $t \in J$. We denote by x_t the element of C_0 defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r,0]$.

To the best of our information, even if various results for fuzzy differential equations (FDEs) have been established until now, results for FDEs with fractional order are rarely seen, papers [1, 2, 3] related to it only. The plan of the present paper is to establish some simple criteria for the existence and uniqueness of solution of the problem (1.1). The paper is structured as follows. In Section 2, we present some preliminaries and lemmas. In Section 3, we discuss the existence of solution for problem (1.1).

2. Prerequisites

Let $P_k(\mathbb{R})$ be the family of all nonempty compact convex subsets of $\mathbb{R}^n$. For $A, B \in P_k(\mathbb{R}^n)$, the Hausdorff metric is defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\quad \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

A fuzzy set in $\mathbb{R}^n$ is a function with domain $\mathbb{R}^n$ and values in $[0,1]$, that is, an element of $[0,1]^{\mathbb{R}^n}$.

Let $u \in [0,1]^{\mathbb{R}^n}$, the α-level set is

$$[u]^\alpha = \{x \in \mathbb{R}^n | u(x) \geq \alpha\}, \quad \alpha \in (0,1].$$

By E^n, we denote the all upper-semi continuous, normal fuzzy convex sets with $[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact.

Let $d : E^n \times E^n \to [0,+\infty)$ be defined by

$$d(u,v) = \sup\{d_H([u]^\alpha,[v]^\alpha)\,|\,\alpha \in [0,1]\}.$$

Then, (E^n,d) is a complete metric space. We define $\widehat{0} \in E^n$ as $\widehat{0}(0) = 1$ if $x = 0$ and $\widehat{0} = 0$ if $x \neq 0$.

Definition 2.1. *[4] The ψ-type R-L fractional integral of order α for a function $f : [a,b] \to E^n$ is defined by*

$$I_{a^+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^b \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1} f(s)ds, \quad \alpha > 0.$$

When $a = 0$, we write $I^{\alpha;\psi} f(t)$.

Definition 2.2. *[4] For a function $f : [a,b] \to E^n$, the ψ-type R-L derivative of fractional order $\alpha > 0$ is defined by*

$$\mathscr{D}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1} f(s)ds,\ n = [\alpha]+1.$$

Denote by $C(J,E^n)$ the set of all continuous mapping from J to E^n the metric on $C(J,E^n)$ is defined by $H(u,v) = \sup_{t\in J} d\,(u(t),v(t))$. And we metricize C_0 by setting

$$H_0(x,y) = \max\{d(x,y), y(t) | t \in [-r,0]\},$$

for all $x,y \in C_0$. Set $X = \{x | x \in C([-r,1],E^n), \mathscr{D}^\beta x \in C(J,E^n), \text{ and } x(t) = \phi(t),\ t \in [-r,0]\}$, the metric on X will be defined later.

3. Main Result

Theorem 3.1. *Assume that $f : J \times D_r \times E^n \to E^n$ and there exist positive constant K,L such that*

$$d\left(f(t,u,\mathscr{D}^{\beta;\psi}u), f(t,v,\mathscr{D}^{\beta;\psi}v)\right) \leq KH_0(u,v) + Ld\left(\mathscr{D}^{\beta;\psi}u, \mathscr{D}^{\beta;\psi}v\right)$$

for all $t \in J$ and all $u,v \in C_0$. Then

$$\left(\frac{K}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha-\beta+1)}\right)\left(1 + \frac{1+(\psi(\zeta))^\alpha}{1-(\psi(\zeta))^{\alpha-1}}\right) < 1,$$

implies that the problem (1.1) has a unique fuzzy solution on $[-r,1]$.

Proof. The metric H on X is defined by

$$H(u,v) = K \max_{t\in[-r,1]} d(u(t),v(t)) + L \max_{t\in[0,1]} d(\mathscr{D}^{\beta;\psi}u, \mathscr{D}^{\beta;\psi}v),$$

$u,v \in X$. Then (X,H) is a complete metric space.

Transform the problem into a fixed point problem. It is clear that the solutions of problem (1.1) are fixed points of the problem $F : X \to X$ defined by

$$F(x)(t) = \begin{cases} \phi(t), \quad t \in [-r,0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\psi(t)-\psi(s))^{\alpha-1} f(s,x_s,\mathscr{D}^{\beta;\psi}x(s))ds \\ +\frac{1}{\Gamma(\alpha)(1-(\psi(\zeta))^{\alpha-1})} \int_0^\zeta \psi'(\psi(\zeta)-\psi(s))^{\alpha-1} f(s,x_s,\mathscr{D}^{\beta;\psi}x(s))ds \\ -\frac{1}{\Gamma(\alpha)(1-(\psi(\zeta))^{\alpha-1})} \int_0^1 \psi'(\psi(1)-\psi(s))^{\alpha-1} f(s,x_s,\mathscr{D}^{\beta;\psi}x(s))ds,\ t \in (0,1). \end{cases}$$

For $u,v \in X$, then

$$d(Fu(t),Fv(t)) = 0, \quad t \in [-r,0], \tag{3.1}$$

and for $t \in J$, we have

$$\begin{aligned}
&d\,(Fu(t),Fv(t))\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds\\
&+\frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)(1-(\psi(\zeta))^{\alpha-1})}\int_0^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\alpha-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds\\
&+\frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)(1-(\psi(\zeta))^{\alpha-1})}\int_0^1 \psi'(s)(\psi(1)-\psi(s))^{\alpha-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds\\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left[Kd(u_s(\theta),v_s(\theta))+Ld(\mathscr{D}^{\beta;\psi}u(s),\mathscr{D}^{\beta;\psi}v(s))\right]ds\right.\\
&+\frac{(\psi(t))^{\alpha-1}}{(1-(\psi(\zeta))^{\alpha-1})}\int_0^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\alpha-1}\left[Kd(u_s(\theta),v_s(\theta))+Ld(\mathscr{D}^{\beta;\psi}u(s),\mathscr{D}^{\beta;\psi}v(s))\right]ds\\
&\left.+\frac{(\psi(t))^{\alpha-1}}{(1-(\psi(\zeta))^{\alpha-1})}\int_0^1 \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left[Kd(u_s(\theta),v_s(\theta))+Ld(\mathscr{D}^{\beta;\psi}u(s),\mathscr{D}^{\beta;\psi}v(s))\right]ds\right)\\
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}ds+\frac{(\psi(t))^{\alpha-1}}{(1-(\psi(\zeta))^{\alpha-1})}\int_0^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\alpha-1}ds\right.\\
&\left.+\frac{(\psi(t))^{\alpha-1}}{(1-(\psi(1))^{\alpha-1})}\int_0^1 \psi'(s)(\psi(\zeta)-\psi(s))^{\alpha-1}ds\right)H(u,v)\\
&\leq \frac{1}{\Gamma(\alpha+1)}\sup_{t\in J}\left((\psi(t))^{\alpha}+\frac{(\psi(\zeta))^{\alpha}(\psi(t))^{\alpha-1}}{1-(\psi(\zeta))^{\alpha-1}}+\frac{(\psi(t))^{\alpha-1}}{1-(\psi(\zeta))^{\alpha-1}}\right)H(u,v)
\end{aligned}$$

and similarly,

$$\begin{aligned}
&d\left(\mathscr{D}^{\beta;\psi}Fu(t),\mathscr{D}^{\beta;\psi}Fv(t)\right)\\
&\leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\beta-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds\\
&+\frac{(\psi(t))^{\alpha-\beta-1}}{(1-(\psi(\zeta))^{\alpha-1})}\int_0^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\alpha-\beta-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds\\
&+\frac{(\psi(t))^{\alpha-\beta-1}}{(1-(\psi(\zeta))^{\alpha-1})}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-\beta-1} d\left(f(s,u_s,\mathscr{D}^{\beta;\psi}u(s)), f(s,v_s,\mathscr{D}^{\beta;\psi}v(s))\right) ds\\
&\leq \frac{1}{\Gamma(\alpha-\beta+1)}\sup_{t\in J}\left((\psi(t))^{\alpha-\beta}+\frac{(\psi(\zeta))^{\alpha}(\psi(t))^{\alpha-\beta-1}}{(1-(\psi(\zeta))^{\alpha-1})}+\frac{(\psi(t))^{\alpha-\beta-1}}{(1-(\psi(\zeta))^{\alpha-1})}\right)H(u,v).
\end{aligned}$$

Therefore, for each $t \in J$, we have

$$H(Fu,Fv) \leq \left(\frac{K}{\Gamma(\alpha+1)}+\frac{L}{\Gamma(\alpha-\beta+1)}\right)\left(1+\frac{1+(\psi(\zeta))^{\alpha}}{1-(\psi(\zeta))^{\alpha-1}}\right)H(u,v).$$

So, F is contraction and thus F has a unique fixed point x on X, then $x(t)$ is the unique solution to problem (1.1) on $[-r,1]$. □

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Author's contributions

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