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RESEARCH ARTICLE

GRAPH-DIRECTED FRACTAL IMAGE COMPRESSION

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ABSTRACT

Fractal image compression is a method to compress images, which significantly reduces the storage to keep data of the images, using partial iterated function systems. In this work, we generalize the classical fractal image compression method to the graph-directed case.

Keywords: Image compression, Fractals, Graph-directed fractals

1. INTRODUCTION

Fractal geometry is an exciting subject of mathematics that has many applications to real life. One of them is the so-called fractal image compression which we deal with in this article. The method of fractal image compression is based on iterated function systems which are introduced by the pioneering work of Hutchinson in 1981, [7] (the term "iterated function systems" is suggested by Barnsley). In 80's the iterated function systems were applied to model natural shapes in computer graphics. Barnsley and Sloan introduced an excellent technique the so-called "fractal image compression" to compress images, their compression ratio was 1/10000 ([1],[2],[3]). First, they divided the image that they wanted to compress into parts that are self-similar and do not intersect with each other. For each part, using the Collage Theorem, they found an associated iterated function system whose attractor is close to that part of the image. The idea behind the algorithm is to keep the data of the associated IFSs instead of the data of the image itself, which reduces the amount of storage needed. In the decoding process, each part uses the chaos game procedure to the associated IFS. By combining the attractors of these associated IFSs one obtains an image close to the original image. However, their algorithm requires someone manual intervention at least in dividing the image into parts and the encoding algorithm takes lots of time.

Jacquin introduced a new algorithm for image compression, which is fully automated in 1989 (see [8] for more details). His algorithm, based on recurrent IFS, has disadvantages on the quality of decompressed image and the time needed for compression. There are some works to handle these problems in the last decades such as Fisher's work, mainly based on Jacquin's method, so called HV-partitioning [6]. Higher quality image in a shorter time can be obtained via this new method. After years, Fisher and his coworkers improve new techniques such as quadtree technique, triangular technique, etc. to improve the quality of compression and manage the time problem.

In this work, we generalize the notion of fractal image compression to the graph-directed case. So, we first give a small brief for iterated function systems (IFS) and graph-directed IFS, which can be considered as a generalization of classical IFSs.

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1.1. Iterated Function Systems

Let $(X,\ d)$ be a complete metric space and $\{w_i\colon X \to X \mid i = 1,2,\dots,n\}$ be a family of contractions with contractivity factors $0 < r_i < 1$. The system $\{X; w_1, w_2, \dots, w_n\}$ is called an iterated function system. For a given IFS $\{X; w_1, w_2, \dots, w_n\}$, one can define a map

$$W\colon \mathcal{H}(X) \to \mathcal{H}(X), \qquad W(A) = \bigcup_{i=1}^{n} w_i(A)$$

where $\mathcal{H}(X)$ is the set of non-empty compact subsets of X which is also complete metric space with the Hausdorff metric. Then W is a contraction on $\mathcal{H}(X)$ and thus, by the Banach fixed point theorem, there exists a unique non-empty compact set $A \subset X$ such that $W(A) = A$. The fixed point A of W is called the attractor of the IFS.

An attractor of an IFS can be considered as a metric space, a finite union of scaled copies of itself. One can consider several metric spaces each of which is a finite union of scaled copies of themselves. Such metric spaces are called graph-directed fractals which we now summarize shortly.

Let $\{(X_\alpha, d_\alpha) \mid \alpha = 1,\dots,N\}$ be a finite collection of complete metric spaces. Let

$$w_k^{\alpha,\beta}\colon X_\beta \to X_\alpha$$

be contractions with contractivity ratios $0 < r_k^{\alpha,\beta} < 1$ $(\alpha, \beta = 1, \dots, N$ and $k = 1,2,\dots,K^{\alpha,\beta})$. The system $\left\{X_\alpha; w_k^{\alpha,\beta}\right\}$ is called a graph-directed iterated function system (GIFS) (see [10] for more details).

For simplicity, set $N = 2$. Define the operator W as

$$W\colon\ \mathcal{H}(X_1) \times \mathcal{H}(X_2) \to \mathcal{H}(X_1) \times \mathcal{H}(X_2)$$

$$W(U,V) = \left(\bigcup_{k=1}^{K^{1,1}} w_k^{1,1}(U) \cup \bigcup_{k=1}^{K^{1,2}} w_k^{1,2}(V), \bigcup_{k=1}^{K^{2,1}} w_k^{2,1}(U) \cup \bigcup_{k=1}^{K^{2,2}} w_k^{2,2}(V) \right).$$

This map is also a contraction on the complete metric space $\mathcal{H}(X_1) \times \mathcal{H}(X_2)$ (concerning the maximum Hausdorff metric) and thus there exists a pair of subsets $A^1 \subseteq X_1$ and $A^2 \subseteq X_2$ such that $W(A^1, A^2) = (A^1, A^2)$ ([5]). These subsets are called attractors of the system $\left\{X_\alpha; w_k^{\alpha,\beta}\right\}$ and can be expressed as follows:

$$A^1 = \bigcup_{k=1}^{K^{1,1}} w_k^{1,1}(A^1) \cup \bigcup_{k=1}^{K^{1,2}} w_k^{1,2}(A^2)$$

$$A^2 = \bigcup_{k=1}^{K^{2,1}} w_k^{2,1}(A^1) \cup \bigcup_{k=1}^{K^{2,2}} w_k^{2,2}(A^2)$$

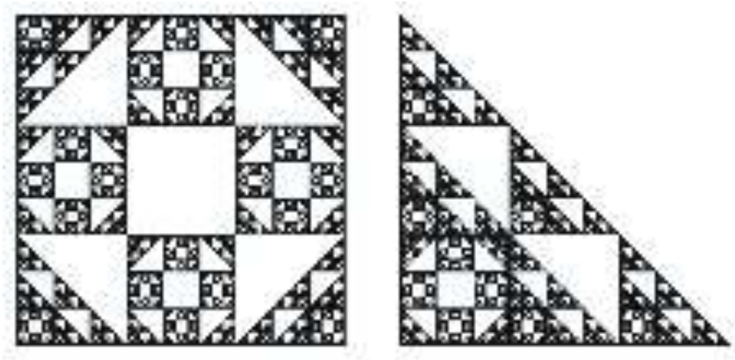

Figure 1. An example of graph-directed fractals ([4])

In the next subsection we give a small brief of the fractal image compression method:

1.2. Mathematical Background of Classical Fractal Image Compression

In this section, we overview the fractal image compression (FIC) method and explain the mathematical basis of FIC.

We can regard a grayscale image on the unit square $I^2 = [0,1] \times [0,1]$ as the graph of a measurable, bounded function $f: I^2 \to [0,1]$. Then the space of all images I^2 will be the space of graphs of bounded measurable functions on the unit square

$$\mathcal{M} := \{graf(f) \mid f: I^2 \to [0,1]\}$$

where we measure the distance between two given images $graf(f)$ and $graf(g)$ by the supremum distance between the functions f and g, that is

$$d_\infty(graf(f), graf(g)) = sup\{|f(x) - g(x)| \, x \in I^2\}.$$

The metric space $(\mathcal{M}, d_\infty)$ is a complete metric space.

Remark 1.1. For simplicity, we will denote a function and its graph by the same notation when it does not cause confusion.

Let f be an image on I^2. The real problem is to find a contractive map $W: \mathcal{M} \to \mathcal{M}$ such that $W(f) = f$. This condition which is hard to satisfy, can be softened in the following way: Instead of finding W satisfying $W(f) = f$, one may seek a contraction map whose fixed point is close enough to the original image f. More precisely, the conditions are

i. (Contractivity of W) $d_\infty(W(g_1), W(g_2)) \leq r\, d_\infty(g_1, g_2)$ for some $0 < r < 1$, for all $g_1,\ g_2 \in \mathcal{M}$,

ii. (Closeness to f) $d_\infty(f, W(f))$ is as small as possible for the original image f.

Using the triangular inequality repeatedly and the contractivity of W one can obtain

$$d_\infty\big(f, W^n(f_0)\big) \leq \frac{1}{1-r} d_\infty\big(f, W(f)\big) + r^n d_\infty(f, f_0) \tag{1.1}$$

for an arbitrary initial image f_0. If one can find a map W satisfying the conditions i and ii, by the inequality (1.1) one can get an upper bound for the distance $d_\infty(f, W^n(f_0))$ for an arbitrary initial image

f_0. When the number of iterations increased, since $r < 1$, the second term on the right hand side of the inequality (1.1) gets smaller, thus the upper bound decreases. Although the distance $d_\infty(f, W(f))$ desired as small as possible, if r is close to 1, the first term can be large and the resulting compression may not be good enough. If the value of r is much less than 1 good compression ratios can be achieved.

Now we summarize the encoding process of fractal image compression and sketch how to find a contraction map W satisfying the above conditions i and ii.

First, we take a partition of I^2 into so-called range blocks $R_1, R_2, \dots, R_n$ such that $R_i \cap R_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} R_i = I^2$. These subsets are usually taken as squares of equal size (say $p \times p$ pixels).

Then, for each R_i it should be find a domain D_i (which is usually be taken as squares of the size $2p \times 2p$ pixels) and a map $w_i: D_i \times I \to R_i \times I$ in the form

$$w_i(x,y,z) = \begin{pmatrix} a_i & b_i & 0 \\ c_i & d_i & 0 \\ 0 & 0 & s_i \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \\ o_i \end{pmatrix} \tag{1.2}$$

$$= (a_i x + b_i y + e_i, c_i x + d_i y + f_i, s_i z + o_i)$$

such that the distance between $(R_i \times I) \cap graf(f)$ and $w_i((D_i \times I) \cap graf(f))$ is small enough. The map w_i has a spatial part (the first two components) and an image part (the third component) which are independent from each other. In the image part s_i and o_i determine contrast and brightness of the map respectively. Once we find a domain D_i for a range R_i, the coefficients of the spatial parts are described, so, it is enough to store simply the locations of R_i and D_i instead of storing a_i, b_i, c_i, d_i, e_i, f_i separately. Using the least square method, one can find s_i and o_i which minimizes $R = \sum_{i=1}^{N} (s_i p_i + o_i - q_i)^2$ where p_i and q_i are the pixel values of the original image f over the squares D_i and R_i respectively and N is the number of pixels of the range R_i. For detailed information and formulas for s_i and o_i see [6], [9].

Remark 1.2. Note that there correspond 2×2 pixels in D_i for each pixel of R_i and the value of p_i can be taken as the value of one of that pixels or the average value of them.

Finding the most suitable domain D_i for a range R_i is crucial part of the method and it is time-consuming. For each range R_i, it is searched all of the domains and computed the values of s and o for each domain D. Then the distance between $(R_i \times I) \cap graf(f)$ and $w_i(D \times I) \cap graf(f))$ is calculated and if this distance is less than the initially determined threshold value then a proper domain is found. In the case of that there is no domain with distance less than the threshold value, there are some methods the so-called quadtree or HV-partitioning (see [6] for details).

After choosing D_i for each range R_i one can define the operator

$$W: \mathcal{M} \to \mathcal{M}, \qquad W(f) := w_1(f) \cup w_2(f) \cup \cdots \cup w_n(f).$$

Since the ranges are in the form of a partition of I^2 then the result is again an image over I^2.

Definition 1.3. *Let* $w: \mathbb{R}^3 \to \mathbb{R}^3, w(x,y,z) = \big(w^1(x,y), w^2(x,y), w^3(x,y,z)\big)$. *If* w *satisfies*

$$d\big(w(x,y,z_1), w(x,y,z_2)\big) \leq r \cdot d((x,y,z_1),(x,y,z_2))$$

where d *is the Euclidean metric and* $0 < r < 1$, *then* w *is called a* $z-$*contraction.*

Note that the first two components of a z −contraction depend only on x, y and thus the inequality in the definition equivalent to

$$|w^3(x,y,z_1) - w^3(x,y,z_2)| \leq r \cdot |z_1 - z_2|.$$

Theorem 1.4. *If $w_1, w_2, \dots, w_n$ are z-contractions then $W = \bigcup_{i=1}^{n} w_i$ is a contraction on $\mathcal{M}$ with the supremum metric (see [6]).*

The above theorem guarantees that there is a fixed point of W (say f') and this fixed point is close to the original image f by the inequality (1.1). Starting from arbitrary initial image f_0 iterations will converges to f'.

Remark 1.5. For a color image, a pixel value can be defined by a triple (R,G,B) where R, G, B are red, green, and blue levels respectively, each in the range [0,1]. Then, one can apply the above procedure and compress a color image by considering its red, green and blue levels separately and obtain three different iterated function system. To decode the system and find the approximation of the original color image, each system's attractors are found and then combined.

We illustrate the above procedure by the following example.

Example 1.6. *Consider an image of 256×256 pixels and divide it into non-overlapping range blocks of size 4×4 pixels. For each range block R, the procedure searches all of $249 \cdot 249 = 62001$ possible domain blocks of 8×8 pixels, compares the distance between $(R \times I) \cap graf(f)$ and $w\big((D_R \times I) \cap graf(f)\big)$ where w is a map defined as in (1.2) and finds a domain block D_R which minimize this distance. The transformed images of the domain blocks via four reflections and four rotations (mapping the domain block into itself isometrically) are also considered at this stage, so, to find the domain block D_R with minimal distance, the procedure needs to make $8 \cdot 249 \cdot 249 = 496008$ comparison. After a suitable domain block is found, the coefficients s_i (for contrast) and o_i (for brightness) are computed using least square method.*

The storage to keep data of the spatial part of the map w is $8 + 8 + 3$ bits. The first 16 bits are needed to keep position of the domain block and last 3 bits stores which transformation is used. Contrast and brightness data are stored in 7 bits and 5 bits respectively. So, the storage needed for one range block is 31 bits. Since the number of range block is $\left(\frac{256}{4}\right)^2 = 4096$ the total amount of storage to keep data of the image is 126976 bits= 15872 bytes≈ $15{,}5$ Kb. See Figure 2 for an application of classical fractal image compression of 256×256 pixels image.

Figure 2. An application of classical fractal image compression

2. GENERALIZATIONS OF CLASSICAL FRACTAL IMAGE COMPRESSION

We first generalize the notion of fractal image compression on I^2 to an arbitrary set X and then to the graph-directed case.

2.1. Fractal Image Compression on an Arbitrary Set

Let X be a set, denote the function space on X to $[0,1]$ by $F(X)$, i.e. $F(X) = \{f | f: X \to [0,1]\}$. This function space $F(X)$ is also a complete metric space with the metric

$$d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$

by the completeness of the interval [0,1].

Consider a partition $\mathcal{R}$ of X, i.e. a collection of sets $R_1, R_2, \ldots, R_N$ such that $R_i \cap R_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{N} R_i = X$. Now we can state the following theorem:

Theorem 2.1. *Let X be a set and $\mathcal{R} = \{R_i\}_{i=1}^{N}$ be a partition of X. Let $w_i: D_i \to R_i$ be a bijection where $D_i \subset X$, $i = 1,2, \ldots, N$. Define the operator*

$$\begin{array}{llll} W: & F(X) & \to & F(X) \\ & f & \mapsto & W(f): X \to [0,1] \\ & & & \quad x \mapsto W(f)(x) = \varphi_i(f(w_i^{-1}(x))), x \in R_i \end{array}$$

where φ_i is a contraction on [0,1] with contractivity factor r_i for $i = 1,2, \ldots, N$. Then W is a contraction with contractivity factor $r = \max\{r_i \mid 1 \leq i \leq N\}$ on the function space $F(X)$.

Proof: Let $f, g \in F(X)$ and $r = \max\{r_i \mid 1 \leq i \leq N\}$. We need to show that

$$d_\infty\big(W(f), W(g)\big) \leq r\, d_\infty(f, g).$$

$$\begin{aligned} d_\infty(W(f), W(g)) &= \sup\{\, |W(f)(x) - W(g)(x)| : x \in X\} \\ &= \max_{1 \leq i \leq N} \{\sup\{\, |W(f)(x) - W(g)(x)| : x \in R_i\}\} \\ &= \max_{1 \leq i \leq N} \big\{\sup\{\, |\varphi_i(f(w_i^{-1}(x))) - \varphi_i(g(w_i^{-1}(x)))| : x \in R_i\}\big\} \\ &\leq \max_{1 \leq i \leq N} \big\{\sup\{\, r_i\ |f(w_i^{-1}(x)) - g(w_i^{-1}(x))| : x \in R_i\}\big\} \\ &\leq r \max_{1 \leq i \leq N} \big\{\sup\{\, |f(w_i^{-1}(x)) - g(w_i^{-1}(x))| : x \in R_i\}\big\} \\ &\leq r\, \{\sup\{|f(x) - g(x)| : x \in X\}\ \} \\ &= r\, d_\infty(f, g). \end{aligned}$$

which completes the proof.

We show that W is a contraction on the function space $F(X)$, thus by the Banach Fixed Point Theorem, there exists a unique $f \in F(X)$ such that $W(f) = f$.

2.2. Graph Directed Fractal Image Compression

Let $X_1, X_2, \ldots, X_N$ be arbitrary sets and $F(X_i) = \{f \mid f: X_i \to [0,1]\}$ be the function spaces for each $i = 1,2, \ldots, N$. Since each $F(X_i)$ is complete then the product space $F(X_1) \times F(X_2) \times \cdots \times F(X_N)$ is also complete concerning the metric

$$d\big((f_1, \ldots, f_N), (f_1', \ldots, f_N')\big) = \max\{d_\infty(f_1, f_1'), \ldots, d_\infty(f_N, f_N')\}.$$

We now state the main theorem of this paper, which enables us to compress several images simultaneously:

Theorem 2.2. *Let $\{X_i\}_{i=1}^N$ be a collection of sets and $\mathcal{R}_i$ be a partition of the set X_i for each i. For each $i = 1, \ldots, N$ and $R \in \mathcal{R}_i$, let $D_R \subset X_{j(R)}$ for some $j(R) \in \{1, \ldots, N\}$ and $w_R: D_R \to R$ be a bijection. Define the operator*

$$\begin{array}{lcl} W: F(X_1) \times \cdots \times F(X_N) & \to & F(X_1) \times \cdots \times F(X_N) \\ (f_1, f_2, \ldots, f_N) & \mapsto & (g_1, g_2, \ldots, g_N) \end{array}$$

such that

$$\begin{array}{l} g_i:\ X_i \to [0,1] \\ \qquad x \mapsto \varphi_R\left(f_{j(R)}\left(w_R^{-1}(x)\right)\right) \qquad\qquad , x \in R \subset X_i \end{array}$$

where φ_R is a contraction on [0,1] with contractivity factor s_R. Then W is a contraction with contractivity factor $r = \max_{1 \le i \le N}\{\max_{R \in \mathcal{R}_i}\{s_R\}\}$.

Proof: Let $(f_1, \ldots, f_N)$ and $(f_1{}', \ldots, f_{N'})$ be two different arbitrary elements in $F(X_1) \times \cdots \times F(X_N)$ and $(g_1, \ldots, g_N) = W(f_1, \ldots, f_N)$ and $(g_1{}', \ldots, g_{N'}) = W(f_1{}', \ldots, f_{N'})$.

$$d\big(W(f_1, \ldots, f_N), W(f_1', \ldots, f_N')\big) = d\big((g_1, \ldots, g_N), (g_1', \ldots, g_N')\big)$$

$$= \max\{d_\infty(g_1, g_1'), \ldots, d_\infty(g_N, g_{N'})\}$$

$$= \max\left\{\sup_{x \in X_1}\{|g_1(x) - g_1'(x)|\}, \ldots, \sup_{x \in X_N}\{|g_N(x) - g_{N'}(x)|\}\right\}$$

$$= \max\left\{\max_{R \in \mathcal{R}_1}\left\{\sup_{x \in R}\{|g_1(x) - g_1{}'(x)|\}\right\}, \ldots, \max_{R \in \mathcal{R}_N}\left\{\sup_{x \in R}\{|g_N(x) - g_{N'}(x)|\}\right\}\right\}$$

$$= \max_{1 \le i \le N}\left\{\max_{R \in \mathcal{R}_i}\left\{\sup_{x \in R}\left\{|\varphi_R(f_{j(R)}(w_R^{-1}))(x) - \varphi_R(f'{}_j(R)(w_R^{-1}))(x)|\right\}\right\}\right\}$$

$$\le \max_{1 \le i \le N}\left\{\max_{R \in \mathcal{R}_i}\left\{\sup_{x \in R}\left\{s_R \cdot |f_{j(R)}(w_R^{-1})(x) - f'{}_{j(R)}(w_R^{-1})(x)|\right\}\right\}\right\}$$

$$\le \max_{1 \le i \le N}\left\{\max_{R \in \mathcal{R}_i}\left\{s_R \cdot \sup_{y \in X_{j(R)}}\left\{|f_{j(R)}(y) - f'{}_{j(R)}(y)|\right\}\right\}\right\}$$

$$\leq \max_{1\leq i\leq N}\left\{\max_{R\in\mathcal{R}_i}\{s_R\}\cdot\max_{R\in\mathcal{R}_i}\{d_\infty(f_{j(R)},f'_{j(R)})\}\right\}$$

$$\leq \max_{1\leq i\leq N}\left\{\max_{R\in\mathcal{R}_i}\{s_R\}\right\}\cdot d((f_1,\dots,f_N),(f_1{}',\dots,f_{N'}))$$

$$\leq r\cdot d((f_1,\dots,f_N),(f_1{}',\dots,f_{N'}))$$

which completes the proof.

Theorem 2.2 allows us to compress several images in one process. In the following, we give a sketch how to compress using graph directed fractal image compression with $N = 2$.

Let $X_1 = X_2 = I^2$ and $(f_1, f_2) \in F(I^2) \times F(I^2)$. We first take partitions $\mathcal{R}_1$ and $\mathcal{R}_2$ for the images f_1, f_2 respectively. In theoretically it is possible to take the each set of the partitions as an arbitrary set, in the application we take these partitions as rectangles (say squares with size $p \times p$ pixels). For each square R of the partition $\mathcal{R}_i, i = 1,2$ we have to find a domain of size $2p \times 2p$ pixels square $D_R \subset X_{j(R)}$ among all possible squares in the images f_1 and f_2, and a z-contraction map w_R which minimize the distance $d_\infty\big(f_i \cap R, w_R(D_R) \cap f_{j(R)}\big)$, for $i = 1,2$.

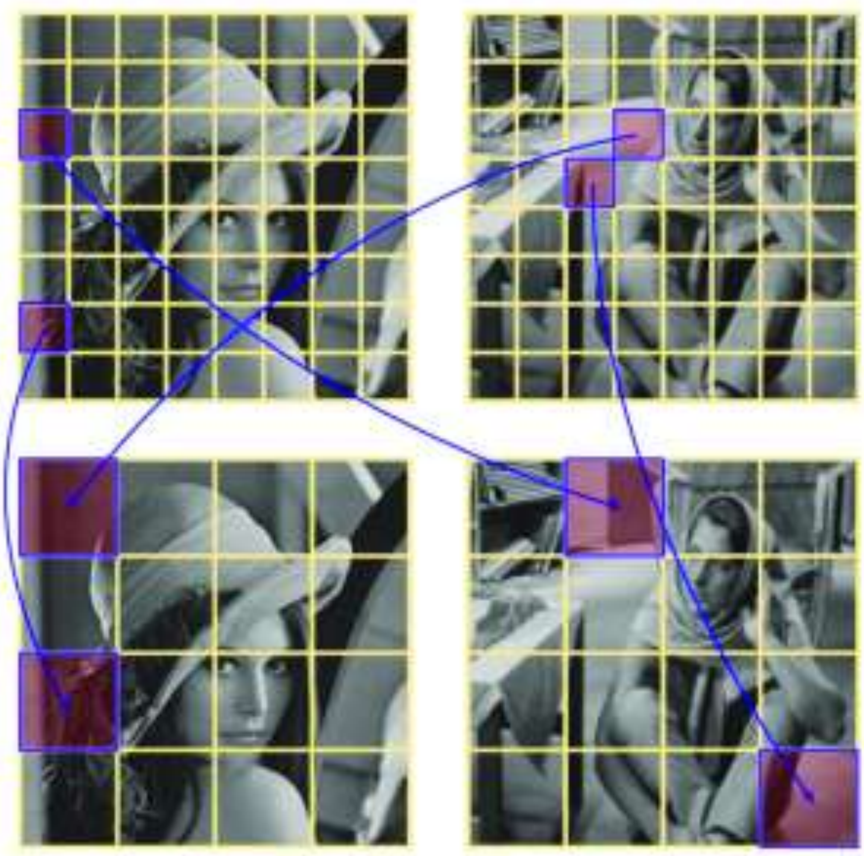

Figure 3: Searching domain block for a range block in graph-directed case

Example 2.3. *Consider two images* f_1, f_2 *of* 256×256 *pixels and divide them into non-overlapping range blocks of size* 4×4 *pixels. For each range block* R *in* $f_i, i = 1,2$, *the procedure searches all of* $2 \cdot 249 \cdot 249 = 124002$ *possible domain blocks of* 8×8 *pixels, compares the distance between* $(R \times I) \cap graf(f_i)$ *and* $w\big((D_R \times I) \cap graf(f_j)\big), (j = 1,2)$, *where* w *is a map defined as in (1.2) and* *finds a domain block D_R in f_j which minimize this distance. The transformed images of the domain blocks via four reflections and four rotations are also considered at this stage, so, to find the domain block D_R with minimal distance, the procedure needs to make* $2 \cdot 8 \cdot 249 \cdot 249 = 992016$ *comparison. After a suitable domain block is found, the coefficients s_i (for contrast) and o_i (for brightness) are described.*

The storage to keep data of the spatial part of the map w is $1 + 8 + 8 + 3$ bits. The first bit stores the information in which image the domain block lies, the next 16 bits are needed to keep position of the domain block in that image and last 3 bits stores which transformation is used. Contrast and brightness data are stored in 7 bits and 5 bits respectively. So, the storage needed for one range block is 32 bits. Since the number of range block is $2 \cdot \left(\frac{256}{4}\right)^2 = 8192$ the total amount of storage to keep data of the image is 262144 bits= 32768 bytes ≈ 32 Kb. See Figure 4 for an application of graph-directed fractal image compression of two images with sides 256×256 pixels. PSNR (Peak signal-to-noise ratio) values of the images obtained in step 10 are calculated as (for the upper image) 19,5 dB and (for the lower image) 24,33 dB respectively.

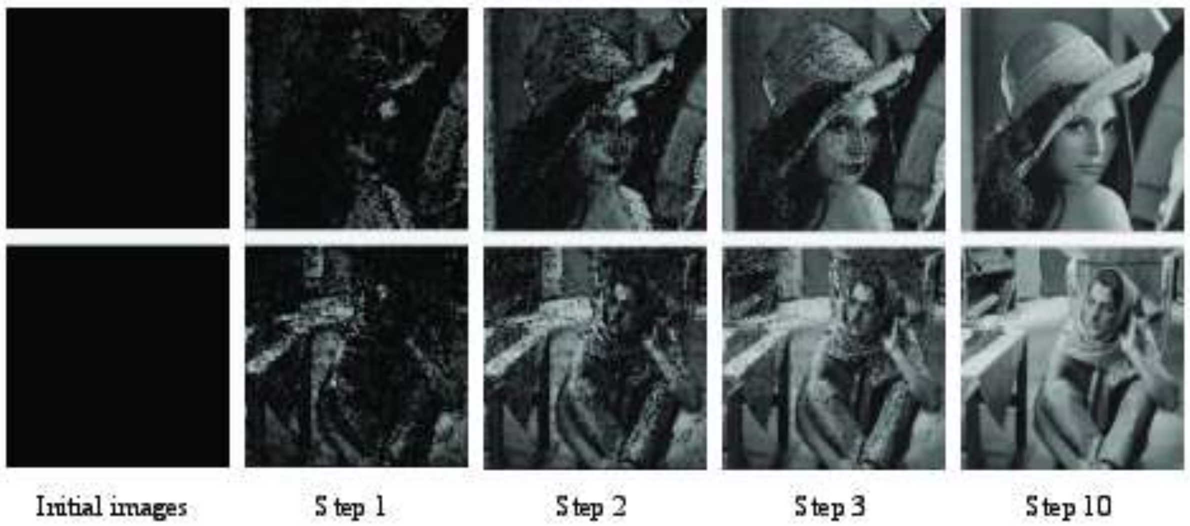

Figure 4. An application of graph-directed fractal image compression

In graph directed case it is possible to compress several pictures at the same procedure while you have to compress every image individually in the classical case.

However, the storage needed is a little bit more in the graph-directed case. If there is n images to compress, we need only p −bits more for each range block (to code in which image the corresponding domain block lies) where $2^{p-1} < n \leq 2^p$. Also, required time to compress n images in graph directed case is n times more than the classical case. But theoretically, in the graph-directed case, the compression quality increases since more domain blocks are searched and a closer domain block is found for each range block.

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CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

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