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On Some Classes of Generalized Recurrent α-Cosymplectic Manifolds

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Abstract. In this paper, we concentrate on hyper generalized φ-recurrent α-cosymplectic manifolds and quasi generalized φ-recurrent α-cosymplectic manifolds and obtain some significant characterizations which classify such manifolds.

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1. Introduction

The theory of the manifold is one of the most comprehensive and notable fields of the studies of differential geometry since manifolds explain space in terms of simpler and easily understandable structures. It is well known that there are many special classes of the manifolds with different structures and names. One of them is almost contact metric manifolds. These manifolds have contributed to many disciplines such as string and knot theory, thermodynamics and fluid mechanics, optics, control systems, hydrodynamics, molecular biology and heat flow. Hence, many geometers have given their attention to such manifolds recently.

Over the past few years, several various kinds of almost contact metric manifolds have been investigated and studied widely. One important kind of them is almost cosymplectic manifolds, which were introduced by Goldberg and Yano. The most basic examples of such manifolds are the products of almost Kähler manifolds and the real R line or the circle S^1. Almost cosymplectic manifolds are almost contact metric manifolds whose structure tensor fields satisfy $d\eta = d\Phi = 0$, where Φ is the fundamental 2-form of the manifold given by $\Phi(.,.) = g(.,\varphi)$. If $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, then the manifold is called almost Kenmotsu. Also, as it is well known that normal almost cosmplectic (respectively almost Kenmotsu) manifolds are cosymplectic (respectively Kenmotsu) manifolds. As a generalization of almost Kenmotsu manifolds, an almost α-Kenmotsu manifold is an almost contact metric manifold M along with $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, where α is a non-zero real number. If we unify almost cosymplectic and almost α-Kenmotsu manifold, then we obtain a new class of almost contact metric manifolds, which is known as almost α-cosymplectic manifold. Such a manifold is defined by the following formula [7]

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi.$$

It is obvious that, if an almost α-cosymplectic manifold is normal, then it is called an α-cosymplectic manifold. For $\alpha \in \mathbb{R}$, an α-cosymplectic manifold is a cosymplectic manifold and α-Kenmotsu manifold accordingly as $\alpha = 0$ and $\alpha \neq 0$, respectively. Several mathematicians have studied the properties of almost α-cosymplectic and α-cosymplectic

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manifolds from the topological and geometrical aspects in literature survey. For further information related to such manifolds, we refer to [1, 5, 7–10, 15, 16] and references therein.

The present paper is on a special class of almost contact metric manifolds, named as α-cosymplectic manifolds. Here, we deal with the concepts of hyper generalized φ-recurrent and quasi generalized φ-recurrent for α-cosymplectic manifolds. The present paper is created as follows. After introduction section, in section 2, we give some essential definitions and notions about almost contact metric manifolds. In the next section, we give our main results which are obtained in this paper.

2. Preliminaries

In this section, we shall give some essential notions and formulas which will be used later [2] and [4].

An almost contact metric manifold M of dimension $(2n+1)$ is a differentiable manifold equipped with an almost contact metric structure (φ, ξ, η, g) on M such that ξ is a vector field (which is so-called the characteristic vector field), η is the g-dual of ξ, φ is a tensor field of type $(1,1)$ on M and g is a Riemannian metric satisfying the following relations [2]:

$$\varphi^2 U = -U + \eta(U)\xi, \qquad \eta(\xi) = 1 \tag{2.1}$$

and

$$g(\varphi U, \varphi T) = g(U,T) - \eta(U)\eta(T),$$

for any $U, T \in \Gamma(TM)$. The above relations imply

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0$$

and

$$g(\varphi U, T) = -g(U, \varphi T), \qquad \eta(U) = g(U, \xi),$$

for any $U, T \in \Gamma(TM)$. An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ becomes a contact metric manifold if the fundamental 2-form Φ of M satisfies

$$\Phi(T,U) = d\eta(T,U),$$

where

$$\Phi(T,U) = g(T, \varphi U)$$

and

$$d\eta(T,U) = \frac{1}{2}\Big\{T\eta(U) - U\eta(T) - \eta([T,U])\Big\}.$$

Also, an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ together with the tensor field N_φ is called a normal contact metric manifold such that

$$N_\varphi + 2d\eta \otimes \xi = 0.$$

Here, N_φ is the Nijenhuis tensor field of φ and is given by

$$N_\varphi(T,U) = [\varphi T, \varphi U] + \varphi^2[T,U] - \varphi[T, \varphi U] - \varphi[\varphi T, U],$$

for all $T, U \in \Gamma(TM)$. For an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ if the following condition

$$(\nabla_T \varphi)U = \alpha(g(\varphi T, U)\xi - \eta(U)\varphi T) \tag{2.2}$$

is satisfied, then $(M, \varphi, \xi, \eta, g)$ is called an α-cosymplectic manifold. Here, ∇ indicates the Levi-Civita connection on M. From (2.2) we get

$$\nabla_T \xi = \alpha\Big(T - \eta(T)\xi\Big), \tag{2.3}$$

for any $T \in \Gamma(TM)$. For an α-cosymplectic manifold $(M, \varphi, \xi, \eta, g)$, we also have

$$\begin{aligned} R(T,U)\xi &= \alpha^2(\eta(T)U - \eta(U)T), \quad (2.4)\\ R(T,\xi)U &= \alpha^2(g(T,U)\xi - \eta(U)T),\\ R(\xi,T)U &= \alpha^2(\eta(U)T - g(T,U)\xi), \quad (2.5)\\ R(T,\xi)\xi &= \alpha^2(\eta(T)\xi - T),\\ S(T,\xi) &= -2n\alpha^2\eta(T), \quad (2.6)\\ S(\xi,\xi) &= -2n\alpha^2,\\ Q\xi &= -2n\alpha^2, \quad (2.7) \end{aligned}$$

where S and R represent the Ricci tensor field and the Riemann curvature tensor of M, respectively and Q is the Ricci operator defined by $S(U, W) = g(QU, W)$.

Now, we recall some definitions from [3, 11–13] as follows:

A Riemannian manifold (M, g) is called a generalized Ricci-recurrent and super generalized Ricci-recurrent manifold, respectively if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies

$$(\nabla_T S)(U, F) = A(T)S(U, F) + B(T)g(U, F)$$

and

$$(\nabla_T S)(U, F) = \upsilon(T)S(U, F) + \pi(T)g(U, F) + \psi(T)\eta(U)\eta(F),$$

for any $T, U, F \in \Gamma(TM)$, where A, B and υ, π, ψ are non-vanishing 1-forms. If $B = 0$, then it is called a Ricci-recurrent manifold. If $\pi = \psi$, then it is called a quasi generalized Ricci-recurrent manifold.

3. Main Results

In [14], the authors introduced the notion of hyper generalized φ-recurrent for Sasakian manifolds and investigated the properties of such manifolds. Afterward, the notion of hyper generalized φ-recurrent for (k, μ)-contact metric manifolds was defined by authors in [6]. Motivated by these works, we study the concept of hyper generalized φ-recurrent for α-cosymplectic manifolds.

Now, we give the following definition analogous to the hyper generalized φ-recurrent Sasakian manifolds [14]:

Definition 3.1. An α-cosymplectic manifold is called a hyper generalized φ-recurrent α-cosymplectic manifold if its Riemannian curvature tensor R satisfies

$$\varphi^2((\nabla_W R)(T, U)F) = A(W)R(T, U)F + B(W)H(T, U)F, \quad (3.1)$$

for any $T, U, F, W \in \Gamma(TM)$, where A and B are two non-vanishing 1-forms such that $A(T) = g(T, \mu_1)$, $B(T) = g(T, \mu_2)$ and the tensor H is given by

$$H(T, U)F = S(U, F)T - S(T, F)U + g(U, F)QT - g(T, F)QU. \quad (3.2)$$

Here, μ_1 and μ_2 are vector fields associated with 1-forms A and B, respectively. If the 1-form B vanishes, then (3.1) turns into the notion of φ-recurrent manifold.

Now, we can give our main theorem.

Theorem 3.2. *Let $(M, \varphi, \xi, \eta, g)$ be a hyper generalized φ-recurrent α-cosymplectic manifold. Then,*
i) M is ricci recurrent if the scalar curvature is zero everywhere on M.
ii) M is generalized ricci recurrent if the scalar curvature is non-zero everywhere on M.

Proof. Since M is a hyper generalized φ-recurrent α-cosymplectic manifold, using (2.1) and (3.1) we have

$$-(\nabla_W R)(T, U)F + \eta\Big((\nabla_W R)(T, U)F\Big)\xi = A(W)R(T, U)F + B(W)H(T, U)F, \quad (3.3)$$

for any $T, U, F, W \in \Gamma(TM)$. Taking inner product of (3.3) with an arbitrary vector field Y gives

$$-g((\nabla_W R)(T, U)F, Y) + \eta\Big((\nabla_W R)(T, U)F\Big)\eta(Y) = A(W)g(R(T, U)F, Y) + B(W)g(H(T, U)F, Y). \quad (3.4)$$

Let $\{e_1, e_2, ..., e_{2n+1} = \xi\}$ be an orthonormal basis (which is called φ-basis) of the tangent space T_pM, at each point $p \in M$. If we set $T = Y = e_j$ in (3.4) and sum over j ($j = 1, 2, ..., 2n+1$), we obtain

$$-(\nabla_W S)(U,F) + \sum_{j=1}^{2n+1} \eta\Big((\nabla_W R)(e_j, U)F\Big)\eta(e_j) = A(W)S(U,F) + B(W)\sum_{j=1}^{2n+1} g(H(e_j, U)F, e_j). \tag{3.5}$$

Making use of (3.2), one immediately has

$$g(H(T,U)F,Y) = S(U,F)g(T,Y) - S(T,F)g(U,Y) + g(U,F)S(Y,T) - g(T,F)S(U,Y). \tag{3.6}$$

Considering φ-basis and setting $T = Y = e_j$ in (3.6) provide

$$\sum_{j=1}^{2n+1} g(H(e_j,U)F,e_j) = (2n-1)S(U,F) + rg(U,F). \tag{3.7}$$

Also, we have

$$\sum_{j=1}^{2n+1} \eta\Big((\nabla_W R)(e_j,U)F\Big)\eta(e_j) = g\Big((\nabla_W R)(\xi,U)F,\xi\Big). \tag{3.8}$$

It follows from (2.3), (2.4) and keeping in mind the relation $g\Big((\nabla_W R)(T,U)F,Y\Big) = -g\Big((\nabla_W R)(T,U)Y,F\Big)$, we derive that

$$g\Big((\nabla_W R)(\xi,U)F,\xi\Big) = 0. \tag{3.9}$$

With the help of (3.7), (3.8) and (3.9), the equation (3.5) takes the form

$$-(\nabla_W S)(U,F) = A(W)S(U,F) + B(W)\Big(2n-1)S(U,F) + rg(U,F)\Big) \tag{3.10}$$

and hence,

$$(\nabla_W S)(U,F) = \phi(W)S(U,F) + \delta(W)g(U,F),$$

where $\phi(W) = -\Big(A(W) + (2n-1)B(W)\Big)$ and $\delta(W) = -rB(W)$. Consequently, a hyper generalized φ-recurrent α-cosymplectic manifold M becomes a generalized ricci recurrent (respectively ricci recurrent) if the scalar curvature is non-zero (respectively zero) everywhere on M. Thus, the proof is completed. □

The next theorem presents an important characterization about hyper generalized φ-recurrent α-cosymplectic manifolds.

Theorem 3.3. *Let $(M, \varphi, \xi, \eta, g)$ be a hyper generalized φ-recurrent α-cosymplectic manifold with non-vanishing scalar curvature. Then, either M becomes a φ-recurrent cosymplectic manifold or M has constant scalar curvature.*

Proof. Taking into account φ-basis and putting $U = F = e_j$ in (3.10), we find that

$$W(r) = -r\Big(A(W) + 4nB(W)\Big). \tag{3.11}$$

Putting $F = \xi$ in (3.10), we write

$$-(\nabla_W S)(U,\xi) = A(W)S(U,\xi) + B(W)\Big(2n-1)S(U,\xi) + rg(U,\xi)\Big) \tag{3.12}$$

such that

$$(\nabla_W S)(U,\xi) = \nabla_W S(U,\xi) - S(\nabla_W U,\xi) - S(U,\nabla_W \xi),$$

for any $U, W \in \Gamma(TM)$. By virtue of (2.3) and (2.6), we get

$$(\nabla_W S)(U,\xi) = -2n\alpha^3 g(U,W) - \alpha S(U,W). \tag{3.13}$$

Substituting (3.13) into (3.12), we arrive at

$$2n\alpha^3 g(U,W) + \alpha S(U,W) = -\Big(2n\alpha^2 A(W)\eta(U) + 2n(2n-1)\alpha^2 B(W)\eta(U) - rB(W)\eta(U)\Big). \tag{3.14}$$

Also, replacing U by ξ in (3.14) and keeping in mind (2.6) we deduce that

$$2n\alpha^2 A(W) + 2n(2n-1)\alpha^2 B(W) - rB(W) = 0, \tag{3.15}$$

from which it follows that

$$\alpha\Big(2n\alpha^2 g(U,W) + S(U,W)\Big) = 0. \tag{3.16}$$

This implies that either $\alpha = 0$ or $S(U,W) = -2n\alpha^2 g(U,W)$.

If $\alpha = 0$, then from (3.15) we get that $rB(W) = 0$. Since the scalar curvature r is non-vanishing, we have that $B(W) = 0$. In this case, the manifold M becomes a φ-recurrent cosymplectic manifold.

If $\alpha \neq 0$, then we have $S(U,W) = -2n\alpha^2 g(U,W)$. Contracting over U and W, we obtain that $r = -2n(2n+1)\alpha^2$. Using this value in (3.15) gives $2n\alpha^2 A(W) + 8n\alpha^2 B(W) = 0$. Due to α being non-zero, we get that $A(W) + 4nB(W) = 0$. Therefore, from (3.11) we infer that the scalar curvature r of M is constant. This completes the proof. □

As an immediate consequence of Theorem 3.3, we can state the following:

Corollary 3.4. *Let (M,φ,ξ,η,g) be a hyper generalized φ-recurrent α-Kenmotsu manifold with non-vanishing scalar curvature. Then, the 1-forms A and B are related by the following identity*

$$A(W) + 4nB(W) = 0. \tag{3.17}$$

Now, we can give another important theorem.

Theorem 3.5. *Let (M,φ,ξ,η,g) be a hyper generalized φ-recurrent α-Kenmotsu manifold with non-vanishing scalar curvature. Then, M is a space of constant curvature -α^2.*

Proof. From (2.3) and (2.4), we have that

$$(\nabla_W R)(T,U)\xi = \alpha^3(g(T,W)U - g(U,W)T) - \alpha R(T,U)W, \tag{3.18}$$

for any $T,U,W \in \Gamma(TM)$. Applying φ^2 on both sides of (3.18) and by means of (2.1), (2.5), after a straight forward computation we acquire

$$\varphi^2(\nabla_W R)(T,U)\xi = \alpha^3(g(U,W)T - g(T,W)U) + \alpha R(T,U)W. \tag{3.19}$$

Substituting F for ξ in (3.1) and using the equalities (2.4), (2.6), (2.7), we obtain

$$\begin{aligned} \varphi^2((\nabla_W R)(T,U)\xi) \;=\; & \alpha^2 A(W)\Big(\eta(T)U - \eta(U)T\Big) + B(W)\Big\{2n\alpha^2\Big(\eta(T)U - \eta(U)T\Big) \\ & + \eta(U)QT - \eta(T)QU\Big\}. \end{aligned} \tag{3.20}$$

On the other hand, since M is a hyper generalized φ-recurrent α-Kenmotsu manifold, namely $\alpha \neq 0$, it follows from (3.16) that one can see

$$S(U,W) = -2n\alpha^2 g(U,W), \tag{3.21}$$

which by removing W on both sides of (3.21) yields $QU = -2n\alpha^2 U$. Using this fact in (3.20), we get

$$\varphi^2((\nabla_W R)(T,U)\xi) = \alpha^2(A(W) + 4nB(W))\Big(\eta(T)U - \eta(U)T\Big),$$

which together with (3.17) gives

$$\varphi^2\Big((\nabla_W R)(T,U)\xi\Big) = 0. \tag{3.22}$$

Therefore, by combining (3.19) and (3.22) yields

$$\alpha\Big\{\alpha^2\Big(g(U,W)T - g(T,W)U\Big) + R(T,U)W\Big\} = 0. \tag{3.23}$$

Owing to α being non-zero, the equation (3.23) turns into

$$R(T,U)W = -\alpha^2\Big(g(U,W)T - g(T,W)U\Big).$$

This implies that M is a space of constant curvature -α^2. Hence, we get the requested result. □

Definition 3.6. An α-cosymplectic manifold is called a quasi generalized φ-recurrent α-cosymplectic manifold if its Riemannian curvature tensor R satisfies

$$\varphi^2((\nabla_W R)(T,U)F) = C(W)R(T,U)F + D(W)G(T,U)F, \tag{3.24}$$

for any $T, U, F, W \in \Gamma(TM)$, where C and D are two non-vanishing 1-forms such that $C(T) = g(T, \rho_1)$, $D(T) = g(T, \rho_2)$ and the tensor G is given by

$$G(T,U)F = g(U,F)T - g(T,F)U + g(U,F)\eta(T)\xi - g(T,F)\eta(U)\xi + \eta(U)\eta(F)T - \eta(T)\eta(F)U. \tag{3.25}$$

Here, ρ_1 and ρ_2 are vector fields associated with 1-forms C and D, respectively. If the 1-form D vanishes, then (3.24) turns into the notion of φ-recurrent manifold.

Now, we shall give an important theorem.

Theorem 3.7. *Let $(M, \varphi, \xi, \eta, g)$ be a quasi generalized φ-recurrent α-cosymplectic manifold. Then,*
i) M is a super generalized ricci recurrent manifold.
ii) M is either a φ-recurrent α-cosymplectic manifold or M is a Ricci symmetric manifold with $n \neq 2$.

Proof. Using the same method as in the proof of the Theorem 3.2, we obtain that

$$-(\nabla_W S)(U,F) = C(W)S(U,F) + D(W)\Big((2n+1)g(U,F)\Big) + D(W)\Big((2n-1)\eta(U)\eta(F)\Big), \tag{3.26}$$

for any $U, F, W \in \Gamma(TM)$, which implies that M is a super generalized ricci recurrent manifold. Again it follows from the method that we have used in Theorem 3.3, we get

$$-W(r) = rC(W) + 2n(2n+3)D(W) \tag{3.27}$$

and

$$D(W) = \frac{\alpha^2}{2}C(W). \tag{3.28}$$

Also, taking $F = \xi$ in (3.26) and keeping in mind (3.28) provide

$$\alpha\Big(2n\alpha^2 g(U,W) + S(U,W)\Big) = 0.$$

Now, if $\alpha = 0$, then from (3.28) one has $D(W) = 0$. Therefore, M becomes a $\varphi-$recurrent $\alpha-$cosymplectic manifold. If $\alpha \neq 0$, then we have that $S(U,W) = -2n\alpha^2 g(U,W)$. Contracting over U and W gives $r = -2n(2n+1)\alpha^2$. Using this value in (3.27) and in view of (3.28), we find that n$\alpha^2(n-2)C(W) = 0$. In this case, either $n = 2$ or $C(W) = 0$. Because of $n \neq 2$, $C(W) = 0$. This case implies $D(W) = 0$. Thus, the manifold M is Ricci symmetric. This result ends the proof of Theorem. □

Now, let us assume that a quasi generalized φ-recurrent α-cosymplectic manifold M is quasi generalized Ricci-recurrent. In this case, by definition of quasi generalized Ricci-recurrent manifold and from (3.26), we have that $2n + 1 = 2n - 1$. This is impossible. Hence, we can give the following.

Corollary 3.8. *A quasi generalized φ-recurrent α-cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ is never quasi generalized Ricci-recurrent manifold.*

Corollary 3.9. *Let $(M, \varphi, \xi, \eta, g)$ be a quasi generalized φ-recurrent α-cosymplectic manifold. Then, the 1-forms C and D are related by the identity (3.28). Also, these 1-forms are co-directional.*

Now, we shall give an important theorem.

Theorem 3.10. *Let $(M, \varphi, \xi, \eta, g)$ be a quasi generalized φ-recurrent α-cosymplectic manifold. Then, the vector field ρ_1 and the Ricci tensor S of M satisfy*

$$S(W,\rho_1) = \Big(\frac{r}{2} + \frac{\alpha^2}{2}(2n^2+n-1)\Big)g(W,\rho_1) - \frac{\alpha^2}{2}(2n-1)\eta(\rho_1)\eta(W),$$

for any $W \in \Gamma(TM)$.

Proof. It follows from (2.1), (3.24) and (3.25) that we have

$$(\nabla_W R)(T,U)F = \eta\Big((\nabla_W R)(T,U)F\Big)\xi - C(W)R(T,U)F - D(W)\Big(g(U,F)T - g(T,F)U + g(U,F)\eta(T)\xi - g(T,F)\eta(U)\xi + \eta(U)\eta(F)T - \eta(T)\eta(F)U\Big), \tag{3.29}$$

for any $T,U,F,W \in \Gamma(TM)$. Cyclically rearranging W,T and U in (3.29) and utilizing from second Bianchi identity, we obtain

$$C(W)R(T,U)F + D(W)\Big(g(U,F)T - g(T,F)U + g(U,F)\eta(T)\xi - g(T,F)\eta(U)\xi + \eta(U)\eta(F)T - \eta(T)\eta(F)U\Big) + C(T)R(U,W)F + D(T)\Big(g(W,F)U - g(U,F)W + g(W,F)\eta(U)\xi - g(U,F)\eta(W)\xi + \eta(W)\eta(F)U - \eta(U)\eta(F)W\Big) + C(U)R(W,T)F + D(U)\Big(g(T,F)W - g(W,F)T + g(T,F)\eta(W)\xi - g(W,F)\eta(T)\xi + \eta(T)\eta(F)W - \eta(W)\eta(F)T\Big) = 0. \tag{3.30}$$

Operating inner product of (3.30) with Z yields

$$C(W)g(R(T,U)F,Z) + D(W)\Big(g(U,F)g(T,Z) - g(T,F)g(U,Z) + g(U,F)\eta(T)\eta(Z) - g(T,F)\eta(U)\eta(Z) + \eta(U)\eta(F)g(T,Z) - \eta(T)\eta(F)g(U,Z)\Big) + C(T)g(R(U,W)F,Z) + D(T)\Big(g(W,F)g(U,Z) - g(U,F)g(W,Z) + g(W,F)\eta(U)\eta(Z) - g(U,F)\eta(W)\eta(Z) + \eta(W)\eta(F)g(U,Z) - \eta(U)\eta(F)g(W,Z)\Big) + C(U)g(R(W,T)F,Z) + D(U)\Big(g(T,F)g(W,Z) - g(W,F)g(T,Z) + g(T,F)\eta(W)\eta(Z) - g(W,F)\eta(T)\eta(Z) + \eta(T)\eta(F)g(W,Z) - \eta(W)\eta(F)g(T,Z)\Big) = 0, \tag{3.31}$$

for any $Z \in \Gamma(TM)$. Considering φ-basis and putting $U = F = e_j$ in (3.31), we deduce that

$$C(W)S(T,Z) + D(W)\Big((2n+1)g(T,Z) + (2n-1)\eta(T)\eta(Z)\Big) - C(T)S(W,Z) - D(T)\Big((2n+1)g(W,Z) + (2n-1)\eta(W)\eta(Z)\Big) - C(R(W,T)Z) + D(T)\Big(g(W,Z) + \eta(W)\eta(Z)\Big) - D(W)\Big(g(T,Z) + \eta(T)\eta(Z)\Big) + D(\xi)\eta(T)g(W,Z) - D(\xi)\eta(W)g(T,Z)\Big) = 0. \tag{3.32}$$

Again, considering $\varphi-$basis and setting $T = Z = e_j$ in (3.32) we find that

$$S(W,\rho_1) = \frac{r}{2}g(W,\rho_1) + \Big(2n^2 + n - 1\Big)g(W,\rho_2) - (2n-1)\eta(\rho_2)\eta(W),$$

which together with (3.28) gives

$$S(W,\rho_1) = \Big(\frac{r}{2} + \frac{\alpha^2}{2}(2n^2 + n - 1)\Big)g(W,\rho_1) - \frac{\alpha^2}{2}(2n-1)\eta(\rho_1)\eta(W),$$

for any $W \in \Gamma(TM)$. This is the desired result. Thus, the proof is completed. □

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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