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A New Game Value Approach for Infinite Interval Matrix Games

Aykut OR[1], Gönül Selin SAVAŞKAN*[1]

Abstract

The purpose of this paper is to determine when and under which conditions the solution and game value of the infinite interval matrix games will exist. Firstly, the concept of a reasonable solution defined in interval matrix games was extended to infinite interval matrix games. Then, the solution and game value were characterized by using sequences of interval numbers defined by Chiao ["Fundamental Properties of Interval Vector Max-Norm", Tamsui Oxf J Math Sci, 18(2):219-233, 2002.] and their concept of convergence of interval numbers. Considering that each row or column of the payoff matrix is a sequence of interval numbers, we assume that each row converges to the same interval number $\tilde{\alpha} = [\alpha_l, \alpha_r]$ and each column to the same interval number $\tilde{\beta} = [\beta_l, \beta_r]$. In a conclusion, the existence of the solution of $\tilde{G}$ is shown.

Keywords: infinite matrix games, interval matrix games, reasonable solution

1. INTRODUCTION

Game theory is a branch of mathematics that studies the interaction between players with mathematical methods, in which all parties engage in a game to maximize the payoffs (for a specific goal) making rational decisions (Barron, 2006). John von Neumann laid the mathematical foundations of game theory with the Minimax Theorem in 1928. Moreover, "Game Theory and Economics Behavior" known as the first book in the field of game theory, was published by von Neumann and Morgenstern in 1944. Different mathematical concepts such as matrix, differential equation, and graphs are used to create mathematical model of the game.

On the other hand, the games are classified according to payoff functions, players' numbers, and strategy sets. In this context, games played with two players in which one's loss is equal to the other's gain and the payoff function is represented by a matrix called matrix games (Owen, 1995).

Generally, in matrix games, all payoff matrix entries are supposed to be certain. However, we often encounter cases with imprecise information such as interval matrices or uncertain environments in real-life situations.

In this paper, we count infinite interval matrix games, a generalized form of interval matrix games, allowing players to have infinite strategies. As in the finite games, $\tilde{g}_{ij}$ is the payoff of player I when players I and II choose ith and jth pure strategies, respectively.

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A mixed strategy for Player I, in this case, will be a sequence $(x_1, x_2, \dots)$ satisfying

$$\sum_{i=1}^{\infty} x_i = 1, x_i \geq 0.$$

Similarly, for player II will be a sequence $(y_1, y_2, \dots)$ defined similarly (Owen, 1995). The game value in mixed strategies is denoted by $h(x, y)$, i.e.,

$$h(x,y) = \sum_{i,j=1}^{\infty} x_i \, \tilde{g}_{ij} y_j.$$

Marchi (1967) found a necessary and sufficient condition for the existence of game value in an infinite matrix game. However, it wasn't easy to verify this condition. Tijs (1975), Cegielski (1991) ve Mendez Naya (1996, 1998, 2001) obtained different results regarding the existence of game value in infinite matrix games. Li et al. (2012) defined the "reasonable solution" and notions of solution for infinite matrix games, developing a bi-objective linear programming method to solve the game. In this paper, we account for a reasonable solution and notions of solution for infinite interval matrix games. Also, we present sufficient conditions for an exact or reasonable solution by using the convergence of sequences of rows or columns.

2. PRELIMINARIES

In this section, we refer to the definition, theorem, and the notions concerning interval matrices (Moore, 1979; Sengupta and Pal, 1997, 2000, 2009) and concerning interval matrix games (Collins and Hu, 2008; Nayak and Pal, 2009). The most extensive work on interval analysis was done by Moore (1979). First of all, we introduced the definitions used in interval matrix games.

2.1. Interval Numbers

Definition 2.1.1 [8] An interval $\tilde{a}$ is the bounded, closed subset of the real number set $\mathbb{R}$, defined by

$$\tilde{a} = [a_L, a_R] = \{x \in \mathbb{R}: a_L \leq x \leq a_R\},$$

where $a_L, a_R \in \mathbb{R}$ and $a_L \leq a_R$. All interval number set is denoted by $\mathbb{R}^i$. If $a_L = a_R$, then $\tilde{a}$ is a degenerate interval.

The following definition describes the basic arithmetic operations for intervals.

Definition 2.1.2 [8] Let $\tilde{a} = [a_L, a_R]$ and $\tilde{b} = [b_L, b_R]$ be two interval numbers, and α is a real number, then

(1) $\tilde{a} = \tilde{b} \Leftrightarrow a_L = b_L$ ve $a_R = b_R$

(2) $\tilde{a} + \tilde{b} \Leftrightarrow [a_L + b_L, a_R + b_R]$

(3) $\tilde{a} - \tilde{b} \Leftrightarrow [a_L - b_R, a_R - b_L]$

(4) $\tilde{a}.\tilde{b} \Leftrightarrow [\min S, \max S]$ such that

$$S = \{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}$$

(5) $\tilde{a}/\tilde{b} \Leftrightarrow [\min K, \max K]$ where

$$K = \{\frac{a_L}{b_L}, \frac{a_L}{b_R}, \frac{a_R}{b_L}, \frac{a_R}{b_R}\} \text{ such that } 0 \notin \tilde{b}$$

(6) For $\alpha \in \mathbb{R}, (\alpha \neq 0)$

$$\alpha\tilde{a} = \begin{cases} [\alpha a_L, \alpha a_R], \alpha \geq 0 \\ [\alpha a_R, \alpha a_L], \alpha < 0. \end{cases}$$

The midpoint, and the radius of the interval $\tilde{a}$ was defined by Moore (1979) as follows respectively,

$$m(\tilde{a}) = \frac{a_L + a_R}{2}, \qquad \omega(\tilde{a}) = \frac{a_R - a_L}{2}.$$

The ordering between intervals is a difficult problem. Firstly, Moore (1979) introduced an order relation to order two discrete interval numbers. Then, Ishibuchi and Tanaka (1990) defined different order relations, which cannot strictly explain ranking interval numbers. Then, an index was defined by Sengupta et al. (2001) and Sengupta et al. (2009), which can order two intervals in terms of value.

Definition 2.1.3 [17] Let $\mathbb{R}^i$ denote the set of all closed intervals on the real line $\mathbb{R}$. Then, an acceptability function $\phi: \mathbb{R}^i \times \mathbb{R}^i \to [0, \infty)$ is defined by

$$\phi(\tilde{a} \prec \tilde{b}) = \frac{m(\tilde{b}) - m(\tilde{a})}{\omega(\tilde{a}) + \omega(\tilde{b})}$$

where $\omega(\tilde{a}) + \omega(\tilde{b}) \neq 0$. $\phi(\tilde{a} \prec \tilde{b})$ may be interpreted as the grade of acceptability of the "second interval to be superior to the first interval".

The grade of acceptability of $\phi(\tilde{a} \prec \tilde{b})$ can be further classified and interpreted based on comparison interval $\tilde{b}$ with respect to the interval $\tilde{a}$ as follows:

$$\phi(\tilde{a} \prec \tilde{b}) = \begin{cases} 0 & m(\tilde{a}) = m(\tilde{b}) \\ > 0, < 1 & m(\tilde{a}) < m(\tilde{b}) \text{ ve } a_R > b_L \\ \geq 1 & m(\tilde{a}) < m(\tilde{b}) \text{ ve } a_R \leq b_L. \end{cases}$$

(1) If $\phi(\tilde{a} \prec \tilde{b}) = 0$, then the premiss "$\tilde{a}$ inferior to $\tilde{b}$" is not accepted.
(2) If $0 < \phi(\tilde{a} \prec \tilde{b}) < 1$, then the interpreter accepts the premiss $\tilde{a} \prec \tilde{b}$ with different grades of satisfaction ranging from zero to one.
(3) If $\phi(\tilde{a} \prec \tilde{b}) \geq 1$, the interpreter is satisfied with the premiss $\tilde{a} \prec \tilde{b}$.

For each interval number, the terms supremum and infimum are defined further below:

Definition 2.1.4 For each $\tilde{a}, \tilde{b} \in \mathbb{R}^i$

$$\sup\{\tilde{a}, \tilde{b}\} = [\sup\{a_L, b_L\}, \sup\{a_R, b_R\}]$$

$$\inf\{\tilde{a}, \tilde{b}\} = [\inf\{a_L, b_L\}, \inf\{a_R, b_R\}].$$

2.2. Interval Matrix Games

Let us consider an interval matrix game. Suppose that $S_I = \{I_1, I_2, \dots, I_m\}$ and $S_{II} = \{II_1, II_2, \dots, II_n\}$ are the sets of pure strategies for Players I and II. If Players I and II choose their pure strategies $I_i \in S_I$ and $II_j \in S_{II}$, respectively, then the payoff for Player I is stated by $\tilde{g}_{ij} = [g_{ijL}, g_{ijR}]$. Interval matrix G is denoted as follows:

$$\begin{array}{c} \\ I_1 \\ I_2 \\ \vdots \\ I_m \end{array} \begin{array}{c} \begin{array}{cccc} II_1 & II_2 & \cdots & II_n \end{array} \\ \begin{bmatrix} [g_{11L}, g_{11R}] & [g_{12L}, g_{12R}] & \cdots & [g_{1nL}, g_{1nR}] \\ [g_{21L}, g_{21R}] & [g_{22L}, g_{22R}] & \cdots & [g_{2nL}, g_{2nR}] \\ \vdots & \vdots & \cdots & \vdots \\ [g_{m1L}, g_{n1R}] & [g_{m2L}, g_{m2R}] & \cdots & [g_{mnL}, g_{mnR}] \end{bmatrix} \end{array}$$

X and Y below represent sets of mixed strategies for Players I and II,

$$X = \left\{ x = (x_1, x_2, \dots, x_m) \mid \sum_{i=1}^{m} x_i = 1, x_i \geq 0, (i = 1,2, \dots, m) \right\}$$

$$Y = \left\{ y = (y_1, y_2, \dots, y_n) \mid \sum_{j=1}^{n} y_i = 1, y_j \geq 0, (j = 1,2, \dots, n) \right\}.$$

Taking into account (2) and (6) of Definition 2.1.2, the payoff function of Player I can be given as follows:

$$h(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{a}_{ij} x_i y_j = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ijL} x_i y_j, \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ijR} x_i y_j \right],$$

which is an interval (Li et al., 2012).

Definition 2.2.1 [6] Let G be an interval matrix. The lower and upper values of the game are defined below

$$V_L = \max_{1 \leq i \leq m} \min_{1 \leq j \leq n} \{[a_{ijL}, a_{ijR}]\}$$

$$V_U = \min_{1 \leq j \leq n} \max_{1 \leq i \leq m} \{[a_{ijL}, a_{ijR}]\}.$$

If $[a_{krL}, a_{krR}] = V_L = V_U$, then (k, r) element of interval matrix G is called an equilibrium point and $[a_{krL}, a_{krR}]$ is called the game value in pure strategies.

Example 2.2.2 Let us consider an interval matrix game given by G

$$G = \begin{matrix} & II_1 & II_2 & II_3 & II_4 \\ \begin{matrix} I_1 \\ I_2 \\ I_3 \end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} [-12,-1] & [-10,5] & [-6,5] & [-8,5] \\ [-11,6] & [-3,5] & [1,6] & [6,7] \\ [-5,-4] & [-3,3] & [-4,0] & [-5,5] \end{pmatrix}} \end{matrix}.$$

$g_{23} = [1,6]$ is the minimum value of the second row and the maximum value of the third column at the same time, namely

$$\max\min g_{ij} = \min\max g_{ij} = [1,6].$$

It means that [1,6] is the game value.

Example 2.2.3 Let us consider the game given by

$$G = \begin{matrix} & II_1 & II_2 \\ \begin{matrix} I_1 \\ I_2 \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} [-1,1] & [1,3] \\ [0,2] & [-2,0] \end{pmatrix}} \end{matrix}$$

Here, we compute

$$\min\max g_{ij} = [-1,1]$$

$$\max\min g_{ij} = [0,2].$$

Then, we get the following,

$$\max\min g_{ij} \neq \min\max g_{ij}.$$

Now, in pure strategies, there cannot always be the game value of the interval matrix game. In this case, Li et al. (2012) defined the reasonable solution and the concepts of solution for interval matrix games, as follows.

Definition 2.2.4 [6] Let $\tilde{v} = [v_L, v_R]$ and $\tilde{w} = [w_L, w_R]$ be interval numbers on $\mathbb{R}$. Assume that there exist $x^* \in X$ and $y^* \in Y$. Then, $(x^*, y^*, \tilde{v}, \tilde{w})$ is called a reasonable solution of the interval matrix game G if $\forall\, x \in X, y \in Y$; $(x^*, y^*, \tilde{v}, \tilde{w})$ satisfies both conditions following

$$x^* G y^T \geq \tilde{v} \text{ and } x G {y^*}^{\mathsf{T}} \leq \tilde{w}.$$

If $(x^*, y^*, \tilde{v}, \tilde{w})$ is a reasonable solution for the interval matrix game G then $\tilde{v}$ is called a reasonable value for Player I and $\tilde{w}$ is called reasonable value for Player II. In this case, x^* and y^* represent reasonable strategies for Player I and II, respectively. In other words, a reasonable solution for the interval matrix game is not an exact solution, which is conceptually defined below.

Definition 2.2.5 [6] Let V and W be the sets of reasonable values $\tilde{v}$ *and* $\tilde{w}$ for Players I and II, respectively. Assume that there exist $\tilde{v}^* \in V, \tilde{w}^* \in W$. If there does not exist any $\tilde{v}' \in V (\tilde{v}' \neq \tilde{v}^*)$ and $\tilde{w}' \in W (\tilde{w}' \neq \tilde{w}^*)$ such that they satisfy either

$$\tilde{v}^* \preccurlyeq \tilde{v}' \quad or \quad \tilde{w}^* \succcurlyeq \tilde{w}',$$

then $(x^*, y^*, \tilde{v}^*, \tilde{w}^*)$ is called a solution of the interval matrix game G, with x^* and y^* being optimal strategies for Player I and II. $\tilde{v}^*$ is called the value of interval matrix game G for Player I and similarly, $\tilde{w}^*$ is for Player II.

2.3. A Sequence of Interval Numbers

Definition 2.3.1 [3] Let $\tilde{x}_i = \{\tilde{x}: \tilde{x}_{iL} \leq x \leq \tilde{x}_{iR}, \tilde{x}_{iL},\ \tilde{x}_R \in \mathbb{R}\}$ be a real closed interval. Then, an ordered n-tuple of intervals

$$\begin{aligned} \tilde{x} &= (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \\ &= ([x_{1L}, x_{1R}], [x_{2L}, x_{2R}], \ldots, [x_{nL}, x_{nR}]) \end{aligned}$$

is called an n-dimensional interval vector.

Definition 2.3.2 [8] Let f be a function defined below

$$\begin{aligned} f: \mathbb{N} &\rightarrow \mathbb{R}^i \\ n &\rightarrow f(n) \coloneqq \tilde{x}_k \end{aligned}$$

Then, f is called a sequence of interval numbers and denoted by $(\tilde{x}) = (\tilde{x}_k)$. Also $\tilde{x}_k$ is the kth term of $(\tilde{x}_k)$.

Let $\mathbb{R}^i$ be the sets of all intervals and $\tilde{a}, \tilde{b} \in \mathbb{R}^i$. The distance between $\tilde{a}$ and $\tilde{b}$ intervals are represented by

$$d(\tilde{a},\tilde{b}) = max\{|a_L - b_L|, |a_R - b_R|\}.$$

Moreover, d is a metric on $\mathbb{R}^i$ and $(\mathbb{R}^i, d)$ is a metric space (Moore, 1979).

Definition 2.3.3 [3] A sequence $(\tilde{x}_k)$ of $(\mathbb{R}^i, d)$ is said to be convergent to the interval number $\tilde{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer k_ε such that $d(\tilde{x}_k, \tilde{x}_0) < \varepsilon$ for all $k \geq k_\varepsilon$ and denoted by

$$\lim_{k\to\infty} \tilde{x}_k = \tilde{x}_0.$$

Thus,

$$\lim_{k\to\infty} \tilde{x}_k = \tilde{x}_0 \Leftrightarrow \lim_{k\to\infty} x_{kL} = x_{0L} \text{ ve } \lim_{k\to\infty} x_{kR} = x_{0R}.$$

Theorem 2.3.4 [3] Let $(\tilde{x}_k)$ be a convergent sequence of intervals with limit interval $\tilde{x}_0$ then

$$\lim_{k\to\infty} \tilde{x}_k = \tilde{x}_0 \Leftrightarrow \lim_{k\to\infty} d(\tilde{x}_k, \tilde{x}_0) = 0.$$

Example 2.3.5 Let us consider a sequence of interval $(\tilde{x}_k) = \left(\left[\frac{1}{k}, 1^k\right]\right)$ and denote its convergent to the interval number [0,1].

$$\begin{aligned}\lim_{k\to\infty} d(\tilde{x}_k, \tilde{x}_0) &= \lim_{k\to\infty} \max\{|x_{kL} - x_{0L}|, |x_{kR} - x_{0R}|\} \\ &= \lim_{k\to\infty} \max\left\{\left|\frac{1}{k} - 0\right|, |1-1|\right\} \\ &= \lim_{k\to\infty} \frac{1}{k} = 0\end{aligned}$$

Thereby, the sequence of interval $(\tilde{x}_k)$ is convergent to the interval number [0,1].

2.4. Infinite Interval Matrix Games

In this paper, we have considered finite interval matrix games till this part. In this section, we focus on the case where the pure strategy sets of both players have infinite elements. We investigate the concepts of game value and solution of infinite interval matrix games generalized to finite interval matrices. Given $\tilde{g}_{ij} = [\tilde{g}_{ijL}, \tilde{g}_{ijR}]$ denotes the payoff of Player I when Player I chooses $i\ th$ strategy and Player II chooses $j\ th$ strategy.

Let $\tilde{G} = [g_{ij}]_{i,j\in\mathbb{N}}$ be an infinite interval matrix. Also denoted by

$$\tilde{G} = [g_{ij}]_{i,j\in\mathbb{N}} = \begin{bmatrix} [g_{11L}, g_{11R}] & [g_{12L}, g_{12R}] & \cdots \\ [g_{21L}, g_{21R}] & [g_{22L}, g_{22R}] & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

where the set of strategies of both players is defined by

$$S = \{(x_1, x_2, \ldots): \sum_{i=1}^{\infty} x_i = 1,\ \ x_i \geq 0,\ \ \forall i \in \mathbb{N}\}$$

Naya (2001). The payoff function of Player I is given below,

$\forall\ (x, y) \in S \times S,$

$$\tilde{h}(x,y) = X\tilde{G}Y^T = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_i\,[g_{ijL}, g_{ijR}]y_j^T.$$

Definition 2.4.1 [11] Let $\tilde{G} = [g_{ij}]_{i,j\in\mathbb{N}}$ be the payoff matrix of the infinite interval matrix game. For each $(x, y) \in S \times S$, the superior and inferior values of $\tilde{G}$, $\overline{V}(\tilde{G})$ and $\underline{V}(\tilde{G})$ respectively, are defined by

$$\overline{V}(\tilde{G}) = \inf_{y\in S}\sup_{x\in S} \tilde{h}(x,y)$$

and

$$\underline{V}(\tilde{G}) = \sup_{x\in S}\inf_{y\in S} \tilde{h}(x,y),$$

respectively.

Assume that the payoff matrix is denoted by

$$\tilde{G}_{i'} = (g_{ij})_{1\leq i\leq i';1\leq j<\infty}$$

in which the Player I has a finite strategy but Player II has infinite strategy while the payoff matrix is stated by

$$\tilde{G}^{j'} = (g_{ij})_{\substack{1\leq i<\infty \\ 1\leq j\leq j'}}$$

in which the rows are infinite, and the columns are finite. Such games are called semi-infinite interval matrix games (Tijs, 1975).

3. RESULTS

This section contains the results obtained and a new definition and theorem based on the approach in this paper.

Definition 3.1 Let $\tilde{v} = [v_l, v_r]$ and $\widetilde{w} = [w_l, w_r]$ be elements of $\mathbb{R}^i$. Assume that $x^*, y^* \in S$ and $(x^*, y^*, \tilde{v}, \widetilde{w})$ is a reasonable solution to the infinite interval matrix game $\tilde{G}$. If $(x^*, y^*, \tilde{v}, \widetilde{w})$ satisfies both the conditions

$$x^* \tilde{G} y^T \succcurlyeq \tilde{v} \text{ and } x \tilde{G} {y^*}^T \preccurlyeq \widetilde{w}, \forall x, y \in S \, .$$

If $(x^*, y^*, \tilde{v}, \widetilde{w})$ is a reasonable solution to the infinite interval matrix game $\tilde{G}$, then $\tilde{v}$ and $\widetilde{w}$ are called reasonable values for each player, where x^* is a reasonable strategy of Player I and y^* is a reasonable strategy for Player II.

Let V and W be the sets of reasonable values for Players I and II, respectively, which does not imply a solution for the infinite interval matrix game. Consequently, the solution to the infinite interval matrix game is defined below.

Definition 3.2 Given that there are $\tilde{v}^* \in V$ and $\widetilde{w}^* \in W$. If there are no $\tilde{v}' \in V (\tilde{v}' \neq \tilde{v}^*)$ and $\widetilde{w}' \in W (\widetilde{w}' \neq \widetilde{w}^*)$ such that they satisfy

$$\tilde{v}^* \preccurlyeq \tilde{v}' \; or \; \widetilde{w}^* \succcurlyeq \widetilde{w}'$$

then $(x^*, y^*, \tilde{v}^*, \widetilde{w}^*)$ is called a solution of the infinite interval matrix game $\tilde{G}$, x^* and $y^*$are optimal strategies for Players I and II, respectively, $\tilde{v}^*$ is called the value of infinite interval matrix game $\tilde{G}$ for Player I and $\widetilde{w}^*$is called the value of the game for Player II.

Definition 3.3 Let $\tilde{G} = \left(\tilde{g}_{ij}\right)_{i,j \in \mathbb{N}}$ be a bounded infinite interval matrix. Limit superior of row sequences and limit inferior of column sequences of an infinite interval matrix are described as follows:

$$T_i = \inf_{m \geq 1} \sup_{j \geq m} \tilde{g}_{ij} \, , \forall j \in \mathbb{N}, T = (T_i)_{i \in \mathbb{N}}$$

and

$$K_j = \sup_{m \geq 1} \inf_{i \geq m} \tilde{g}_{ij} \, , \forall i \in \mathbb{N}, K = \left(K_j\right)_{j \in \mathbb{N}} \, ,$$

respectively.

Theorem 3.4 Let $\tilde{v} = \inf_{j \in \mathbb{N}} K_j$ be infimum of K_j and $\widetilde{w} = \sup_{i \in \mathbb{N}} T_i$ be supremum of T_i, $x^*, y^* \in S$ and $(x^*, y^*, \tilde{v}, \widetilde{w})$ be a reasonable solution. If $\tilde{G} = \left(\tilde{g}_{ij}\right)_{i,j \in \mathbb{N}}$ is a bounded infinite interval matrix such that

$$\tilde{v} \succ \widetilde{w},$$

then $\tilde{G}$ does not have a value.

Proof. First,

$$\inf x \tilde{G} y^T \preccurlyeq xT, \forall x \in S.$$

Suppose otherwise. Then, there exist $z \in S$ and $\epsilon > 0$ such that

$$\inf_{y \in S} z \tilde{G} y^T \succ zT + \epsilon.$$

Therefore,

$$z \tilde{G} e_j^T \prec zT + \epsilon \Rightarrow z\left(T - \tilde{G} e_j^T\right) \prec -\epsilon, \forall j \in \mathbb{N}.$$

But this is a contradiction.

Hence,

$$\inf_{y \in S} x \tilde{G} y^T \preccurlyeq xT, \forall x \in S.$$

Thereby,

$$\sup_{x \in S} \inf_{y \in S} x \tilde{G} y^T \preccurlyeq \sup xT = \sup_{i \in \mathbb{N}} T_i = \widetilde{w}$$
$$\Rightarrow V_L\left(\tilde{G}\right) \preccurlyeq \widetilde{w}.$$

Similarly,

$$V^U\left(\tilde{G}\right) \succcurlyeq \tilde{v}.$$

Thus, game $\tilde{G}$ does not have a value.

Example 3.5 Payoff matrix $\tilde{G}$ is given as follows

$$\tilde{G} = (\tilde{g}_{ij})_{i,j\in\mathbb{N}} = \begin{cases} [-1,1] & i < j, \\ [0,0] & i = j, \\ [1,1] & i > j. \end{cases}$$

Given Theorem 3.4, the fact that $\tilde{G}$ has no value is an immediate result, since it is obvious that for this matrix $T_i = [-1,1]$ for all $i, j \in \mathbb{N}$ and $K_j = [1,1]$.

Theorem 3.6 Let $\tilde{G} = (\tilde{g}_{ij})_{i,j\in\mathbb{N}}$ be a bounded infinite interval matrix game such that all its column converges to $\tilde{\alpha} = [\alpha_l, \alpha_r]$ and all its rows converge to $\tilde{\beta} = [\beta_l, \beta_r]$. If $\tilde{\alpha} \preccurlyeq \tilde{\beta}$ then,

$$\tilde{\alpha} \preccurlyeq \underline{V}(\tilde{G}) \preccurlyeq \overline{V}(\tilde{G}) \preccurlyeq \tilde{\beta}.$$

Proof. Let us consider $\tilde{G} = (\tilde{g}_{ij})_{i,j\in\mathbb{N}}$ as a separate payoff matrix taking into account the lower and upper values of the intervals, as follows:

$$\tilde{G} = [g_{ij}]_{i,j\in\mathbb{N}} = \begin{bmatrix} [g_{11L}, g_{11R}] & [g_{12L}, g_{12R}] & \cdots \\ [g_{21L}, g_{21R}] & [g_{22L}, g_{22R}] & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$G_L = [g_{ijL}]_{i,j\in\mathbb{N}} = \begin{bmatrix} g_{11L} & g_{12L} & \cdots \\ g_{21L} & g_{22L} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$G_R = [g_{ijR}]_{i,j\in\mathbb{N}} = \begin{bmatrix} g_{11R} & g_{12R} & \cdots \\ g_{21R} & g_{22R} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

where G_L and G_R are infinite interval matrices whose entries are real numbers. Assume that each of G_L and G_R is an infinite matrix game and in the payoff matrix G_L the sequence of rows (or columns) converges to the real number β_L (or α_L). Similarly, in the payoff matrix G_R the sequence of rows (or columns) converges to the real number β_R (or α_R).

Since $\tilde{\alpha} \preccurlyeq \tilde{\beta}$

then,

$$\alpha_L \leq \beta_L \; and \; \alpha_R \leq \beta_R.$$

Therefore,

$$\alpha_L \leq V_L(G_L) \leq V^U(G_L) \leq \beta_L$$

$$\alpha_R \leq V_L(G_R) \leq V^U(G_R) \leq \beta_R.$$

the above inequalities hold (Naya, 2001). Given these inequalities when the nested intervals are ignored

$$[\alpha_L, \alpha_R] \preccurlyeq [V_L(G_L), V_L(G_R)] \preccurlyeq [V^U(G_L), V^U(G_R)] \preccurlyeq [\beta_L, \beta_R]$$

the above inequality is obtained.

On the other hand, it means that

$$\tilde{\alpha} \preccurlyeq \underline{V}(\tilde{G}) \preccurlyeq \overline{V}(\tilde{G}) \preccurlyeq \tilde{\beta}.$$

Namely, the proof is completed.

4. CONCLUSIONS

The concepts of game value and solution of the game are two main characteristics of the research-related matrix games. Currently, interval matrix games, which play a crucial part in dealing with uncertainties in the real world, are gaining attention. Studies on interval matrix games and their solution methods play an essential role in the game theory since their applicability to daily life practices is high. Consequently, within the scope of this paper, we extend an interval matrix game to an infinite interval matrix game, and we state some definitions and theorems for finite interval matrix games. The results obtained from the present work will significantly contribute to future works since the infinite interval matrix game is an emerging topic. After this work, research on the convergence of either row or column sequence to a different interval number for a matrix game with infinite payoffs is an open problem.

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The Declaration of Conflict of Interest/ Common Interest

No conflict of interest or common interest has been declared by the authors.

Authors' Contribution

The authors contributed equally to the study and approved the last version of the manuscript.

The Declaration of Ethics Committee Approval

This study does not require ethics committee permission or any special permission.

The Declaration of Research and Publication Ethics

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