



Local fractional Elzaki transform and its application to local fractional differential equations

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Abstract — The objective of our work is to couple the Elzaki transform method and the local fractional derivative which is called local fractional Elzaki transform, where we have provided important results of this transformation as local fractional Laplace-Elzaki duality, Elzaki transform of the local fractional derivative and the local fractional integral and the local fractional convolution, also we have presented the properties of some special functions with the local fractional derivative sense. The Elzaki transform was applied to solve some linear local fractional differential equations in order to obtain non-differentiable analytical solutions. The results of the solved examples show the effectiveness of the proposed method.

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1. Introduction

The integral transformations play a crucial role in the resolution of ordinary differential equations, partial differential equations and in the resolution of integral differential equations with integer order or fractional order. They are also widely used by engineers to solve the linear differential equations, the systems of linear partial differential equation, and determine the transfer function of a linear system.

The Laplace transform method [1], the Fourier transform method [2], the Hankel transform method [3], and the Mellin transform method [4] are among the most well-known transformations. Other transformations that have recently appeared include for example the Sumudu transform method [5], the Natural transform method [6], the Elzaki transform method [7–10], the Aboodh transform method [11], the ZZ-transform method [12], the Shehu transform method [13], and others.

Tarig M. Elzaki et al. developed the Elzaki transform method in 2011 from the classical Fourier integral [7], based on the Elzaki transform's mathematical simplicity and fundamental features. This transform has been used by many researchers to facilitate the process of solving the ordinary differential equations without and with variable coefficients [14–17], the higher order differential equations [18], the integro-differential

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equations [19–22] and to the partial differential equations [23–27] in the time domain. Several works have been made by using the Elzaki transform method and its combination with other methods to solve the linear and nonlinear differential equations with the concept of fractional operator, for example the fractional order differential equation arising in RLC electrical circuit, the fractional Porous Medium equation, the fractional higher dimensional initial boundary value problems [28–32].

In this work, we will based on the notion of local fractional operator by presented some important results and properties of the local fractional Elzaki transform and we will extend and applied it to solve the linear differential equations with local fractional derivative. We supported our work, with illustrative examples showing how to apply this transformation to the differential equations on Cantor sets.

2. Basic of local fractional calculus

We present the basic definitions and theorems of local fractional derivative, local fractional integral, local fractional Taylor’s series and local fractional Laplace transform method.

Definition 2.1. [33–35] Let $f(x) \in C_\sigma(a, b)$, so

$$|f(x) - f(x_0)| < \epsilon^\sigma, \quad 0 < \sigma \leq 1, \tag{2.1}$$

with $|x - x_0| < \delta$, for $\epsilon, \delta \in \mathbb{R}_+^*$. $f(x)$ is local fractional continuous.

Definition 2.2. [33–35] The local fractional derivative of $f(x)$ of order σ at $x = x_0$ is

$$f^{(\sigma)}(x) = \left. \frac{d^\sigma f}{dx^\sigma} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Delta^\sigma (f(x) - f(x_0))}{(x - x_0)^\sigma}, \quad 0 < \sigma \leq 1, \tag{2.2}$$

where

$$\Delta^\sigma (f(x) - f(x_0)) \cong \Gamma(1 + \sigma) [(f(x) - f(x_0))]. \tag{2.3}$$

The local fractional partial differential operator of order σ was given by

$$\frac{\partial^\sigma w(x_0, t)}{\partial t^\sigma} = \lim_{x \rightarrow x_0} \frac{\Delta^\sigma (w(x_0, t) - w(x_0, t_0))}{(t - t_0)^\sigma}, \tag{2.4}$$

where

$$\Delta^\sigma (u(x_0, t) - u(x_0, t_0)) \cong \Gamma(1 + \sigma) [u(x_0, t) - u(x_0, t_0)]. \tag{2.5}$$

Definition 2.3. [33–35] The local fractional integral of $f(x)$ of order σ is

$$\begin{aligned} {}_a I_b^{(\sigma)} f(x) &= \frac{1}{\Gamma(1 + \sigma)} \int_a^b f(t) (dt)^\sigma \\ &= \frac{1}{\Gamma(1 + \sigma)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\sigma, \end{aligned} \tag{2.6}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \Delta t_1, \Delta t_2, \dots\}$ and $[t_j, t_{j+1}]$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

Definition 2.4. [35–37] The local fractional Laplace transform of $f(x)$ of order σ is given by

$$L_\sigma \{f(x)\} = F_\sigma(s) = \frac{1}{\Gamma(1+\sigma)} \int_0^\infty E_\sigma(-s^\sigma x^\sigma) f(x) (dx)^\sigma. \tag{2.7}$$

If $LF L_\sigma \{f(x)\} = F_\sigma(s)$, so the inverse formula of (2.7) is

$$f(x) = L_\sigma^{-1} \{F_\sigma(s)\} = \frac{1}{(2\pi)^\sigma} \int_{\beta-i\infty}^{\beta+i\infty} E_\sigma(s^\sigma x^\sigma) F_\sigma(s) (ds)^\sigma, \tag{2.8}$$

$s^\sigma = \beta^\sigma + i^\sigma \infty^\sigma$, i^σ is the fractal imaginary unit and $\text{Re}(s) = \beta > 0$.

Theorem 2.5. [34] If $L_\sigma \{f(x)\} = F_\sigma(s)$, then one has

$$L_\sigma \{f^{(\sigma)}(x)\} = s^\sigma L_\sigma \{f(x)\} - f(0). \tag{2.9}$$

Proof.

see [34]

Theorem 2.6. [34] If $L_\sigma \{f(x)\} = F_\sigma(s)$, so

$$L_\sigma \{ {}_0I_x^\sigma f(x) \} = \frac{1}{s^\sigma} L_\sigma \{f(x)\}. \tag{2.10}$$

Proof.

see [34]

Theorem 2.7. [34] If $L_\sigma \{f(x)\} = F_\sigma(s)$ and $L_\sigma \{g(x)\} = G_\sigma(s)$, then one has

$$L_\sigma \{f(x) * g(x)\} = F_\sigma(z) G_\sigma(z), \tag{2.11}$$

where

$$f(x) * g(x) = \frac{1}{\Gamma(1+\sigma)} \int_0^\infty f(t) g(x-t) (dt)^\sigma. \tag{2.12}$$

Proof.

see [34]

Theorem 2.8. [38] Let $f(x) \in C_\sigma[a, b]$, so there is a function

$$\Pi(x) = {}_aI_x^{(\sigma)} f(x),$$

its derivative with respect to $(dx)^\rho$ is

$$\frac{d^\sigma \Pi(x)}{(dx)^\sigma} = f(x), \quad a \leq x \leq b.$$

Proof.

see [38]

3. Main Result

Now, we present the local fractional Elzaki transform method (*LFE_σ*) and some properties are discussed. If there is a new transform operator $LFE_{\sigma} : f(x) \rightarrow T_{\sigma}(v)$, namely

$$LFE_{\sigma} \{f(x)\} = LFE_{\sigma} \left\{ \sum_{k=0}^{\infty} a_k x^{k\sigma} \right\} = \sum_{k=0}^{\infty} \Gamma(1+k\sigma) a_k v^{k\sigma+2\sigma}. \tag{3.1}$$

For example if $f(x) = E_{\sigma}(i^{\sigma} x^{\sigma})$, we obtain

$$LFE_{\sigma} \{E_{\sigma}(i^{\sigma} x^{\sigma})\} = LFE_{\sigma} \left\{ \sum_{k=0}^{\infty} \frac{i^{k\sigma} x^{k\sigma}}{\Gamma(1+k\sigma)} \right\} \tag{3.2}$$

$$= \sum_{k=0}^{\infty} i^{k\sigma} v^{k\sigma+2\sigma}, \tag{3.3}$$

and if $f(x) = \frac{x^{\sigma}}{\Gamma(1+\sigma)}$, we get

$$LFE_{\sigma} \left\{ \frac{x^{\sigma}}{\Gamma(1+\sigma)} \right\} = v^{3\sigma}. \tag{3.4}$$

These results can be generalized by this definition.

Definition 3.1. The local fractional Elzaki transform of $f(x)$ of order σ is defined as

$$LFE_{\sigma} \{f(x)\} = T_{\sigma}(v) = \frac{1}{\Gamma(1+\sigma)} v^{\sigma} \int_0^{\infty} E_{\sigma}\left(-\frac{x^{\sigma}}{v^{\sigma}}\right) f(x) (dx)^{\sigma}, \quad 0 < \sigma \leq 1. \tag{3.5}$$

The inverse transformation can be obtained as follows

$$LFE_{\sigma}^{-1} \{ T_{\sigma}(v) \} = f(x). \tag{3.6}$$

Theorem 3.2. (linearity).

If $LFE_{\sigma} \{f(x)\} = F_{\sigma}(v)$ and $LFE_{\sigma} \{g(x)\} = G_{\sigma}(v)$, then one has

$$LFE_{\sigma} \{ \lambda f(x) + \mu g(x) \} = \lambda F_{\sigma}(v) + \mu G_{\sigma}(v), \tag{3.7}$$

where λ and μ are constant.

Proof.

Using formula (3.5), we obtain

$$\begin{aligned}
 LFE_{\sigma} \{ \lambda f(x) + \mu g(x) \} &= \frac{v^{\sigma}}{\Gamma(1+\sigma)} \int_0^{\infty} E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) \{ \lambda f(x) + \mu g(x) \} (dx)^{\sigma} \\
 &= \frac{v^{\sigma}}{\Gamma(1+\sigma)} \int_0^{\infty} \left[E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) (\lambda f(x)) + E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) (\mu g(x)) \right] (dx)^{\sigma} \\
 &= \lambda \frac{v^{\sigma}}{\Gamma(1+\sigma)} \int_0^{\infty} E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) f(x) (dx)^{\sigma} + \mu \frac{v^{\sigma}}{\Gamma(1+\sigma)} \int_0^{\infty} E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) g(x) (dx)^{\sigma} \\
 &= \lambda LFE_{\sigma} \{ f(x) \} + \mu LFE_{\sigma} \{ g(x) \}.
 \end{aligned}$$

This ends the proof.

Theorem 3.3. (local fractional Elzaki-Laplace and Laplace-Elzaki duality).

If $L_{\sigma} \{ f(x) \} = F_{\sigma}(s)$ and $LFE_{\sigma} \{ g(x) \} = T_{\sigma}(v)$, then one has

$$LFE_{\sigma} \{ f(x) \} = v^{\sigma} F_{\sigma} \left(\frac{1}{v} \right). \tag{3.8}$$

$$L_{\sigma} \{ f(x) \} = s^{\sigma} T_{\sigma} \left(\frac{1}{s} \right). \tag{3.9}$$

Proof.

We show the formula (3.8). Using the formula (3.5) gives

$$\begin{aligned}
 LFE_{\sigma} \{ f(x) \} &= v^{\sigma} \frac{1}{\Gamma(1+\sigma)} \int_0^{\infty} E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) f(x) (dx)^{\sigma} \\
 &= v^{\sigma} \frac{1}{\Gamma(1+\sigma)} \int_0^{\infty} E_{\sigma} \left(-\left(\frac{1}{v} \right)^{\sigma} x^{\sigma} \right) f(x) (dx)^{\sigma} \\
 &= v^{\sigma} F_{\sigma} \left(\frac{1}{v} \right).
 \end{aligned}$$

Proof of the formula (3.9). We have

$$T_{\sigma}(v) = v^{\sigma} F_{\sigma} \left(\frac{1}{v} \right).$$

By substituting $v = \frac{1}{s}$, we obtain

$$T_{\sigma} \left(\frac{1}{s} \right) = \frac{1}{s^{\sigma}} F_{\sigma}(s),$$

then

$$F_{\sigma}(s) = s^{\sigma} T_{\sigma} \left(\frac{1}{s} \right),$$

therefore, we get

$$L_{\sigma} \{ f(x) \} = s^{\sigma} T_{\sigma} \left(\frac{1}{s} \right).$$

This ends the proof.

Theorem 3.4. (Elzaki transform of local fractional derivative).

If $LFE_\sigma \{f(x)\} = T_\sigma(v)$, so

$$LFE_\sigma \{D_{0+}^\sigma f(x)\} = \frac{1}{v^\sigma} T_\sigma(v) - v^\sigma f(0), \quad 0 < \sigma \leq 1, \tag{3.10}$$

and

$$LFE_\sigma \{D_{0+}^{n\sigma} f(x)\} = \frac{1}{v^{n\sigma}} T_\sigma(v) - \sum_{k=0}^{n-1} v^{(k-n+2)\sigma} f^{(k\sigma)}(0), \quad 0 < \sigma \leq 1. \tag{3.11}$$

Proof.

We proof the formula (3.10). Using the formula (3.8) and the integration by parts [41], we get the following

$$\begin{aligned} LFE_\sigma \{f^{(\sigma)}(x)\} &= v^\sigma F_\sigma \left(\frac{1}{v} \right) = \frac{v^\sigma}{\Gamma(1+\sigma)} \int_0^\infty E_\sigma \left(-\frac{x^\sigma}{v^\sigma} \right) f^{(\sigma)}(x) (dx)^\sigma \\ &= \frac{v^\sigma}{\Gamma(1+\sigma)} \left([-\Gamma(1+\sigma) f(0)] + \frac{1}{v^\sigma} \lim_{t \rightarrow \infty} \int_0^t E_\sigma \left(-\frac{x^\sigma}{v^\sigma} \right) f(x) (dx)^\sigma \right) \\ &= -v^\sigma f(0) + \frac{1}{v^\sigma} \left(\frac{v^\sigma}{\Gamma(1+\sigma)} \int_0^\infty E_\sigma \left(-\frac{x^\sigma}{v^\sigma} \right) f(x) (dx)^\sigma \right) \\ &= \frac{1}{v^\sigma} T_\sigma(v) - v^\sigma f(0). \end{aligned}$$

To demonstrate the validity of (3.11), we use the mathematical induction.

If $n = 1$ and according to (3.11), we have

$$LFE_\sigma \{D_{0+}^\sigma f(x)\} = \frac{1}{v^\sigma} T_\sigma(v) - v^\sigma f(0),$$

so, according to (3.10), we note that the formula holds when $n = 1$.

Assume inductively that the formula holds for n , we get

$$LFE_\sigma \{D_{0+}^{n\sigma} f(x)\} = \frac{1}{v^{n\sigma}} T_\sigma(v) - \sum_{k=0}^{n-1} v^{(k-n+2)\sigma} f^{(k\sigma)}(0). \tag{3.12}$$

It remains to show that (3.11) is true for $n + 1$. Let $D_{0+}^{n\sigma} f(x) = g(x)$ and according to (3.10) and (3.12), we have

$$\begin{aligned} LFE_\sigma \left[D_{0+}^{(n+1)\sigma} f(x) \right] &= LFE_\sigma \left[D_{0+}^\sigma g(x) \right] = \frac{1}{v^\sigma} G_\sigma(v) - v^\sigma g(0) \\ &= \frac{1}{v^\sigma} \left[\frac{1}{v^{n\sigma}} T_\sigma(v) - \sum_{k=0}^{n-1} v^{(k-n+2)\sigma} f^{(k\sigma)}(0) \right] - v^\sigma g(0) \\ &= \frac{1}{v^{(n+1)\sigma}} T_\sigma(v) - \sum_{k=0}^{n-1} v^{(k-n+1)\sigma} f^{(k\sigma)}(0) - v^\sigma f^{(n\sigma)}(x) \\ &= \frac{1}{v^{(n+1)\sigma}} T_\sigma(v) - \sum_{k=0}^n v^{(k-n+1)\sigma} f^{(k\sigma)}(0). \end{aligned}$$

Therefore the formula (3.11) is true for $n + 1$. Thus by the principle of mathematical induction, for all $n \geq 1$ the formula (3.11) holds.

Theorem 3.5. (Elzaki transform of local fractional integral).

If $LFE_\sigma \{f(x)\} = T_\sigma(v)$, so

$$LFE_\sigma \{ {}_0I_x^{(\sigma)} f(x) \} = v^\sigma T_\sigma(v). \tag{3.13}$$

Proof.

Let $H(x) = {}_0I_x^{(\sigma)} f(x)$. According to the (theorem 3.2.9, [38]), we get

$$D_{0+}^\sigma H(x) = f(x), \tag{3.14}$$

and $H(0) = 0$.

Taking the local fractional Elzaki transform on both sides of (3.14), we have

$$LFE_\sigma \{ D_{0+}^\sigma H(x) \} = LFE_\sigma \{ f(x) \}.$$

Which give

$$\frac{1}{v^\sigma} LFE_\sigma \{ H(x) \} = T_\sigma(v),$$

because $H(0) = 0$, and $LFE_\sigma \{ f(x) \} = T_\sigma(v)$.

Thus we get

$$LFE_\sigma \{ {}_0I_x^{(\sigma)} f(x) \} = v^\sigma T_\sigma(v).$$

Theorem 3.6. (local fractional convolution).

If $LFE_\sigma \{f(x)\} = T_\sigma(v)$ and $LFE_\sigma \{g(x)\} = G_\sigma(v)$,

then one has

$$LFE_\sigma \{ (f(x) * g(x))_\sigma \} = \frac{1}{v^\sigma} T_\sigma(v) G_\sigma(v), \tag{3.15}$$

where

$$(f(x) * g(x))_\sigma = \frac{1}{\Gamma(1+\sigma)} \int_0^\infty f(\theta) g(r-\theta) (dr)^\sigma.$$

Proof.

The Laplace transform of fractional order of the function $(f(x) * g(x))_\sigma$ is

$$L_\sigma \{(f(x) * g(x))_\sigma\} = L_\sigma \{f(x)\} L_\sigma \{g(x)\}.$$

Using the formula (3.8) gives

$$\begin{aligned} LFE_\sigma \{(f(x) * g(x))_\sigma\} &= \nu^\sigma L_\sigma \{f(x) * g(x)\} \\ &= \nu^\sigma L_\sigma \{f(x)\} L_\sigma \{g(x)\} \\ &= \frac{1}{\nu^\sigma} (\nu^\sigma L_\sigma \{f(x)\} \nu^\sigma L_\sigma \{g(x)\}) \\ &= \frac{1}{\nu^\sigma} T_\sigma(\nu) G_\sigma(\nu). \end{aligned}$$

This completes the proof.

3.1. Local fractional Elzaki transform of some special functions

In the all following results, we relied on the formula (3.5), and some of the results found in the references [39–41].

1) If $f(x) = 1$, we get

$$\begin{aligned} LFE_\sigma \{1\} &= \frac{1}{\Gamma(1+\sigma)} \nu^\sigma \int_0^\infty E_\sigma \left(-\frac{x^\sigma}{\nu^\sigma} \right) (dx)^\sigma \\ &= \nu^\sigma \lim_{t \rightarrow \infty} \left[-\nu^\sigma E_\sigma \left(-\frac{x^\sigma}{\nu^\sigma} \right) \right]_0^t \\ &= \nu^{2\sigma}. \end{aligned}$$

2) If $f(x) = \frac{x^\sigma}{\Gamma(1+\sigma)}$ ($0 < \sigma \leq 1$), by using the integration by parts [41], we obtain the following

$$\begin{aligned} LFE_\sigma \left\{ \frac{x^\sigma}{\Gamma(1+\sigma)} \right\} &= \frac{1}{\Gamma(1+\sigma)} \nu^\sigma \int_0^\infty E_\sigma \left(-\frac{x^\sigma}{\nu^\sigma} \right) \frac{x^\sigma}{\Gamma(1+\sigma)} (dx)^\sigma \\ &= \frac{\nu^\sigma}{\Gamma(1+\sigma)} \lim_{t \rightarrow \infty} \left(\int_0^t \left(-\nu^\sigma E_\sigma \left(-\frac{x^\sigma}{\nu^\sigma} \right) \right)^{(\sigma)} \frac{x^\sigma}{\Gamma(1+\sigma)} (dx)^\sigma \right) \\ &= \nu^\sigma \lim_{t \rightarrow \infty} \left(\left[-\nu^\sigma E_\sigma \left(-\frac{x^\sigma}{\nu^\sigma} \right) \frac{x^\sigma}{\Gamma(1+\sigma)} \right]_0^t + \frac{\nu^\sigma}{\Gamma(1+\sigma)} \int_0^t E_\sigma \left(-\frac{x^\sigma}{\nu^\sigma} \right) (dx)^\sigma \right) \\ &= \lim_{t \rightarrow \infty} \left(\nu^{2\sigma} \left[-\nu^\sigma E_\sigma \left(-\frac{x^\sigma}{\nu^\sigma} \right) \right]_0^t \right). \end{aligned}$$

Because $\lim_{t \rightarrow \infty} \left[-\nu^\sigma E_\sigma \left(-\frac{x^\sigma}{\nu^\sigma} \right) \frac{x^\sigma}{\Gamma(1+\sigma)} \right]_0^t = 0$.

Therefore,

$$\begin{aligned} LFE_\sigma \left\{ \frac{x^\sigma}{\Gamma(1+\sigma)} \right\} &= \lim_{t \rightarrow \infty} \left(v^{2\sigma} \left[-v^\sigma E_\sigma \left(-\frac{x^\sigma}{v^\sigma} \right) \right]_0^t \right) \\ &= v^{3\sigma}. \end{aligned}$$

3) If $f(x) = E_\sigma((ax)^\sigma)$, using the formula (3.5) gives

$$\begin{aligned} LFE_\sigma \{E_\sigma((ax)^\sigma)\} &= \frac{v^\sigma}{\Gamma(1+\sigma)} \int_0^\infty E_\sigma\left(-\frac{x^\sigma}{v^\sigma}\right) E_\sigma((ax)^\sigma) (dx)^\sigma \\ &= \frac{v^\sigma}{\Gamma(1+\sigma)} \int_0^\infty E_\sigma\left(-\frac{1-(av)^\sigma}{v^\sigma} x^\sigma\right) (dx)^\sigma \\ &= v^\sigma \lim_{t \rightarrow \infty} \left[\frac{-v^\sigma}{1-(av)^\sigma} E_\sigma\left(-\frac{1-(av)^\sigma}{v^\sigma} x^\sigma\right) \right]_0^t \\ &= \frac{v^{2\sigma}}{1-(av)^\sigma}. \end{aligned}$$

4) If $f(x) = \frac{x^\sigma}{\Gamma(1+\sigma)} E_\sigma((ax)^\sigma)$, by using the definition, and the integration by parts [41], we have

$$\begin{aligned} LFE_\sigma \left\{ \frac{x^\sigma}{\Gamma(1+\sigma)} E_\sigma((ax)^\sigma) \right\} &= \frac{v^\sigma}{\Gamma(1+\sigma)} \int_0^\infty E_\sigma\left(-\frac{x^\sigma}{v^\sigma}\right) \frac{x^\sigma}{\Gamma(1+\sigma)} E_\sigma((ax)^\sigma) (dx)^\sigma \\ &= \frac{v^\sigma}{\Gamma(1+\sigma)} \lim_{t \rightarrow \infty} \int_0^t \left(\frac{-v^\sigma}{1-(av)^\sigma} E_\sigma\left(-\frac{1-(av)^\sigma}{v^\sigma} x^\sigma\right) \right)^{(\sigma)} \frac{x^\sigma}{\Gamma(1+\sigma)} (dx)^\sigma \\ &= v^\sigma \lim_{t \rightarrow \infty} \left[\frac{-v^\sigma}{1-(av)^\sigma} E_\sigma\left(-\frac{1-(av)^\sigma}{v^\sigma} x^\sigma\right) \frac{x^\sigma}{\Gamma(1+\sigma)} \right]_0^t \\ &\quad + \frac{v^\sigma}{\Gamma(1+\sigma)} \lim_{t \rightarrow \infty} \int_0^t \frac{v^\sigma}{1-(av)^\sigma} E_\sigma\left(-\frac{1-(av)^\sigma}{v^\sigma} x^\sigma\right) (dx)^\sigma \\ &= v^\sigma \lim_{t \rightarrow \infty} \left[\frac{-v^{2\sigma}}{(1-(av)^\sigma)^2} E_\sigma\left(-\frac{1-(av)^\sigma}{v^\sigma} x^\sigma\right) \right]_0^t, \end{aligned}$$

because $\lim_{t \rightarrow \infty} \left[\frac{-v^\sigma}{1-(av)^\sigma} E_\sigma\left(-\frac{1-(av)^\sigma}{v^\sigma} x^\sigma\right) \frac{x^\sigma}{\Gamma(1+\sigma)} \right]_0^t = 0$.

Therefore, we get

$$LFE_\sigma \left\{ \frac{x^\sigma}{\Gamma(1+\sigma)} E_\sigma((ax)^\sigma) \right\} = \frac{v^{3\sigma}}{(1-(av)^\sigma)^2}. \tag{3.16}$$

5) If $f(x) = \sin_\sigma((ax)^\sigma)$ ($0 < \sigma \leq 1$), by using the formula (3.5), we get

$$\begin{aligned} LFE_\sigma \{ \sin_\sigma((ax)^\sigma) \} &= \frac{v^\sigma}{\Gamma(1+\sigma)} \int_0^\infty E_\sigma\left(-\frac{x^\sigma}{v^\sigma}\right) \frac{E_\sigma(i^\sigma(ax)^\sigma) - E_\sigma(-i^\sigma(ax)^\sigma)}{2i^\sigma} (dx)^\sigma \\ &= \frac{1}{2i^\sigma} \frac{v^\sigma}{\Gamma(1+\sigma)} \int_0^\infty \left[E_\sigma\left(-\frac{1-(av)^\sigma i^\sigma}{v^\sigma} x^\sigma\right) - E_\sigma\left(-\frac{1+(av)^\sigma i^\sigma}{v^\sigma} x^\sigma\right) \right] (dx)^\sigma \\ &= \frac{v^\sigma}{2i^\sigma} \lim_{t \rightarrow \infty} \left[-\frac{v^\sigma}{1-(av)^\sigma i^\sigma} E_\sigma\left(-\frac{1-(av)^\sigma i^\sigma}{v^\sigma} x^\sigma\right) + \frac{v^\sigma}{1+(av)^\sigma i^\sigma} E_\sigma\left(-\frac{1+(av)^\sigma i^\sigma}{v^\sigma} x^\sigma\right) \right]_0^t. \end{aligned}$$

After calculations, we find

$$LFE_{\sigma} \{ \sin_{\sigma}((ax)^{\sigma}) \} = \frac{a^{\sigma} v^{3\sigma}}{1 + (av)^{2\sigma}}. \tag{3.17}$$

6) If $f(x) = \cos_{\sigma}((ax)^{\sigma})$ ($0 < \sigma \leq 1$), the use of the formula (3.5) gives

$$\begin{aligned} LFE_{\sigma} \{ \cos_{\sigma}((ax)^{\sigma}) \} &= \frac{v^{\sigma}}{\Gamma(1 + \sigma)} \int_0^{\infty} E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) \frac{E_{\sigma}(i^{\sigma}(ax)^{\sigma}) + E_{\sigma}(-i^{\sigma}(ax)^{\sigma})}{2} (dx)^{\sigma} \\ &= \frac{1}{2} \frac{v^{\sigma}}{\Gamma(1 + \sigma)} \int_0^{\infty} \left[E_{\sigma} \left(-\frac{1 - (av)^{\sigma} i^{\sigma}}{v^{\sigma}} x^{\sigma} \right) + E_{\sigma} \left(-\frac{1 + (av)^{\sigma} i^{\sigma}}{v^{\sigma}} x^{\sigma} \right) \right] (dx)^{\sigma} \\ &= \frac{v^{\sigma}}{2} \lim_{t \rightarrow \infty} \left[-\frac{v^{\sigma}}{1 - (av)^{\sigma} i^{\sigma}} E_{\sigma} \left(-\frac{1 - (av)^{\sigma} i^{\sigma}}{v^{\sigma}} x^{\sigma} \right) - \frac{v^{\sigma}}{1 + (av)^{\sigma} i^{\sigma}} E_{\sigma} \left(-\frac{1 + (av)^{\sigma} i^{\sigma}}{v^{\sigma}} x^{\sigma} \right) \right]_0^t. \end{aligned}$$

After calculations, we find

$$LFE_{\sigma} \{ \cos_{\sigma}((ax)^{\sigma}) \} = \frac{v^{2\sigma}}{1 + (av)^{2\sigma}}. \tag{3.18}$$

7) If $f(x) = \sinh_{\sigma}((ax)^{\sigma})$ ($0 < \sigma \leq 1$), we obtain

$$\begin{aligned} LFE_{\sigma} \{ \sinh_{\sigma}((ax)^{\sigma}) \} &= \frac{v^{\sigma}}{\Gamma(1 + \sigma)} \int_0^{\infty} E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) \frac{E_{\sigma}((ax)^{\sigma}) - E_{\sigma}(- (ax)^{\sigma})}{2} (dx)^{\sigma} \\ &= \frac{1}{2} \frac{v^{\sigma}}{\Gamma(1 + \sigma)} \int_0^{\infty} \left[E_{\sigma} \left(-\frac{1 - (av)^{\sigma}}{v^{\sigma}} x^{\sigma} \right) - E_{\sigma} \left(-\frac{1 + (av)^{\sigma}}{v^{\sigma}} x^{\sigma} \right) \right] (dx)^{\sigma} \\ &= \frac{v^{\sigma}}{2} \lim_{t \rightarrow \infty} \left[-\frac{v^{\sigma}}{1 - (av)^{\sigma}} E_{\sigma} \left(-\frac{1 - (av)^{\sigma}}{v^{\sigma}} x^{\sigma} \right) + \frac{v^{\sigma}}{1 + (av)^{\sigma}} E_{\sigma} \left(-\frac{1 + (av)^{\sigma}}{v^{\sigma}} x^{\sigma} \right) \right]_0^t. \end{aligned}$$

By performing simple operations, we find

$$LFE_{\sigma} \{ \sinh_{\sigma}((ax)^{\sigma}) \} = \frac{a^{\sigma} v^{3\sigma}}{1 - (av)^{2\sigma}}. \tag{3.19}$$

8) If $f(x) = \cosh_{\sigma}((ax)^{\sigma})$ ($0 < \sigma \leq 1$), we obtain

$$\begin{aligned} LFE_{\sigma} \{ \cosh_{\sigma}((ax)^{\sigma}) \} &= \frac{v^{\sigma}}{\Gamma(1 + \sigma)} \int_0^{\infty} E_{\sigma} \left(-\frac{x^{\sigma}}{v^{\sigma}} \right) \frac{E_{\sigma}((ax)^{\sigma}) + E_{\sigma}(- (ax)^{\sigma})}{2} (dx)^{\sigma} \\ &= \frac{1}{2} \frac{v^{\sigma}}{\Gamma(1 + \sigma)} \int_0^{\infty} \left[E_{\sigma} \left(-\frac{1 - (av)^{\sigma}}{v^{\sigma}} x^{\sigma} \right) + E_{\sigma} \left(-\frac{1 + (av)^{\sigma}}{v^{\sigma}} x^{\sigma} \right) \right] (dx)^{\sigma} \\ &= \frac{v^{\sigma}}{2} \lim_{t \rightarrow \infty} \left[-\frac{v^{\sigma}}{1 - (av)^{\sigma}} E_{\sigma} \left(-\frac{1 - (av)^{\sigma}}{v^{\sigma}} x^{\sigma} \right) - \frac{v^{\sigma}}{1 + (av)^{\sigma}} E_{\sigma} \left(-\frac{1 + (av)^{\sigma}}{v^{\sigma}} x^{\sigma} \right) \right]_0^t. \end{aligned}$$

By performing simple operations, we get

$$LFE_{\sigma} \{ \cosh_{\sigma}((ax)^{\sigma}) \} = \frac{v^{2\sigma}}{1 - (av)^{2\sigma}}. \tag{3.20}$$

4. Illustrative examples

Finally, we'll use the local fractional Elzaki transform to solve the local fractional differential equations that have been suggested.

Example 4.1. We consider the local fractional differential equation of order σ

$$\frac{d^\sigma U(x)}{dx^\sigma} - U(x) = 1, \quad 0 < \sigma \leq 1, \tag{4.1}$$

with the initial condition $U(0) = 0$.

On both sides of (4.1), the local fractional Elzaki transform yields

$$\frac{1}{v^\sigma} LFE_\sigma \{U(x)\} - v^\sigma U(0) - LFE_\sigma \{U(x)\} = LFE_\sigma \{1\}. \tag{4.2}$$

Then

$$\left(\frac{1}{v^\sigma} - 1\right) LFE_\sigma \{U(x)\} = v^{2\sigma}. \tag{4.3}$$

Which give

$$\begin{aligned} LFE_\sigma \{U(x)\} &= \frac{v^{3\sigma}}{1 - v^\sigma} \\ &= \frac{v^{2\sigma}}{1 - v^\sigma} - v^{2\sigma}. \end{aligned} \tag{4.4}$$

We get (4.4) by doing the inverse transformation on both sides

$$U(x) = E_\sigma(-x^\sigma) - 1, \tag{4.5}$$

which is the exact solution of the equation (4.1). E_σ is the Mittag-Leffler function.

Example 4.2. Next, we consider the following local fractional differential equation of order σ , ($0 < \sigma \leq 1$)

$$\frac{d^\sigma U(x)}{dx^\sigma} - 2U(x) = 4, \tag{4.6}$$

with the initial condition

$$U(0) = 0. \tag{4.7}$$

Taking local fractional Elzaki transform on both sides of (4.6), we have

$$\frac{1}{v^\sigma} LFE_\sigma \{U(x)\} - 2LFE_\sigma \{U(x)\} = 4v^{2\sigma}. \tag{4.8}$$

By following the same steps as the previous example, we obtain

$$LFE_{\sigma}\{U(x)\} = -2\nu^{2\sigma} + 2\frac{\nu^{2\sigma}}{1-2\nu^{\sigma}}. \quad (4.9)$$

Take the inverse transformation on both sides of (4.9), we get

$$U(x) = 2E_{\sigma}(2x^{\sigma}) - 2. \quad (4.10)$$

The result (4.10) represents the exact solution of (4.6) – (4.7).

Example 4.3. We consider the local fractional differential equation of order 2σ , ($0 < \sigma \leq 1$)

$$\frac{d^{2\sigma}U(x)}{dx^{2\sigma}} + U(x) = -\frac{x^{\sigma}}{\Gamma(1+\sigma)}, \quad (4.11)$$

with to the initial conditions

$$U(0) = 0, U^{(\sigma)}(0) = 0. \quad (4.12)$$

Taking local fractional Elzaki transform on both sides of (4.11), we get

$$\frac{1}{\nu^{2\sigma}} LFE_{\sigma}\{U(x)\} + LFE_{\sigma}\{U(x)\} = -\nu^{3\sigma}. \quad (4.13)$$

We get the same result as the previous example by following the same steps

$$LFE_{\sigma}\{U(x)\} = -\nu^{3\sigma} + \frac{\nu^{3\sigma}}{1+\nu^{2\sigma}}. \quad (4.14)$$

Take the inverse transformation on both sides of (4.14), we get

$$U(x) = \sin_{\sigma}(x^{\sigma}) - \frac{x^{\sigma}}{\Gamma(1+\sigma)}, \quad (4.15)$$

so this result (4.15) represents the exact solution to the equation (4.11) – (4.12).

5. Conclusion

The idea that we presented in this paper is based on combining the Elzaki transform with the local fractional derivative, where we present some important results of this combination which called: Local fractional Elzaki transform of the Mittag-Leffler function, the hyperbolic sine and the hyperbolic cosine in fractal space and also we provide its properties of some non-differentiable functions were presented. This method was used to solve several linear local fractional differential equations, where we have seen that the non-differential solutions are precise and these results lead us to say that the local fractional Elzaki transform is powerful and effective in solving this type of equation, and thus can be applied to other linear local fractional partial differential equations with variable coefficients, to the linear systems of differential equations and to the other linear problems in Cantor sets.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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