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AVD Proper Edge Coloring of Some Cycle Related Graphs

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Abstract — The adjacent vertex-distinguishing proper edge-coloring is the minimum number of colors required for a proper edge-coloring of G, in which no two adjacent vertices are incident to edges colored with the same set of colors. The minimum number of colors required for an adjacent vertex-distinguishing proper edge-coloring of G is called the adjacent vertex-distinguishing proper edge-chromatic index. In this paper, we compute adjacent vertex-distinguishing proper edge-chromatic index of Anti-prism, sunflower graph, double sunflower graph, triangular winged prism, rectangular winged prism and Polygonal snake graph.

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1. Introduction

The terminology and notations we refer to Bondy and Murthy [4]. Let G be a finite, simple, undirected and connected graph. Let $\Delta(G)$ denote the maximum degree of G. A *proper edge-coloring* σ is a mapping from $E(G)$ to the set of colors such that any two adjacent edges receive distinct colors. For any vertex v of G, let $S_\sigma(v)$ denote the set of the colors of all edges incident to v. A proper edge-coloring σ is said to an *adjacent vertex-distinguishing* (AVD) if $S_\sigma(u) \neq S_\sigma(v)$, for every adjacent vertices u and v. The minimum number of colors required for an adjacent vertex-distinguishing proper edge-coloring of G, denoted by $\chi'_{as}(G)$, is called the *adjacent vertex-distinguishing proper edge-chromatic index (AVD proper edge-chromatic index)* of G. Thus, $\chi'_{as}(G) \geq \chi'(G)$.

Conjecture 1.1. [11] For any connected graph G ($|V(G)| \geq 6$), there is $\chi'_{as}(G) \leq \Delta(G) + 2$. If H is a subgraph of G, it is interesting that $\chi'_{as}(H) \leq \chi'_{as}(G)$ is not always true.

Let $K_{m,n}$ be the complete bipartite graph, then $\chi'_{as}(K_{2,3}) = 3$ and $K_{2,3} - e$ for any edge, then $\chi'_{as}(K_{2,3} - e) = 4$. Deletion of an edge of a graph may also decrease the coloring number of the graph. Let $n \geq 3$, then $\chi'_{as}(K_{1,n}) = n$ and $\chi'_{as}(K_{1,n} - e) = n - 1$.

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In [11] Zhang et al. proved: if G has n components G_i, $1 \le i \le n$, with at least three vertices in each, then $\chi'_{as}(G) = max_{1\le i\le n}\{\chi'_{as}(G_i)\}$. So we consider only connected graphs. For a tree T with $|V(T)| \ge 3$, if any two vertices of maximum degree are non-adjacent, then $\chi'_{as}(T) = \Delta(T)$. If T has two vertices of maximum degree which are adjacent, then $\chi'_{as}(T) = \Delta(T) + 1$. For cycle C_n we have $\chi'_{as}(C_n) = 3$, for $n \equiv 0\ (mod3)$, otherwise $\chi'_{as}(C_n) = 4$ for $n \not\equiv 0\ (mod3)$ and $n \ne 5$, $\chi'_{as}(C_n) = 5$, for $n = 5$. For the complete bipartite graph $K_{m,n}$ for $1 \le m \le n$, we have $\chi'_{as}(K_{m.n}) = n$ if $m < n$, and $\chi'_{as}(K_{m.n}) = n + 2$ if $m = n \ge 2$. For the complete graph K_n $(n \ge 3)$, we have $\chi'_{as}(K_n) = n$ for $n \equiv 1\ (mod\ 2)$; $\chi'_{as}(K_n) = n + 1$ for $n \equiv 0\ (mod\ 2)$. If G is a graph which has two adjacent maximum degree vertices, then $\chi'_{as}(G) \ge \Delta(G) + 1$. If G is a graph such that the degree of any two adjacent vertices is different, then $\chi'_{as}(G) = \Delta(G)$. In [9] Shiu proved: for $n \ge 3$, we have $\chi'_{as}(F_n) = n$, if $n = 3,4$ and $\chi'_{as}(F_{n-1}) = n - 1$, for $n \ge 5$. For $n \ge 3$, we have $\chi'_{as}(W_n) = 5$, if $n = 3$, and $\chi'_{as}(W_n) = n$, for $n \ge 4$. In [7] Hatami prove that if G is a graph with no isolated edges and maximum degree $\Delta(G) > 10^{20}$, then $\chi'_{as} \le \Delta + 300$. In [2] Balister et al. proved: if G is a k-chromatic graph with no isolated edges, then $\chi'_{as}(G) \le \Delta(G) + O(\log k)$. In [1] Axenovich et al. obtained upper bound for adjacent vertex-distinguishing edge-colorings of graphs. In [3] Baril et al. obtained exact values for adjacent vertex-distinguishing edge-coloring of meshes. In [5] Bu et al. finding adjacent vertex-distinguishing edge-colorings of planar graphs with girth at least six. In [6] Chen et al. obtained adjacent vertex-distinguishing proper edge-coloring of planar bipartite graphs with $\Delta = 9,10\ or\ 11$.

In this paper, we compute adjacent vertex-distinguishing edge-chromatic index of Anti- prism, sunflower graph, double sunflower graph, triangular winged prism, rectangular winged prism and Polygonal snake graph.

Observation 1.1. If a connected graph G contains two adjacent vertices of degree $\Delta(G)$, then $\chi'_t(G) \ge \Delta(G) + 1$.

Observation 1.2. If G is a graph such that the degree of any two adjacent vertices is different, then $\chi'_{as}(G) = \Delta(G)$.

2. AVD Proper Edge-chromatic Index of Anti-prism Graph, Sunflower Graph, Double Sunflower Graph, Triangular Winged Prism and Rectangular Winged Prism

In this section, The AVD proper edge-chromatic index of Anti-prism graph, Sunflower graph, Double Sunflower graph, Triangular winged prism and Rectangular winged prism graph will be discussed. We have the following results.

2.1. AVD Proper Edge-chromatic Index of Anti-prism Graph

If $C_n \square K_2, n \ge 3$, is called prism graph, where $\square$ is Cartesian product, and it is denoted by D_n

By an Anti-prism graph of order n denoted by A_n, we mean a graph obtained from a prism graph D_n by adding some crossing edges $x_i y_{(i+1)(mod\ n)}$, $i = 1,2, \dots, n$. [10]

Theorem 2.1. $\chi'_{as}(A_n) = 5,\ for\ n \ge 3$.

Proof. Let $C_n = x_1x_2 \dots x_nx_1$, For $n \geq 4$ and $x_1', x_2', \dots, x_n'$ be newly added vertices corresponding to the vertices $x_1, x_2, \dots, x_n$ to form A_n. In A_n, for $i \in \{1,2,\dots,n\}$, let $e_i = x_ix_{i+1}$, $e_i' = x_i'x_{i+1}'$, $f_i = x_ix_i'$, $g_i = x_ix_{i+1}'$, where $x_{n+1} = x_1$, $x_{n+1}' = x_1'$.

Define $\sigma : E(A_3) \to \{1,2,3,4,5\}$ as follows: $\sigma(e_1) = 1, \sigma(e_2) = 4, \sigma(e_3) = 5, \sigma(e_1') = 4, \sigma(e_2') = 5, \sigma(e_3') = 1, \sigma(f_1) = \sigma(f_2) = \sigma(f_3) = 3, \sigma(g_1) = \sigma(g_2) = \sigma(g_3) = 2$. Therefore σ is proper-edge coloring. The induced vertex-color sets are: $S_\sigma(x_1) = \{1,2,3,5\}, S_\sigma(x_2) = \{1,2,3,4\}, S_\sigma(x_3) = \{2,3,4,5\}, S_\sigma(x_1') = \{1,2,3,4\}, S_\sigma(x_2') = \{2,3,4,5\}$, $S_\sigma(x_3') = \{1,2,3,5\}$. Hence σ is an AVD proper edge-coloring A_3. By observation 1.1, $\chi'_{as}(A_3) \geq 5$ and so $\chi'_{as}(A_3) = 5$. Define $\sigma : E(A_4) \to \{1,2,3,4,5\}$ as follows: $\sigma(e_1) = \sigma(e_1') = 1, \sigma(e_2) = \sigma(e_2') = 4, \sigma(e_3) = \sigma(e_3') = 5, \sigma(e_4) = \sigma(e_4') = 3, \sigma(f_1) = \sigma(f_2) = \sigma(f_3) = \sigma(f_4) = 2, \sigma(g_1) = 5, \sigma(g_2) = 3, \sigma(g_3) = 1, \sigma(g_4) = 4$. Therefore σ is proper-edge coloring. The induced vertex-color sets are: $S_\sigma(x_1) = \{1,2,3,5\}, S_\sigma(x_2) = \{1,2,3,4\}, S_\sigma(x_3) = \{1,2,4,5\}, S_\sigma(x_4) = \{2,3,4,5\}, S_\sigma(x_1') = \{1,2,3,4\}, S_\sigma(x_2') = \{1,2,4,5\}, S_\sigma(x_3') = \{2,3,4,5\}, S_\sigma(x_4') = \{1,2,3,5\}$. Hence σ is an AVD proper edge-coloring A_5. By observation 1.1, $\chi'_{as}(A_4) \geq 5$ and so $\chi'_{as}(A_4) = 5$. Define $\sigma: E(A_5) \to \{1,2,3,4,5\}$ as follows: $(e_1) = \sigma(e_1') = 2, \sigma(e_2) = 5, \sigma(e_2') = 3, \sigma(e_3) = 2, \sigma(e_3') = 5, \sigma(e_4) = 3, \sigma(e_4') = 2, \sigma(e_5) = \sigma(e_5') = 5, \sigma(f_1) = 3, \sigma(f_2) = 4 = \sigma(f_3), \sigma(f_4) = 1 = \sigma(f_5), \sigma(g_1) = 1 = \sigma(g_2), \sigma(g_3) = 3, \sigma(g_4) = 4 = \sigma(g_5)$. Therefore σ is proper-edge coloring. The induced vertex-color sets are: $S_\sigma(x_1) = \{1,2,3,5\}, S_\sigma(x_2) = \{1,2,4,5\}, S_\sigma(x_3) = \{2,3,4,5\}, S_\sigma(x_4) = \{1,2,3,4\}, S_\sigma(x_5) = \{1,3,4,5\}, S_\sigma(x_1') = \{2,3,4,5\}, S_\sigma(x_2') = \{1,2,3,4\}, S_\sigma(x_3') = \{1,3,4,5\}, S_\sigma(x_4') = \{1,2,3,5\}, S_\sigma(x_5') = \{1,2,4,5\}$. Hence σ is an AVD proper edge-coloring A_5. By observation 1.1, $\chi'_{as}(A_5) \geq 5$ and so $\chi'_{as}(A_5) = 5$.

For $n \geq 6$, since $\Delta(A_n) = 4$, by observation 1.1. $\chi'_{as}(A_n) \geq 5$. To show $\chi'_{as}(A_n) \leq 5$. we consider five cases and in each case, we first define $\sigma : E(A_n) \to \{1,2,3,4,5\}$ as follows:

For $n \equiv 0 \pmod 3$

For $i \in \{1,2,\dots,n\}$,

$$\sigma(e_i) = \begin{cases} 5 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$\sigma(e_i') = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$\sigma(f_i) = 4, \ \sigma(g_i) = 1$$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$$\text{For } i \in \{1,2,\dots,n\}, S_\sigma(x_i) = \begin{cases} \{1,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,3,4\} & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

$$S_\sigma(x_i') = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

For $n \equiv 1 \pmod 6$

$\sigma(e_1) = 1 = \sigma(e_1')$

For $i \in \{2,3,\dots,n-1\}$, $\sigma(e_i) = \sigma(e'_i) = \begin{cases} 4 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_n) = 3 = \sigma(e'_n)$

$\sigma(f_1) = 4, \sigma(f_2) = 2,$

For $i \in \{3,4,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 5 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 1 & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$\sigma(f_n) = 5,$

$\sigma(g_1) = 5,$

For $i \in \{2,3,\dots,n-2\}$, $\sigma(g_i) = \begin{cases} 3 & \text{if } i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \\ 5 & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$\sigma(g_{n-1}) = 1, \sigma(g_n) = 2.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,3,4,5\}$

For $i \in \{2,3,\dots,n\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

For $i \in \{1,2,\dots,n-1\}$, $S_\sigma(x'_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \end{cases}$

$S_\sigma(x'_n) = \{1,3,4,5\}$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

For $n \equiv 2 \pmod 6$

$\sigma(e_1) = 1 = \sigma(e'_1)$

For $i \in \{2,3,\dots,n-2\}$, $\sigma(e_i) = \sigma(e'_i) = \begin{cases} 4 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_n) = 3 = \sigma(e'_n)$, $\sigma(e_{n-1}) = 5 = \sigma(e'_{n-1})$

$\sigma(f_1) = \sigma(f_2) = 2$

For $i \in \{3,4,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 5 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 1 & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$\sigma(f_n) = 1,$

$\sigma(g_1) = 5,$

For $i \in \{2,3,\dots,n-2\}$, $\sigma(g_i) = \begin{cases} 3 & \text{if } i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \\ 5 & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$\sigma(g_{n-1}) = 2, \sigma(g_n) = 4.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,3,5\}$

For $i \in \{2,3,\dots,n-1\}, S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_n) = \{1,3,4,5\}$

For $i \in \{1,2,\dots,n-2\}, S_\sigma(x_i') = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \end{cases}$

$S_\sigma(x_{n-1}') = \{1,3,4,5\}, S_\sigma(x_n') = \{1,2,3,5\}$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi_{as}'(A_n) = 5$.

For $n \equiv 4 \pmod 6$

$\sigma(e_1) = 1 = \sigma(e_1')$

For $i \in \{2,3,\dots,n-4\}, \sigma(e_i) = \sigma(e_i') = \begin{cases} 4 & \text{if } i \text{ is even} \\ 5 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_n) = 3 = \sigma(e_n'),\ \sigma(e_{n-1}) = 5 = \sigma(e_{n-1}'),$

$\sigma(e_{n-2}) = 1 = \sigma(e_{n-2}'),\ \sigma(e_{n-3}) = 2 = \sigma(e_{n-3}')$

$\sigma(f_1) = \sigma(f_2) = 2$

For $i \in \{3,4,\dots,n-3\}, \sigma(f_i) = \begin{cases} 2 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 1 & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$\sigma(f_n) = 1, \sigma(f_{n-1}) = 4 = \sigma(f_{n-2})$

$\sigma(g_1) = 5,$

For $i \in \{2,3,\dots,n-4\},\ \sigma(g_i) = \begin{cases} 3 & \text{if } i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \\ 2 & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$\sigma(g_{n-3}) = 5, \sigma(g_{n-2}) = 3, \sigma(g_{n-1}) = 2, \sigma(g_n) = 4.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,3,5\},\ S_\sigma(x_2) = \{1,2,3,4\}$

For $i \in \{3,4,\dots,n-3\}, S_\sigma(x_i) = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$S_\sigma(x_n) = \{1,3,4,5\},\ S_\sigma(x_{n-1}) = \{1,2,4,5\},\ S_\sigma(x_{n-2}) = \{1,2,3,4\}$

$S_\sigma(x_1') = \{1,2,3,4\}$

For $i \in \{ 2,3, \dots, n-4\}, S_\sigma(x_i') = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_{n-3}') = \{1,2,3,4\}, S_\sigma(x_{n-2}') = \{1,2,4,5\}, S_\sigma(x_{n-1}') = \{1,3,4,5\}, S_\sigma(x_n') = \{1,2,3,5\}$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

For $n \equiv 5 \pmod 6$

$\sigma(e_1) = 1 = \sigma(e_1')$

For $i \in \{2,3, \dots, n-4\}, \sigma(e_i) = \sigma(e_i') = \begin{cases} 4 & \text{if } i \text{ is even} \\ 5 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_n) = 2 = \sigma(e_n'),\ \sigma(e_{n-1}) = 3 = \sigma(e_{n-1}'),$

$\sigma(e_{n-2}) = 5 = \sigma(e_{n-2}'),\ \sigma(e_{n-3}) = 3 = \sigma(e_{n-3}')$

$\sigma(f_1) = 3, \sigma(f_2) = 2$

For $i \in \{3,4, \dots, n-3\}, \sigma(f_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$\sigma(f_n) = 5, \sigma(f_{n-1}) = 4, \sigma(f_{n-2}) = 1$

$\sigma(g_1) = 5,$

For $i \in \{2,3, \dots, n-4\},\ \sigma(g_i) = \begin{cases} 3 & \text{if } i \equiv 2 \pmod 3 \\ 2 & \text{if } i \equiv 0 \pmod 3 \\ 1 & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$\sigma(g_{n-3}) = 4, \sigma(g_{n-2}) = 2, \sigma(g_{n-1}) = 1, \sigma(g_n) = 4.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,3,5\},\ S_\sigma(x_2) = \{1,2,3,4\}$

For $i \in \{ 3,4, \dots, n-2\}, S_\sigma(x_i) = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 2 \pmod 3 \end{cases}$

$S_\sigma(x_n) = \{2,3,4,5\},\ S_\sigma(x_{n-1}) = \{1,3,4,5\},\ S_\sigma(x_{n-2}) = \{1,2,3,5\}$

$S_\sigma(x_1') = \{1,2,3,4\}$

For $i \in \{ 2,3, \dots, n-4\}, S_\sigma(x_i') = \begin{cases} \{1,2,4,5\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,3,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{2,3,4,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_{n-3}') = \{1,2,3,5\}, S_\sigma(x_{n-2}') = \{1,3,4,5\}, S_\sigma(x_{n-1}') = \{2,3,4,5\}, S_\sigma(x_n') = \{1,2,3,5\}$

Therefore σ is an AVD proper edge-coloring of A_n. Hence, $\chi'_{as}(A_n) = 5$.

2.2. AVD Proper Edge-chromatic Index of Sunflower Graph

By an sun flower graph of order n denoted by SF_n, we mean a graph that is isomorphic to a graph obtained from Anti-prism graph A_n by deleting edges $y_i y_{(i+1)(mod\ n)}, i = 1,2, \dots, n$.

Theorem 2.2. $\chi'_{as}(SF_n) = 5,\ for\ n \geq 4$.

Proof. Let $C_n = x_1 x_2 \dots x_n x_1$ For $n \geq 4$ and $x'_1, x'_2, \dots, x'_n$ be newly added vertices corresponding to the vertices $x_1, x_2, \dots, x_n$ to form SF_n. In SF_n, for $i \in \{1,2, \dots, n\}$, let $e_i = x_i x_{i+1}$, $f_i = x_i x'_i$, and $g_i = x'_i x_{i+1}$, where $x_{n+1} = x_1$.

Define $\sigma : E(SF_3) \to \{1,2,3,4,5\}$ as follows: $(e_1) = 1, \sigma(e_2) = 2, \sigma(e_3) = 5$, $\sigma(f_1) = \sigma(f_2) = \sigma(f_3) = 3$, $\sigma(g_1) = \sigma(g_2) = \sigma(g_3) = 4$. The induced vertex-color sets are: $S_\sigma(x_1) = \{1,3,4,5\}, S_\sigma(x_2) = \{1,2,3,4\}, S_\sigma(x_3) = \{2,3,4,5\}, S_\sigma(x'_1) = S_\sigma(x'_2) = S_\sigma(x'_3) = \{3,4\}$. Therefore σ is an AVD proper edge-coloring SF_n. Hence, $\chi'_{as}(SF_3) = 5$.

For $n \geq 4$, since $\Delta(SF_n) = 4$, by observation 1.1. $\chi'_{as}(SF_n) \geq 5$. To show $\chi'_{as}(SF_n) \leq 5$. we consider two cases first define $\sigma : E(SF_n) \to \{1,2,3,4,5\}$ as follows:

Case 1. If n is even

For $i \in \{1,2, \dots, n\}$

$$\sigma(e_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(f_i) = \begin{cases} 5 & \text{if } i \text{ is odd} \\ 4 & \text{if } i \text{ is even} \end{cases}$$

$\sigma(g_i) = 3$,

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,3\}$

For $i \in \{1,2,3, \dots, n\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,5\} & \text{if } i \text{ is odd} \\ \{1,2,3,4\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(x'_i) = \begin{cases} \{3,5\} & \text{if } i \text{ is odd} \\ \{3,4\} & \text{if } i \text{ is even} \end{cases}$

Therefore σ is an AVD proper edge-coloring of SF_n. Hence, $\chi'_{as}(SF_n) = 5$

Case 2. If n is odd

For $i \in \{1,2, \dots, n-1\}, \sigma(e_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$

$\sigma(e_n) = 5$

$\sigma(f_1) = 4$,

For $i \in \{2,3, \dots, n-1\}$, $\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is even} \\ 5 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(f_n) = 4$,

For $i \in \{1,2, \dots, n\}, \sigma(g_i) = 3$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,3,4,5\}$

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \text{ is even} \\ \{1,2,3,5\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n) = \{2,3,4,5\}$

$S_\sigma(x'_1) = \{3,4\}$

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(x'_i) = \begin{cases} \{3,4\} & \text{if } i \text{ is even} \\ \{3,5\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x'_n) = \{3,4\}$

Therefore σ is an AVD proper edge-coloring of SF_n. Hence, $\chi'_{as}(SF_n) = 5$.

2.3. AVD Proper Edge-chromatic Index of Double Sunflower Graph

By a double sunflower graph of order n denoted by DSF_n, is a graph obtained from the graph SF_n by inserting a new vertex z_i on each edges $x_i x_{i+1}$ and adding edges $y_i z_i$ for each i.

Theorem 2.3. $\chi'_{as}(DSF_n) = 4,\ for\ n \geq 4,$

Proof. Let $C_n = x_1 x_2 \dots x_n x_1$ For $n \geq 4$ and $x'_1, x'_2, \dots, x'_n$ be newly added vertices corresponding to the vertices $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ be newly added vertices corresponding to the sub division of each edge of the cycle C_n to form DSF_n. In DSF_n, for $i \in \{1,2,\dots,n\}$, let $e_i = x_i y_i$, $e'_i = y_i x_{i+1}$ $f_i = x_i x'_i$, $g_i = x'_i x_{i+1}$ and $h_i = x'_i y_i$ where $x_{n+1} = x_1$.

For $n \geq 4$, since $\Delta(DSF_n) = 4$, by observation 1.2. $\chi'_{as}(DSF_n) \geq 4$. To show $\chi'_{as}(DSF_n) \leq 4$.

We consider two cases first define $\sigma : E(DSF_n) \to \{1,2,3,4\}$ as follows:

Case 1. If n is even

For $i \in \{1,2,\dots,n\}$

$$\sigma(e_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(e'_i) = \begin{cases} 2 & \text{if } i \text{ is odd} \\ 4 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(f_i) = \begin{cases} 2 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(g_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(h_i) = \begin{cases} 3 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{1,2,3,\dots,n\}$, $S_\sigma(x_i) = \{1,2,3,4\}$

$S_\sigma(y_i) = \begin{cases} \{1,2,3\} & \text{if } i \text{ is odd} \\ \{2,3,4\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(x_i') = \begin{cases} \{2,3,4\} & \text{if } i \text{ is odd} \\ \{1,2,3\} & \text{if } i \text{ is even} \end{cases}$

Therefore σ is an AVD proper edge-coloring of DSF_n. Hence, $\chi'_{as}(DSF_n) = 4$.

Case 2. If n is odd

$\sigma(e_1) = 1, \sigma(e_1') = 3$

For $i \in \{2,3, \dots, n\}, \sigma(e_i) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(e_i') = 4,$

For $i \in \{1,2, \dots, n\},\ \sigma(f_i) = \begin{cases} 3 & \text{if } i \neq 2 \\ 2 & \text{if } i = 2 \end{cases}$

$\sigma(g_1) = 4,$

For $i \in \{2,3, \dots, n\},\ \sigma(g_i) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$

$\sigma(h_1) = 2, \sigma(h_2) = 3,$

For $i \in \{3,4, \dots, n\},\ \sigma(h_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{1,2, \dots, n\},\ S_\sigma(x_i) = \{1,2,3,4\}$

$S_\sigma(y_1) = \{1,2,3\}, S_\sigma(y_2) = \{1,3,4\}$

For $i \in \{3,4, \dots, n\},\ S_\sigma(y_i) = \{1,2,4\}$

$S_\sigma(x_1') = \{2,3,4\}$

For $i \in \{2,3, \dots, n\},\ S_\sigma(x_i') = \{1,2,3\}$.

Therefore σ is an AVD proper edge-coloring of DSF_n. Hence, $\chi'_{as}(DSF_n) = 4$.

2.4. AVD Proper Edge-chromatic Index of Triangular Winged Prism

By a triangular winged prism of order n denoted by TWP_n, is a graph obtained from the prism graph D_n, by adding some outsider middle vertices z_i on edge $y_i y_{i+1}$ and adding z_i to both vertices y_i and y_{i+1}.

Theorem 2.4. $\chi'_{as}(TWP_n) = 6,\ for\ n \geq 4$.

Proof. Let $C_n = x_1 x_2 \dots x_n x_1$ For $n \geq 4$, $x_1', x_2', \dots, x_n'$ and $y_1, y_2, \dots, y_n$ be newly added vertices corresponding to the vertices $x_1, x_2, \dots, x_n$ to form TWP_n. In TWP_n, for $i \in \{1,2, \dots, n\}$, let $e_i = x_i x_{i+1}$, $e_i' = x_i' x_{i+1}'$, $f_i = x_i x_i'$, $g_i = x_i' y_i$ and $h_i = x_{i+1}' y_i$, where $x_{n+1} = x_1$, $x_{n+1}' = x_1'$.

For $n \geq 4$, since $\Delta(TWP_n) = 5$, by observation 1.1. $\chi'_{as}(TWP_n) \geq 6$. To show $\chi'_{as}(TWP_n) \leq 6$. we consider two cases first define $\sigma : E(TWP_n) \to \{1,2,3,4,5,6\}$ as follows:

Case 1. If n is even

For $i \in \{1,2,\dots,n\}$

$$\sigma(e_i) = \sigma(e_i') = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(g_i) = 5$$

$$\sigma(h_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 6 & \text{if } i \text{ is even} \end{cases}$$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{1,2,3,\dots,n\}$, $S_\sigma(x_i) = \begin{cases} \{1,3,4\} & \text{if } i \text{ is odd} \\ \{1,2,3\} & \text{if } i \text{ is even} \end{cases}$

$$S_\sigma(x_i') = \begin{cases} \{1,3,4,5,6\} & \text{if } i \text{ is odd} \\ \{1,2,3,4,5\} & \text{if } i \text{ is even} \end{cases}$$

For $i \in \{1,2,\dots,n\}$, $S_\sigma(y_i) = \begin{cases} \{4,5\} & \text{if } i \text{ is odd} \\ \{5,6\} & \text{if } i \text{ is even} \end{cases}$

Therefore σ is an AVD proper edge-coloring of TWP_n. Hence, $\chi'_{as}(TWP_n) = 6$.

Case 2. If n is odd

For $i \in \{1,2,3,\dots,n-1\}$

$$\sigma(e_i) = \sigma(e_i') = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(e_n) = \sigma(e_n') = 2,$$

For $i \in \{1,2,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$

$\sigma(f_n) = 4,$

For $i \in \{1,2,\dots,n\}$, $\sigma(g_i) = 5,$

For $i \in \{1,2,\dots,n-1\}$, $\sigma(h_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 6 & \text{if } i \text{ is even} \end{cases}$

$\sigma(h_n) = 6.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,4\}$

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3\} & \text{if } i \text{ is even} \\ \{1,3,4\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n) = \{2,3,4\}$

$S_\sigma(x'_1) = \{1,2,4,5,6\}$

For $i \in \{2,3,\dots,n-1\}, S_\sigma(x'_i) = \begin{cases} \{1,2,3,4,5\} & \text{if } i \text{ is even} \\ \{1,3,4,5,6\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x'_n) = \{2,3,4,5,6\}$

For $i \in \{1,2,\dots,n-1\}$, $S_\sigma(y_i) = \begin{cases} \{4,5\} & \text{if } i \text{ is odd} \\ \{5,6\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(y_n) = \{5,6\}$.

Therefore σ is an AVD proper edge-coloring of TWP_n. Hence, $\chi'_{as}(TWP_n) = 6$.

2.5. AVD Proper Edge-chromatic Index of Rectangular Winged Prism Graph

By a rectangular winged prism graph of order n denoted by RWP_n, is a graph obtained from the prism graph D_n, by adding an edge a_ib_i corresponding to the edge y_iy_{i+1} and adding an edge a_i to y_i and b_i to y_{i+1}.

Theorem 2.5. $\chi'_{as}(RWP_n) = 6,\ for\ n \geq 4$.

Proof. Let $C_n = x_1x_2 \dots x_nx_1$, For $n \geq 4$ and $x'_1, x'_2, \dots, x'_n$ be newly added vertices corresponding to the vertices $x_1, x_2, \dots, x_n$. Let $y_1, y_2, \dots, y_n$ and $z_1, z_2, \dots, z_n$ be newly added vertices corresponding to the vertices $x'_1, x'_2, \dots, x'_n$ to form RWP_n. In RWP_n, for $i \in \{1,2,\dots,n\}$, let $e_i = x_ix_{i+1}$, $e'_i = x'_ix'_{i+1}$, $e''_i = y_iz_i$, $f_i = x_ix'_i$, $g_i = x'_iy_i$ and $h_i = x'_{i+1}z_i$, where $x_{n+1} = x_1$, $x'_{n+1} = x'_1$.

For $n \geq 4$, since $\Delta(RWP_n) = 5$, by observation 1.1. $\chi'_{as}(RWP_n) \geq 6$. To show $\chi'_{as}(RWP_n) \leq 6$. we consider two cases first define $\sigma : E(RWP_n) \to \{1,2,3,4,5,6\}$ as follows:

Case 1. If n is even

For $i \in \{1,2,\dots,n\}$

$$\sigma(e_i) = \sigma(e'_i) = \sigma(e''_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$$

$$\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$

$\sigma(g_i) = 5$

$\sigma(h_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 6 & \text{if } i \text{ is even} \end{cases}$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{1,2,3,\dots,n\}, S_\sigma(x_i) = \begin{cases} \{1,3,4\} & \text{if } i \text{ is odd} \\ \{1,2,3\} & \text{if } i \text{ is even} \end{cases}$

$$S_\sigma(x_i') = \begin{cases} \{1,3,4,5,6\} & \text{if } i \text{ is odd} \\ \{1,2,3,4,5\} & \text{if } i \text{ is even} \end{cases}$$

For $i \in \{1,2,\ldots,n\}$, $S_\sigma(y_i) = \begin{cases} \{1,5\} & \text{if } i \text{ is odd} \\ \{3,5\} & \text{if } i \text{ is even} \end{cases}$

For $i \in \{1,2,\ldots,n\}$, $S_\sigma(z_i) = \begin{cases} \{1,4\} & \text{if } i \text{ is odd} \\ \{3,6\} & \text{if } i \text{ is even} \end{cases}$

Therefore σ is an AVD proper edge-coloring of RWP_n. Hence, $\chi'_{as}(RWP_n) = 6$.

Case 2. If *n* is odd

For $i \in \{1,2,\ldots,n-1\}$, $\sigma(e_i) = \sigma(e_i') = \sigma(e_i'') = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$

$\sigma(e_n) = \sigma(e_n') = \sigma(e_n'') = 2,$

For $i \in \{1,2,\ldots,n-1\}$, $\sigma(f_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$

$\sigma(f_n) = 4,$

For $i \in \{1,2,\ldots,n\}$, $\sigma(g_i) = 5,$

For $i \in \{1,2,\ldots,n-1\}$, $\sigma(h_i) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 6 & \text{if } i \text{ is even} \end{cases}$

$\sigma(h_n) = 6.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

$S_\sigma(x_1) = \{1,2,4\}$

For $i \in \{2,3,\ldots,n-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3\} & \text{if } i \text{ is even} \\ \{1,3,4\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n) = \{2,3,4\}$

$S_\sigma(x_1') = \{1,2,4,5,6\}$

For $i \in \{2,3,\ldots,n-1\}$, $S_\sigma(x_i') = \begin{cases} \{1,2,3,4,5\} & \text{if } i \text{ is even} \\ \{1,3,4,5,6\} & \text{if } i \text{ is odd} \end{cases}$

$S_\sigma(x_n') = \{2,3,4,5,6\}$

For $i \in \{1,2,\ldots,n-1\}$, $S_\sigma(y_i) = \begin{cases} \{1,5\} & \text{if } i \text{ is odd} \\ \{3,5\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(y_n) = \{2,5\}$

For $i \in \{1,2,\ldots,n-1\}$, $S_\sigma(z_i) = \begin{cases} \{1,4\} & \text{if } i \text{ is odd} \\ \{3,6\} & \text{if } i \text{ is even} \end{cases}$

$S_\sigma(z_n) = \{2,6\}$

Therefore σ is an AVD proper edge-coloring of RWP_n. Hence, $\chi'_{as}(RWP_n) = 6$.

3. AVD Proper Edge-chromatic Index of Polygonal Snake Graph

In this section, we investigate AVD proper edge-coloring of Polygonal snake graph only. A graph is obtained from a path P_m with vertex set $x_1, x_2, \dots, x_m$ by joining all consecutive vertices by path P_n with vertex set $y_1, y_2, \dots, y_n$ in such a way that merging y_1 with x_i and y_n with x_{i+1}, $i \in \{1,2,\dots,n-1\}$ and so on. Then $P_m(S_n)$, $\forall\, m, n$ is called as polygonal snake graph. [8]

Theorem 3.1. $\chi'_{as}\big(P_m(S_n)\big) = 5,\ for\, m \geq 3, n \geq 5.$

Proof. Let $P_m: x_1x_2 \dots x_m$, For $n \geq 5$, $P_n: y_1y_2 \dots y_n$ be attached to an edge x_ix_{i+1}, $i \in \{1,3,\dots,m-1\}$, m is even, where $x_i = y_1,\ x_{i+1} = y_n$ and $P'_n: y'_1y'_2 \dots y'_n$ be attached to an edge x_ix_{i+1}, $i \in \{2,4,\dots,m-1\}$, m is odd, where $x_i = y'_1, x_{i+1} = y'_n$ to form $P_m(S_n)$. In $P_m(S_n)$, for $i \in \{1,2,\dots,m-1\}$, let $e_i = x_ix_{i+1}$. For $i \in \{1,2,\dots,n-1\}$, $f_i = y_iy_{i+1}$, $f'_i = y'_iy'_{i+1}$.

For $m \geq 3, n \geq 5$, since $\Delta\big(P_m(S_n)\big) = 4$, by observation 1.1. $\chi'_{as}\big(P_m(S_n)\big) \geq 5$. To show $\chi'_{as}\big(P_m(S_n)\big) \leq 5$. we consider five cases and in each case, we first define $\sigma : E\big(P_m(S_n)\big) \to \{1,2,3,4,5\}$ as follows:

Case 1: For $n \equiv 5 \pmod 6$

For $i \in \{1,2,\dots,m-1\}$, $\sigma(e_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3 \\ 4 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

For $i \in \{1,2,\dots,n-1\}$, $\sigma(f_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

$$\sigma(f'_i) = \begin{cases} 2 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i \equiv 2 \pmod 3 \\ 1 & \text{if } i \equiv 0 \pmod 3 \end{cases}$$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(y_i) = \begin{cases} \{1,2\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(y'_i) = \begin{cases} \{2,3\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,3\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,2\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_1) = \{1,3\}$,

For $i \in \{2,3,\dots,m-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,2,3,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_m) = \begin{cases} \{2,4\} & \text{if } m \equiv 3 \pmod 6 \\ \{1,5\} & \text{if } m \equiv 4 \pmod 6 \\ \{2,3\} & \text{if } m \equiv 5 \pmod 6 \\ \{1,4\} & \text{if } m \equiv 0 \pmod 6 \\ \{2,5\} & \text{if } m \equiv 1 \pmod 6 \\ \{1,3\} & \text{if } m \equiv 2 \pmod 6 \end{cases}$

Therefore σ is an AVD proper edge-coloring of $P_m(S_n)$. Hence, $\chi'_{as}(P_m(S_n)) = 5$.

Case 2: For $n \equiv 0 \pmod 6$

For $i \in \{1,2,\dots,m-1\}$, $\sigma(e_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3 \\ 4 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

For $i \in \{1,2,\dots,n-1\}$, $\sigma(f_i) = \sigma(f'_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{2,3,\dots,n-1\}$, $S_\sigma(y_i) = S_\sigma(y'_i) = \begin{cases} \{1,2\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_1) = \{1,3\}$,

For $i \in \{2,3,\dots,m-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,2,3,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(x_m) = \begin{cases} \{2,4\} & \text{if } m \equiv 0 \pmod 3 \\ \{2,5\} & \text{if } m \equiv 1 \pmod 3 \\ \{2,3\} & \text{if } m \equiv 2 \pmod 3 \end{cases}$

Therefore σ is an AVD proper edge-coloring of $P_m(S_n)$. Hence, $\chi'_{as}(P_m(S_n)) = 5$.

Case 3: For $n \equiv 1 \pmod 6$

For $i \in \{1,2,\dots,m-1\}$, $\sigma(e_i) = \begin{cases} 3 & \text{if } i \equiv 1 \pmod 3 \\ 4 & \text{if } i \equiv 2 \pmod 3 \\ 5 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

For $i \in \{1,2,\dots,n-4\}$, $\sigma(f_i) = \sigma(f'_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 2 & \text{if } i \equiv 2 \pmod 3 \\ 3 & \text{if } i \equiv 0 \pmod 3 \end{cases}$

$\sigma(f_{n-3}) = \sigma(f'_{n-3}) = 4, \sigma(f_{n-2}) = \sigma(f'_{n-2}) = 1, \sigma(f_{n-1}) = \sigma(f'_{n-1}) = 2.$

Therefore σ is a proper edge-coloring. It remains to show that σ is an AVD proper edge-coloring. We compare the sets of colors of adjacent vertices of the same degree.

The induced vertex-color sets are:

For $i \in \{2,3,\dots,n-4\}$, $S_\sigma(y_i) = S_\sigma(y'_i) = \begin{cases} \{1,2\} & \text{if } i \equiv 2 \pmod 3 \\ \{2,3\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,3\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$S_\sigma(y_{n-3}) = S_\sigma(y'_{n-3}) = \{3,4\}, S_\sigma(y_{n-2}) = S_\sigma(y'_{n-2}) = \{1,4\}, S_\sigma(y_{n-1}) = S_\sigma(y'_{n-1}) = \{1,2\}$

$S_\sigma(x_1) = \{1,3\}$,

For $i \in \{2,3,\dots,m-1\}$, $S_\sigma(x_i) = \begin{cases} \{1,2,3,4\} & \text{if } i \equiv 2 \pmod 3 \\ \{1,2,4,5\} & \text{if } i \equiv 0 \pmod 3 \\ \{1,2,3,5\} & \text{if } i \equiv 1 \pmod 3 \end{cases}$

$$S_\sigma(x_m) = \begin{cases} \{2,4\} & \text{if } m \equiv 0 \pmod 3 \\ \{2,5\} & \text{if } m \equiv 1 \pmod 3 \\ \{2,3\} & \text{if } m \equiv 2 \pmod 3 \end{cases}$$

Therefore σ is an AVD proper edge-coloring of $P_m(S_n)$. Hence, $\chi'_{as}(P_m(S_n)) = 5$.

Case 4: For $n \equiv 2 \pmod 6$

Proof is similar to case 1. $n \equiv 5 \pmod 6$

Case 5: For $n \equiv 3 \pmod 6$

Proof is similar to case 2. $n \equiv 0 \pmod 6$

Case 6: For $n \equiv 4 \pmod 6$

Proof is similar to case 3. $n \equiv 1 \pmod 6$

4. Conclusion

In this paper, I investigate the AVD proper edge-chromatic index of Anti-prism, sunflower graph, double sunflower graph, triangular winged prism and rectangular winged prism. And I also investigate AVD Proper edge-chromatic index of Polygonal snake graph. The investigation of analogous results for different graphs and different operation of above families of graphs are still open.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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