http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 71, Number 4, Pages 1007–1022 (2022)
DOI:10.31801/cfsuasmas.990670
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:September 6, 2021; Accepted: May 25, 2022

CHEN INVARIANTS FOR RIEMANNIAN SUBMERSIONS AND THEIR APPLICATIONS

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Abstract. In this paper, an optimal inequality involving the delta curvature is exposed. With the help of this inequality some characterizations about the vertical motion and the horizontal divergence are obtained.

1. Introduction

The celebrated divergence theorem states that divergence of a vector field indicates how much the vector spreads out from the certain point. In fluid kinematics, if a vector field X is considered as velocity of a fluid or a gas, then sign of $\operatorname{div}(X)$ describes the expansion or compression of flow. Therefore, the total expansion or compression of flow can be calculated by the help of divergence theorem so divergence is a useful tool to measuring the net flow of fluid diverging from a point or approaching a point. The first phenomenon is called as *horizontal divergence* and the other is called as *horizontal convergence*.

The continuity equation simple states that any matter can either be created or destroyed and implies for the atmosphere that its mass may be redistributed but can never be disappeared. Therefore, this equation gives us that

$$\operatorname{div}(U) = 0 \tag{1}$$

for any vector field $U = (u^1, u^2, u^3)$ on E^3. It can be written from (1) that

$$\operatorname{div}_H(U) + \frac{\partial u^3}{\partial z} = 0, \tag{2}$$

2020 *Mathematics Subject Classification.* 53B20,53C43,53C80.
Keywords. Curvature, Riemannian submersion, horizontal divergence.
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where $\mathrm{div}_H(U)$ is the horizontal divergence of U defined by

$$\mathrm{div}_H(U) = \frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y}. \tag{3}$$

The equation given (2) is also known as the *continuity equation* in literature. Integrating (2), we have

$$\omega(p_1, p_0) \equiv u^3(p_1) - u^3(p_0) = -\int_{p_0}^{p_1} (\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y}) dz, \tag{4}$$

where p_1 and p_0 is some pressure levels on the atmosphere. If we assume that p_0 is the surface pressure then $u^3(p_0) = 0$ and thus we get

$$\omega(p_1) = -\int_{p_0}^{p_1} (\frac{\partial u^1}{\partial x} + \frac{\partial u^2}{\partial y}) dz. \tag{5}$$

This formula tells us that w at a given pressure level is proportional to the integral of the horizontal divergence. Here, $\omega(p_1)$ is called the *vertical motion* at p_1. If $\omega(p) < 0$ at every point p then this statement is called *rising motion*, $\omega(p) > 0$ at every point p then this statement is called *descending motion*, (in this case, divergence is called convergence) in meteorology. There is no divergence and it is clear that there is a local maximum or minimum of w.

Beside these facts , B.-Y. Chen [7] initially introduced a new invariant the so-called delta curvature δ for an n-dimensional Riemannian manifold M by

$$\delta^k(p) = \tau(p) - (\inf \tau(\Pi_k))(p), \tag{6}$$

where $2 \leq k \leq n-1$, $\tau(p)$ is the scalar curvature at $p \in M$ and

$$(\inf \tau(\Pi_k))(p) = \inf\{\tau(\Pi_k) \mid \Pi_k \text{ is a } k\text{-plane section} \subset T_pM\}.$$

Furthermore, he gave a relation involving the delta curvature, the main intrinsic and extrinsic invariants of submanifolds in a real space form (cf. Lemma 3.2 in [7]). Then, this curvature drew attention of many authors and the notion of discovering simple basic relationships between intrinsic and extrinsic invariants of a submanifold becomes one of the most fundamental problems in submanifold theory (cf. [1, 3, 8, 10, 11, 19, 23, 24], etc.). Furthermore, various inequalities and their applications on Riemannian submersions were studied recently in [4, 12, 15, 22].

Apart from isometric immersions and submanifolds theory, Riemannian submersions have played a substantial role in differential geometry since this frame of maps also makes possible to compare geometrical properties between smooth manifolds. Besides the mathematical significance, Riemannian submersions have important physical and engineering aspects. There exist very nice applications of these mappings in the Kaluza-Klein theory [13, 16, 25], in the statical machine learning process [26], in the medical imaging [18], in the statical analysis [6], in the robotic theory [2, 20, 21].

Motivated by these facts, we firstly establish an optimal inequality involving the delta curvature for Riemannian manifolds admitting a Riemannian submersion.

Then, we investigate this inequality for some special cases. Finally, we obtain some results dealing the vertical motion and horizontal divergence.

2. Preliminaries

Let (M,g) be and n dimensional Riemannian manifold with Riemannian metric g. The sectional curvature, denoted $K_M(e_i \wedge e_j)$, of the plane section spanned by orthogonal unit vectors e_i and e_j at $p \in M$ is

$$K(e_i \wedge e_j) \equiv R\left(e_i, e_j, e_j, e_i\right) = R\left(e_j, e_i, e_i, e_j\right), \tag{7}$$

where R is the Riemann curvature tensor. Usually the sectional curvature $K(e_i \wedge e_j)$ is denoted by K_{ij}.

Let $\{e_1, \ldots, e_n\}$ be any orthonormal basis for T_pM. In particular, the Ricci curvature Ric is defined by

$$\mathrm{Ric}\left(X\right) = \sum_{j=1}^{n} K\left(X \wedge e_j\right). \tag{8}$$

for each fixed e_i, $i \in \{1, ..., n\}$ we have

$$\mathrm{Ric}(e_i) = \sum_{j \neq i}^{n} K(e_i \wedge e_j).$$

The scalar curvature $\tau\left(p\right)$ at p is defined by

$$\tau\left(p\right) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j). \tag{9}$$

In particular, for a 2-dimensional Riemannian manifold, the scalar curvature is its Gaussian curvature.

Let Π_k be a k-plane section of T_pM and X a unit vector in Π_k. If $k = n$ then $\Pi_n = T_pM$; and if $k = 2$ then Π_2 is a plane section of T_pM. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of Π_k such that $e_1 = X$. The k-Ricci curvature of Π_k at X, denoted $\mathrm{Ric}_{\Pi_k}(X)$, is defined by [9]

$$\mathrm{Ric}_{\Pi_k}(X) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \cdots + K(e_1 \wedge e_k). \tag{10}$$

Thus for each fixed e_i, $i \in \{1, \ldots, k\}$ we get

$$\mathrm{Ric}_{\Pi_k}(e_i) = \sum_{j \neq i}^{k} K(e_i \wedge e_j) = \sum_{j \neq i}^{k} K_{ij}. \tag{11}$$

We note that an *n-Ricci curvature* $\mathrm{Ric}_{T_pM}(e_i)$ is the usual *Ricci curvature* of e_i, denoted $\mathrm{Ric}\left(e_i\right)$. Thus for any orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and for a fixed $i \in \{1, \ldots, n\}$, we have

$$\mathrm{Ric}_{T_pM}(e_i) \equiv \mathrm{Ric}\left(e_i\right) = \sum_{j \neq i}^{n} K_{ij}.$$

The scalar curvature $\tau(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau(\Pi_k) = \sum_{1\leq i<j\leq k} K(e_i \wedge e_j) = \sum_{1\leq i<j\leq k} K_{ij}, \tag{12}$$

where $\{e_1, ..., e_k\}$ is any orthonormal basis of the k-plane section Π_k. We note that

$$\tau(\Pi_k) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}^{k} K(e_i \wedge e_j) = \frac{1}{2}\sum_{i=1}^{k} \mathrm{Ric}_{\Pi_k}(e_i). \tag{13}$$

Given an orthonormal basis $\{e_1, ..., e_n\}$ for T_pM, $\tau_{1\cdots k}$ will denote the scalar curvature of the k-plane section spanned by $e_1, ..., e_k$.

The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is,

$$\tau(p) = \tau(T_pM).$$

Let (M, g) and $(B, \widetilde{g})$ be m and n dimensional Riemannian manifolds with Riemannian metrics g and $\widetilde{g}$, respectively. A smooth map $\pi : (M, g) \to (B, \widetilde{g})$ is called a *Riemannian submersion* if

i) π has maximal rank.

ii) The differential π_* preserves the lengths of horizontal vectors.

Now, let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemannian submersion. For any $b \in B$, $\pi^{-1}(b)$ is closed r-dimensional submanifold of M. The submanifolds $\pi^{-1}(b)$ are called *fibers*. A vector field tangent to fibers is called *vertical* and a vector field orthogonal to fibers is called *horizontal*. If we put

$$\mathcal{V}_p = \mathrm{kernel}(\pi_*) \tag{14}$$

at a point $p \in M$, then it can be obtained an integrable distribution $\mathcal{V}$ corresponding to the foliation of M determined by the fibres of π. The distribution $\mathcal{V}_p$ is called *vertical space* at $p \in M$.

Let $\mathcal{H}$ be a complementary distribution of $\mathcal{V}$ determined by the Riemannian metric g. For any $p \in M$, the distribution $\mathcal{H}_p = (\mathcal{V}_p)^\perp$ is called *horizontal space* on M [17]. Thus, we have the following orthogonal decomposition:

$$TM = \mathcal{V} \oplus \mathcal{H}. \tag{15}$$

A vector field E on M is called *basic* if it is horizontal and $\pi-$ related to a vector field E_* on B i.e., $\pi_* E_p = E_{*\pi(p)}$ for all $p \in M$. Furthermore, it is known that if E and F are the basic vector fields respectively $\pi-$related to E_* and F_*, one has

$$g(E, F) \;\; = \;\; \widetilde{g}(E_*, F_*) \circ \pi. \tag{16}$$

Let h and v are the projections of $\Gamma(TM)$ onto $\Gamma(\mathcal{H})$ and $\Gamma(\mathcal{V})$, respectively. The *fundamental tensor fields* of π, denoted by A and T, are defined respectively by

$$A_E F = h\nabla_{hE} vF + v\nabla_{hE} hF, \tag{17}$$

$$T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{18}$$

for any $E, F \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M.

Now, let us define the following mappings:

$$\begin{aligned} T^{\mathcal{H}} : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) &\to \Gamma(\mathcal{H}), \\ (U, V) &\to T^{\mathcal{H}}(U, V) = h\nabla_U V, \end{aligned}$$

$$\begin{aligned} T^{\mathcal{V}} : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{H}) &\to \Gamma(\mathcal{V}), \\ (U, X) &\to T^{\mathcal{V}}(U, X) = v\nabla_U X, \end{aligned}$$

and

$$\begin{aligned} A^{\mathcal{H}} : \Gamma(\mathcal{H}) \times \Gamma(\mathcal{V}) &\to \Gamma(\mathcal{H}), \\ (X, U) &\to A^{\mathcal{H}}(X, U) = h\nabla_X U, \end{aligned}$$

$$\begin{aligned} A^{\mathcal{V}} : \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) &\to \Gamma(\mathcal{V}), \\ (X, Y) &\to A^{\mathcal{V}}(X, Y) = v\nabla_X Y, \end{aligned}$$

Then, it is clear from (17) and (18) that $T^{\mathcal{H}}$ is a symmetric operator on $\Gamma(\mathcal{V}) \times \Gamma(\mathcal{V})$ and $A^{\mathcal{V}}$ is an anti-symmetric operator on $\Gamma(\mathcal{H}) \times \Gamma(\mathcal{H})$. If (17) and (18) are taken into account in (15), we can write

$$\nabla_U V = T^{\mathcal{H}}(U, V) + v\nabla_U V, \tag{19}$$
$$\nabla_V X = h\nabla_V X + T^{\mathcal{V}}(V, X), \tag{20}$$
$$\nabla_X U = A^{\mathcal{H}}(X, U) + v\nabla_X U, \tag{21}$$
$$\nabla_X Y = h\nabla_X Y + A^{\mathcal{V}}(X, Y) \tag{22}$$

for any $U, V \in \Gamma(\mathcal{V})$ and $X, Y \in \Gamma(\mathcal{H})$.

Let $\{U_1, \ldots, U_r, X_1, \ldots, X_n\}$ be an orthonormal basis on T_pM, where $\mathcal{V} = \mathrm{Span}\{U_1, \ldots, U_r\}$ and $\mathcal{H} = \mathrm{Span}\{X_1, \ldots, X_n\}$. The mean curvature vector field $\hbar(p)$ of any fibre is defined by

$$\mathcal{N}(p) = \frac{1}{r}\sum_{j=1}^{r} T^{\mathcal{H}}(U_j, U_j). \tag{23}$$

Note that each fiber is a minimal submanifold of M if and only if $\hbar(p) = 0$ for all $p \in M$. Furthermore, each fiber is called *totally geodesic* if both $T^{\mathcal{H}}$ and $T^{\mathcal{V}}$ vanish identically and it is called *totally umbilical* if

$$T^{\mathcal{H}}(U, V) = g\,(U, V)\,\hbar$$

for all $U, V \in \Gamma(\mathcal{V})$.

Now we recall the following Theorem [14]:

Theorem 1. *Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemann submersion. Then the horizontal space $\mathcal{H}$ is an integrable distribution if and only if A vanishes identically.*

Remark 1. *As a consequence of Theorem 1, we see that both $A^{\mathcal{H}}$ and $A^{\mathcal{V}}$ are related to integrability of $\mathcal{H}$, that is, they are identically zero if and only if $\mathcal{H}$ is integrable.*

Let R, $\widetilde{R}$ and $\hat{R}$ are the curvature tensors on M, B and be the collection of all curvature tensors on fibers $\pi^{-1}(b)$ respectively, and $\check{R}(X,Y)Z$ be the horizontal lift of $\widetilde{R}_{\pi(b)}(\pi_{*p}X_b, \pi_{*p}Y_b)Z_b$ at any point $b \in M$ satisfying

$$\pi_*(\check{R}(X,Y)Z) = \widetilde{R}(\pi_*X, \pi_*Y)\pi_*Z.$$

Then, there exist the following relations between these tensors:

$$\begin{aligned} R(U,V,W,G) &= \hat{R}(U,V,W,G) + g\left((T^{\mathcal{H}}(U,G), T^{\mathcal{H}}(V,W)\right) \\ &\quad -g\left(T^{\mathcal{H}}(V,G), T^{\mathcal{H}}(U,W)\right), \qquad (24) \\ R(X,Y,Z,H) &= \check{R}(X,Y,Z,H) - 2g\left(A^{\mathcal{V}}(X,Y), A^{\mathcal{V}}(Z,H)\right) \\ &\quad +g\left(A^{\mathcal{V}}(Y,Z), A^{\mathcal{V}}(X,H)\right) - g\left(A^{\mathcal{V}}(X,Z), A^{\mathcal{V}}(Y,H)\right), \qquad (25) \\ R(X,V,Y,W) &= g\left((\nabla_X T)(V,W), Y\right) + g\left((\nabla_V A)(X,Y), W\right) \\ &\quad -g\left(T^{\mathcal{V}}(V,X), T^{\mathcal{V}}(W,Y)\right) \\ &\quad +g\left(A^{\mathcal{H}}(X,V), A^{\mathcal{H}}(Y,W)\right), \qquad (26) \end{aligned}$$

for any $U,V,W,G \in \Gamma(\mathcal{V})$ and $X,Y,Z,H \in \Gamma(\mathcal{H})$. Note that the above equalities are known as *Gauss–Codazzi equations* for a Riemannian submersion. With the help of Gauss–Codazzi equations, we get the following relations between the sectional curvatures as follows:

$$\begin{aligned} K(U \wedge V) &= \hat{K}(U \wedge V) - \left\|T^{\mathcal{H}}(U,V)\right\|^2 \\ &\quad +g\left(T^{\mathcal{H}}(U,U), T^{\mathcal{H}}(V,V)\right), \qquad (27) \\ K(X \wedge Y) &= \check{K}(\check{X} \wedge \check{Y}) + 3\|A^{\mathcal{V}}(X,Y)\|^2, \qquad (28) \\ K(X \wedge V) &= -g\left((\nabla_X T)(V,V), X\right) + \left\|T^{\mathcal{V}}(V,X)\right\|^2 \\ &\quad - \left\|A^{\mathcal{H}}(X,V)\right\|^2, \qquad (29) \end{aligned}$$

where K, $\hat{K}$ and $\check{K}$ denote the sectional curvatures in M, any fiber $\pi^{-1}(b)$ and the horizontal distribution $\mathcal{H}$, respectively. The scalar curvatures of the vertical and horizontal spaces at a point $p \in M$ are given respectively by

$$\hat{\tau}(p) = \sum_{1 \le i < j \le r} \hat{K}(U_i, U_j) \qquad (30)$$

and

$$\check{\tau}(p) = \sum_{1 \le i < j \le n} \check{K}(X_i, X_j). \qquad (31)$$

Now, we recall the following definition of [5].

Definition 1. *Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemann submersion and X be a horizontal vector field on π. Then, horizontal divergence of X is defined by*

$$div_{\mathcal{H}}(X) = \sum_{i=1}^{n} g(\nabla_{X_i} X, X_i). \tag{32}$$

Lemma 1. *[14] Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemann submersion and $\{U_1, \ldots, U_r\}$ be any orthonormal basis of $\Gamma(\mathcal{V})$. For any $E \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{H})$, we have*

$$g\left(\nabla_E \mathcal{N}, X\right) = \frac{1}{r} \sum_{j=1}^{r} g\left(\left(\nabla_E T\right)\left(U_j, U_j\right), X\right). \tag{33}$$

As a consequence of Lemma 1, we obtain that

$$\mathrm{div}_{\mathcal{H}}(\mathcal{N}) = \frac{1}{r} \sum_{i=1}^{n} \sum_{j=1}^{r} g\left(\left(\nabla_{X_i} T\right)\left(U_j, U_j\right), X_i\right). \tag{34}$$

3. An Optimal Inequality for Riemannian Submersions

We begin this section with the following algebraic lemma:

Lemma 2. *If $n > k \geq 2$ and $a_1, \ldots, a_n, a$ are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-k+1)\left(\sum_{i=1}^{n} a_i^2 + a\right), \tag{35}$$

then

$$2 \sum_{1 \leq i < j \leq k} a_i a_j \geq a,$$

with equality holding if and only if

$$a_1 + a_2 + \cdots + a_k = a_{k+1} = \cdots = a_n.$$

Proof. By the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq (n-k+1)((a_1+a_2+\cdots+a_k)^2+a_{k+1}^2+\cdots+a_n^2). \tag{36}$$

From (35) and (36), we get

$$\sum_{i=1}^{n} a_i^2 + a \leq (a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}^2 + \cdots + a_n^2.$$

The above equation is equivalent to

$$2 \sum_{1 \leq i < j \leq k} a_i a_j \geq a.$$

The equality holds if and only if $a_1 + a_2 + \cdots + a_k = a_{k+1} = \cdots = a_n$. $\square$

Let $\pi : (M,g) \to (B,\widetilde{g})$ be a Riemannian submersion between Riemannian manifolds (M,g) and $(B,\widetilde{g})$. Suppose $\{U_1,\dots,U_r,X_1,\dots,X_n\}$ be an orthonormal basis on T_pM, where $\mathcal{V} = \mathrm{Span}\{U_1,\dots,U_r\}$ and $\mathcal{H} = \mathrm{Span}\{X_1,\dots,X_n\}$. Then, we have

$$\|T^{\mathcal{H}}\|^2 = \sum_{i,j=1}^{r} g\left(T^{\mathcal{H}}(U_i,U_j), T^{\mathcal{H}}(U_i,U_j)\right), \tag{37}$$

$$\|T^{\mathcal{V}}\|^2 = \sum_{i=1}^{r}\sum_{j=1}^{n} g\left(T^{\mathcal{V}}(U_i,X_j), T^{\mathcal{V}}(U_i,X_j)\right), \tag{38}$$

$$\|A^{\mathcal{H}}\|^2 = \sum_{i=1}^{r}\sum_{j=1}^{n} g\left(A^{\mathcal{H}}(X_j,U_i), A^{\mathcal{H}}(X_j,U_i)\right), \tag{39}$$

$$\|A^{\mathcal{V}}\|^2 = \sum_{i,j=1}^{n} g\left(A^{\mathcal{V}}(X_i,X_j), A^{\mathcal{V}}(X_i,X_j)\right). \tag{40}$$

Putting (27) – (29), (34) and (37) – (40) in

$$\tau(p) = \sum_{1\le i<j\le n} [K(U_i,U_j) + K(X_i,U_j) + K(X_i,X_j)],$$

we obtain the following lemma:

Lemma 3. *Let (M,g) and $(B,\widetilde{g})$ be a Riemannian manifolds admitting a Riemannian submersion $\pi : (M,g) \to (B,\widetilde{g})$. For any point $p \in M$, we have*

$$\begin{aligned} 2\tau(p) &= 2\hat{\tau}(p) + 2\check{\tau}(p) + r^2\|\hbar(p)\|^2 - \left\|T^{\mathcal{H}}\right\|^2 + 3\left\|A^{\mathcal{V}}\right\|^2 \\ &\quad - r\ div_{\mathcal{H}}(\hbar(p)) + \|T^{\mathcal{V}}\|^2 - \|A^{\mathcal{H}}\|^2. \end{aligned} \tag{41}$$

Now, we are going to give an optimal inequality involving the $\delta-$curvature for Riemannian manifolds admitting a Riemannian submersion.

Theorem 2. *Let $\pi : (M,g) \to (B,\widetilde{g})$ be a Riemannian submersion. Then, for each point $p \in M$ and each k-plane section $L_k \subset \mathcal{V}_p$ $(r > k \ge 2)$, we have*

$$\begin{aligned} \delta(k) &\le \hat{\tau}(p) - \hat{\tau}(L_k) + \check{\tau}(p) + \frac{r^2(r-k)}{2(r-k+1)}\|\hbar\|^2 - \frac{r}{2}div_{\mathcal{H}}(\hbar(p)) \\ &\quad + \frac{3}{2}\|A^{\mathcal{V}}\|^2 + \frac{1}{2}\|T^{\mathcal{V}}\|^2. \end{aligned} \tag{42}$$

The equality of (42) *holds at* $p \in M$ *if and only if* $A^{\mathcal{H}}$ *vanishes identically and the shape operators* $S_{X_1}, \ldots, S_{X_n}$ *of* $\mathcal{V}_p$ *take forms as follows:*

$$S_{X_1} = \begin{pmatrix} \begin{matrix} T_{11}^1 & 0 & \cdots & 0 \\ 0 & T_{22}^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{kk}^1 \end{matrix} & 0 \\ 0 & \left(\sum_{i=1}^{k} T_{ii}^1\right) I_{r-k} \end{pmatrix}, \tag{43}$$

$$S_{X_s} = \begin{pmatrix} \begin{matrix} T_{11}^s & T_{12}^s & \cdots & T_{1k}^s \\ T_{12}^s & T_{22}^s & \cdots & T_{2k}^s \\ \vdots & \vdots & \ddots & \vdots \\ T_{1k}^s & T_{2k}^s & \cdots & -\sum_{i=1}^{k-1} T_{ii}^s \end{matrix} & 0 \\ 0 & 0_{n-k} \end{pmatrix}, \qquad s \in \{2, \ldots, n\}. \tag{44}$$

Proof. Let L_k be a k-plane section of $\mathcal{V}_p$. We choose an orthonormal basis $\{U_1, \ldots, U_r, X_1, \ldots, X_n\}$ on T_pM such that $\mathcal{V} = \mathrm{Span}\{U_1, \ldots, U_r\}$ and $\mathcal{H} = \mathrm{Span}\{X_1, \ldots, X_n\}$. We write

$$T_{ij}^s = g(T^{\mathcal{H}}(U_i, U_j), X_s) \tag{45}$$

for any $i, j \in \{1, \ldots, r\}$ and $s \in \{1, \ldots, n\}$. Suppose that the mean curvature vector $\hbar(p)$ is in the direction of X_1 and $X_1, ..., X_n$ diagonalize the shape operator S_{X_1}. If we put

$$\begin{aligned} \eta \quad = \quad & 2\tau(p) - 2\hat{\tau}(p) - 2\check{\tau}(p) - \frac{r^2(r-k)}{(r-k+1)}\|\hbar\|^2 + r\ div_{\mathcal{H}}(\hbar(p)) \\ & -3\left\|A^{\mathcal{V}}\right\|^2 - \|T^{\mathcal{V}}\|^2 + \|A^{\mathcal{H}}\|^2 \end{aligned} \tag{46}$$

in (41), it follows that

$$r^2\|\hbar\|^2 = (n-k+1)(\eta + \|T^{\mathcal{H}}\|^2). \tag{47}$$

The equation (47) is equivalent to

$$\left(\sum_{i=1}^{r} T_{ii}^1\right)^2 = (n-k+1)\left(\eta + \sum_{i=1}^{r}\left(T_{ii}^1\right)^2 + \sum_{s=2}^{n}\sum_{i,j=1}^{r}\left(T_{ij}^s\right)^2\right). \tag{48}$$

Applying Lemma 2 to equation (48), we get

$$2\sum_{1 \le i < j \le k} T_{ii}^{n+1} T_{jj}^{n+1} \ge \eta + \sum_{s=2}^{n}\sum_{i,j=1}^{r}\left(T_{ij}^s\right)^2. \tag{49}$$

On the other hand, we have from (41) that

$$\tau\left(L_k\right)=\hat{\tau}\left(L_k\right)+\sum_{1\leq i<j\leq k}T_{ii}^1T_{jj}^1+\sum_{s=2}^{n}\sum_{1\leq i<j\leq k}\left(T_{ii}^sT_{jj}^s-\left(T_{ij}^s\right)^2\right). \tag{50}$$

From (49) and (50), we get

$$\begin{aligned}\tau\left(L_k\right) \quad &\geq \quad \hat{\tau}\left(L_k\right)+\frac{1}{2}\eta+\sum_{s=2}^{n}\sum_{j>k}\{(T_{1j}^s)^2+(T_{2j}^s)^2+\cdots+(T_{kj}^s)^2\}\\ &\quad+\frac{1}{2}\sum_{s=2}^{n}(T_{11}^s+T_{22}^s+\cdots+T_{kk}^s)^2+\frac{1}{2}\sum_{s=2}^{n}\sum_{i,j>k}(T_{ij}^s)^2. \end{aligned}\tag{51}$$

In view of (51), we see that

$$\tau\left(\Pi_k\right)\geq\widetilde{\tau}\left(\Pi_k\right)+\frac{1}{2}\eta. \tag{52}$$

From (47) and (52), we obtain (42).

If the equality case of (42) holds, then we have $A^{\mathcal{H}}$ vanishes identically and

$$\begin{cases} T_{1j}^1=T_{2j}^1=T_{kj}^1=0, \qquad j=k+1,\ldots,r,\\ T_{ij}^s=0, \qquad i,j=k+1,\ldots,r,\\ T_{11}^r+T_{22}^r+\cdots+T_{kk}^r=0 \end{cases}\tag{53}$$

for $s=2,\ldots,n$. Applying Lemma 2, we also have

$$T_{11}^1+T_{22}^1+\cdots+T_{kk}^1=T_{ll}^1, \qquad l=k+1,\ldots,n. \tag{54}$$

Thus, with respect to a suitable orthonormal basis $\{X_1,\ldots,X_m\}$ on $\mathcal{H}_p$, the shape operator of $\mathcal{V}_p$ becomes of the form given by (43) and (44). The proof of the converse part is straightforward. □

In particular case of $k=2$, we have the following:

Corollary 1. *Let $\pi:(M,g)\to(B,\widetilde{g})$ be a Riemannian submersion. Then, for each point $p\in M$ and each plane section $L\subset\mathcal{V}_p$, we have*

$$\begin{aligned}\delta(2) \quad &\leq \quad \hat{\tau}(p)-\hat{K}(L)+\check{\tau}(p)+\frac{r^2(r-2)}{2(r-1)}\|\hbar\|^2-\frac{r}{2}div_{\mathcal{H}}(\hbar)\\ &\quad+\frac{3}{2}\|A^{\mathcal{V}}\|^2+\frac{1}{2}\|T^{\mathcal{V}}\|^2. \end{aligned}\tag{55}$$

The equality of (55) holds at $p\in M$ if and only if $A^{\mathcal{H}}$ vanishes identically and the shape operators $S_{X_1},\ldots,S_{X_n}$ of $\mathcal{V}_p$ take forms

$$S_{X_1}=\begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & (a+b)\,I_{r-2}\end{pmatrix}, \tag{56}$$

$$S_{X_s} = \begin{pmatrix} c_s & d_s & 0 \\ d_s & -c_s & 0 \\ 0 & 0 & 0_{r-2} \end{pmatrix}, \qquad s \in \{2, \ldots, n\}. \tag{57}$$

In particular case of $k = r-1$, we have the following

Corollary 2. *Let $\pi : (M, g) \to (B, \widetilde{g})$ be a Riemannian submersion. For each vertical unit vector U, we have*

$$\mathrm{Ric}_{\mathcal{V}}(U) \le \hat{\mathrm{Ric}}(U) + \check{\tau}(p) + \frac{r^2}{4}\|\hbar\|^2 - \frac{r}{2} div_{\mathcal{H}}(\hbar) + \frac{3}{2}\|A^{\mathcal{V}}\|^2 + \frac{1}{2}\|T^{\mathcal{V}}\|^2. \tag{58}$$

The equality case of (58) *holds for all unit vectors $U \in \mathcal{V}_p$ if and only if $A^{\mathcal{H}}$ vanishes identically and we have either*

(i) *if $r = 2$, π has totally umbilical fibers at $p \in M$,*
(i) *if $r \neq 2$, π has totally geodesic fibers at $p \in M$.*

Proof. Let L_{r-1} be a $(r-1)$-plane section of $\mathcal{V}_p$. We get from Theorem 2 that

$$\begin{aligned} \delta(r-1) \quad \le \quad & \hat{\tau}(p) - \hat{\tau}(L_{r-1}) + \check{\tau}(p) + \frac{r^2}{4}\|\hbar\|^2 - \frac{r}{2} div_{\mathcal{H}}(\hbar) \\ & + \frac{3}{2}\|A^{\mathcal{V}}\|^2 + \frac{1}{2}\|T^{\mathcal{V}}\|^2. \end{aligned} \tag{59}$$

Now, let U be a unit vertical vector field such that $U = U_r$. By a straightforward computation, we obtain (58).

The equality of (59) holds if and only if the forms of shape operators S_{X_s}, $s = 1, \ldots, n$, become

$$S_{X_1} = \begin{pmatrix} T^1_{11} & 0 & \cdots & 0 & 0 \\ 0 & T^1_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T^1_{(r-1)(r-1)} & 0 \\ 0 & 0 & \cdots & 0 & \left(\sum_{i=1}^{r-1} T^1_{ii}\right) \end{pmatrix}, \tag{60}$$

$$S_{X_s} = \begin{pmatrix} T^s_{11} & T^s_{12} & \cdots & T^r_{1(r-1)} & 0 \\ T^s_{12} & T^s_{22} & \cdots & T^s_{2(r-1)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T^s_{1(r-1)} & T^s_{2(r-1)} & \cdots & -\sum_{i=1}^{r-2} T^s_{ii} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad r \in \{2, \ldots, n\}. \tag{61}$$

From (60) and (61), we see that the equality in (58) is valid for a unit vertical vector field $U = U_r$ if and only if

$$\begin{cases} T^s_{rr} = T^s_{11} + T^s_{22} + \cdots + T^s_{(r-1)(r-1)} \\ T^s_{1r} = T^s_{2r} = \cdots = T^s_{(r-1)r} = 0. \end{cases} \tag{62}$$

for $s \in \{1, \ldots, n\}$.

Assuming the equality case of (58) holds for all unit vertical vector fields, in view of (62), for each $s \in \{1, \ldots, n\}$, we have

$$\begin{cases} 2T^s_{ii} = T^s_{11} + T^s_{22} + \cdots + T^s_{rr}, \\ T^s_{ij} = 0, \quad i \neq j \end{cases} \tag{63}$$

for all $i \in \{1, ..., r\}$ and $s \in \{1, \ldots, n\}$. Thus, we have two cases, namely either $r = 2$ or $r \neq 2$. In the first case we see that π has totally umbilical fibers, while in the second case π has totally geodesic fibers. The proof of converse part is straightforward. □

Remark 2. *We note that (58) was also proved in [15] (see Theorem 4.1 in [15]). In Theorem 2, we gave a new proof for this inequality.*

4. Main Conclusions

In this section, we shall present a solution way with the help of differential geometry tools for the following natural problem:

"Which conditions should provide to the horizontal divergence or the convergence receives to the maximum value or minimum value?"

To obtain minimum or maximum values of the vertical motion (or horizontal divergence) it can be considered a Riemannian submersion on E^3 to E^2. Moreover, we can regard to different Riemannian submersions such as a Riemannian submersion on a three dimensional Riemannian manifold to two dimensional Riemannian manifold as

$$\pi : M^3 \to N^2. \tag{64}$$

It can also be considered globally in high dimensional Riemannian manifolds with taking a Riemannian submersion on m-dimensional Riemannian manifold to n-dimensional Riemannian manifold.

Taking into account of the continuity equation and (42), (55) and (58) inequalities, we get some result dealing minimum or maximum values of vertical motion for a manifold admitting a Riemannain submersion.

As a consequence of (42), we obtain the following:

Corollary 3. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion. Then we have*

$$\frac{r}{2}\omega(p) \quad \geq \quad \delta(k) - \frac{r^2(r-k)}{2(r-k+1)}\|\hbar\|^2 - \frac{3}{2}\|A^{\mathcal{V}}\|^2 - \frac{1}{2}\|T^{\mathcal{V}}\|^2. \tag{65}$$

The vertical motion at a point p takes the minimum value if and only if $A^{\mathcal{H}}$ vanishes identically and the matrixes of shape operators of the vertical space of M take the form as (43) *and* (44).

As a consequence of (55), we obtain the followings:

Corollary 4. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion with integrable horizontal distribution. Then we have*

$$\frac{r}{2}\omega(p) \quad \geq \quad \delta(2) - \frac{r^2(r-2)}{2(r-1)}\|\hbar\|^2 - \frac{1}{2}\|T^{\mathcal{V}}\|^2. \tag{66}$$

The vertical motion takes the minimum value if and only if the matrixes of shape operators $S_{x_1}, \ldots, S_{x_n}$ of the vertical space of M take the form as (56) *and* (57).

Corollary 5. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion with totally geodesic leaves and integrable horizontal distribution. Then we have*

$$\frac{r}{2}\omega(p) \quad = \quad \delta(2). \tag{67}$$

From (58), we get the followings:

Corollary 6. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion. For each vertical unit vector U, we have*

$$\frac{r}{2}\omega(p) \geq \mathrm{Ric}_{\mathcal{V}}(U) - \frac{r^2}{4}\|\hbar\|^2 - \frac{3}{2}\|A^{\mathcal{V}}\|^2 - \frac{1}{2}\|T^{\mathcal{V}}\|^2. \tag{68}$$

The equality case of (68) *holds for all unit vectors $U \in \mathcal{V}_p$ if and only if $A^{\mathcal{H}}$ vanishes identically and we have either*

(i) *if $r = 2$, π has totally umbilical fibers at $p \in M$,*
(ii) *if $r \neq 2$, π has totally geodesic fibers at $p \in M$.*

Corollary 7. *Let $\pi : \mathrm{E}^{n+r} \to \mathrm{E}^n$ be a Riemannian submersion with totally geodesic fibers. For each vertical unit vector U, we have*

$$\frac{r}{2}\omega(p) = \mathrm{Ric}_{\mathcal{V}}(U) - \frac{3}{2}\|A^{\mathcal{V}}\|^2. \tag{69}$$

Now we shall mention some examples:

Example 1. *Consider the mapping $\pi : \mathrm{E}^5 \to \mathrm{E}^2$ which is defined by*

$$\pi(x_1, x_2, x_3, x_4, x_5) = \left(\frac{1}{\sqrt{2}}(x_1 + x_2), \frac{1}{\sqrt{2}}(x_3 + x_4)\right).$$

Then, it is clear that π is a Riemannian submersion and the Jacobian of π is equal to

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

The horizontal space and the vertical space are given by

$$\mathcal{H} = Span\{X_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2}, X_2 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_4}\}$$

and

$$\mathcal{V} = Span\{U_1 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}, U_2 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_4}, U_3 = \frac{\partial}{\partial x_5}\},$$

respectively. By a straightforward computation, we get the tensor fields A, T, $\mathrm{Ric}_{\mathcal{V}}$ *vanish and* $\omega(p) = div_{\mathcal{H}}(\hbar) = 0$ *from* (3). *Therefore,* π *is a trivial example satisfying Corollary* 3*-Corollary* 7*.*

Example 2. *(Example 5.1 in [15])*

Let us consider the Remannian submesion $\pi : M \to \mathrm{E}^3$ *defined by*

$$\pi(x_1, x_2, x_3, x_4, x_5) = (x_1 \cos x_3 + x_2 \sin x_3, x_4, x_5)\,,$$

where M *is a non-flat submanifold of* E^5 *such that* $\cot x_3 = \frac{x_1}{x_2}$, $x_2 \neq 0$ *and* $x_3 \in \left(0, \frac{\pi}{2}\right)$. *Here, the horizontal space and the vertical space of* M *are given by*

$$\mathcal{H} = Span\{X_1 = \sin x_3 \frac{\partial}{\partial x_1} + \cos x_3 \frac{\partial}{\partial x_2}, X_2 = \frac{\partial}{\partial x_4}, X_3 = \frac{\partial}{\partial x_5}\}$$

and

$$\mathcal{V} = Span\{U_1 = -\cos x_3 \frac{\partial}{\partial x_1} + \sin x_3 \frac{\partial}{\partial x_2}, U_2 = \frac{\partial}{\partial x_3}\},$$

respectively. By straightforward computations, we have $T^{\mathcal{V}}(U_2, X_1) = -U_1$, $T^{\mathcal{H}}(U_1, U_2) = X_1$ *and the other components of operators* $T^{\mathcal{H}}$, $T^{\mathcal{V}}$, $A^{\mathcal{H}}$, $A^{\mathcal{V}}$ *vanish identically. Moreover, we have* $\hat{\mathrm{Ric}}(U_1) = 1$, $\mathrm{Ric}_{\mathcal{V}}(U_1) = \mathrm{Ric}_{\mathcal{V}}(U_2) = 0$ *and* $\omega(p) = 0$ *from* (3). *Considering these facts, we obtain the left hand side of* 68 *is equal to* 0 *and the right hand side of* (68) *is equal to* -1 *for* $U = U_1$. *This inequality also satisfies for* $U = U_2$. *This shows that the correctness of* (68) *and* π *is an example of Corollary* 6*.*

Author Contribution Statements The authors contributed equally to this study.

Declaration of Competing Interests The authors have no competing interests to declare.

Acknowledgment The authors are thankful to the editor İsmail Gök and reviewers for their constructive comments.

References

[1] Alegre, P., Chen, B. Y., Munteanu, M. I., Riemannian submersions, δ-invariants and optimal inequality, *Ann. Glob. Anal. Geom.*, 42 (2010), 317–331. https://doi.org/10.1007/s10455-012-9314-4

[2] Altafini, C., Redundant robotic chains on Riemannian submersions, *IEEE Robot. Autom.*, 20(2) (2004), 335–340. https://doi.org/10.1109/TRA.2004.824636

[3] Arslan, K., Ezentaş, R., Mihai, I, Murathan C., Özgür, C., B.Y Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds, *Bull. Inst. Math. Acad. Sin.*, 29(3) (2001), 231–242.

[4] Aytimur, H., Özgür, C., Sharp inequalities for anti-invariant Riemannian submersions from Sasakian space forms, *J. Geom. Phys.*, 166 (2021), 104251. https://doi.org/10.1016/j.geomphys.2021.104251

[5] Besse, A. L., Einstein Manifolds, Berlin-Heidelberg-New York, Spinger-Verlag, 1987.

[6] Bhattacharyaa, R., Patrangenarub, V., Nonparametic estimation of location and dispersion on Riemannian manifolds, *J. Statist. Plann. Inference*, 108 (2002), 23–35. https://doi.org/10.1016/S0378-3758(02)00268-9

[7] Chen, B. Y., Some pinching and classification theorems for minimal submanifolds, *Arch. Math.*, 60 (1993), 568–578. https://doi.org/10.1007/BF01236084

[8] Chen, B. Y., A Riemannian invariant and its applications to submanifold theory, *Results Math.*, 27 (1995), 17–28. https://doi.org/10.1007/BF03322265

[9] Chen, B. Y., Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, *Glasgow Math. J.*, 41 (1999), 33–41. https://doi.org/10.1017/S0017089599970271

[10] Chen, B. Y., Pseudo-Riemannian Geometry, δ-Invariants and Applications, World Scientific Publishing, Hackensack, NJ, 2011.

[11] Chen, B. Y., Dillen, F., Verstraelen, L., δ-invariants and their applications to centroaffine geometry, *Differ. Geom. Appl.*, 22, (2005) 341–354. https://doi.org/10.1016/j.difgeo.2005.01.008

[12] Eken Meriç, S., Gülbahar, M., Kılıç, E., Some inequalities for Riemannian submersions, *An. Stiint. Univ. Al. I. Cuza Iasi. Mat.*, 63 (2017), 471-482.

[13] Falcitelli, M., Ianus, S., Pastore, A. M., Visinescu, M., Some applications of Riemannian submersions in physics, *Rev. Roum. Phys.*, 48 (2003), 627–639.

[14] Falcitelli, M., Ianus, S., Pastore, A. M., Riemannian Submersions and Related Topics, World Scientific Company, 2004.

[15] Gülbahar, M., Eken Meriç, S., Kılıç, E., Sharp inequalities involving the Ricci curvature for Riemannian submersions, *Kragujevac J. Math.*, 42(2) (2017). https://doi.org/10.5937/KgJMath1702279G

[16] Kennedy, L. C., Some Results on Einstein Metrics on Two Classes of Quotient Manifolds, PhD thesis, University of California, 2003.

[17] Kobayashi, S., Submersions of CR-submanifolds, *Tohoku Math. J.*, 89 (1987), 95–100. https://doi.org/10.2748/tmj/1178228372

[18] Memoli F., Sapiro G., Thompson P., Implicit brain imaging, *Neuro Image*, 23 (2004), 179–188. https://doi.org/10.1016/j.neuroimage.2004.07.072

[19] Poyraz, N., Yaşar, E., Chen-like inequalities on lightlike hypersurface of a Lorentzian product manifold with quarter-symmetric nonmetric connection, *Kragujevac J. Math.*, 40 (2016), 146–164. https://doi.org/10.5937/kgjmath1602146p

[20] Şahin, B., Riemannian submersions from almost Hermitian manifolds, *Taiwan. J. Math.*, 17 (2013), 629–659. https://doi.org/10.11650/tjm.17.2013.2191

[21] Şahin, B., Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and Their Applications, Academic Press, 2017.

[22] Siddiqui, A. N., Chen inequalities for statistical submersions between statistical manifolds, *Int. J. Geom. Methods Mod. Phys.*, 18 (2021), 2150049. https://doi.org/10.1142/S0219887821500493

[23] Uddin, S., Solamy, F. R., Shahid, M. H., Saloom, A., B.-Y. Chen's inequality for bi-warped products and its applications in Kenmotsu manifolds, *Mediterr. J. Math.*, 15 (2018). https://doi.org/10.1007/s00009-018-1238-1

[24] Vilcu, G. E., On Chen invariants and inequalities in quaternionic geometry, *J. Inequal. Appl.*, 2013:66 (2013). https://doi.org/10.1186/1029-242X-2013-66

[25] Wang, H., Ziller, W., Einstein metrics on principal torus bundles, *J. Differ. Geo.*, 31 (1990), 215–248.

[26] Zhao, H., Kelly, A. R., Zhou, J., Lu, J., Yang, Y. Y., Graph attribute embedding via Riemannian submersion learning. *Comput. Vis. Image Underst.*, 115 (2011), 962–975. https://doi.org/10.1016/j.cviu.2010.12.005