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Research Article

The Möbius Curvature of Bezier Curves

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Abstract

The aim of this study is to observe the Möbius curvature is computed by me as using curvature of Bezier curve is therefore proportional to the differentials of the curvature also correspond to a such as survey properties of Bezier curves. The Möbius curvature of Bezier curve has different value according to the control points. Also when the different cases may ocur, it has different values according to the angle is constant or not.

Keywords: Bezier curves, Curvature, Möbius Curvature.

Bezier Eğrilerinin Möbius Eğriliği

Öz

Bu çalışmanın amacı Bezier eğrilerinin eğriliğini kullanarak hesapladığım Möbius eğrilerini Bezier eğrilerinin özellikleri araştırılmasından dolayı oarantılı olarak buna karşılık gelen eğriliğin diferensiyellerinin incelenmesidir. Bezier eğrisinin Möbius eğriliği kontrol noktalarında farklı değer alır. Yine açının sabit yada değişken olmasına göre de farklı durumlar söz konusu olduğunda değeri değişebilmektedir.

Anahtar Kelimeler: Bezier eğrileri, Eğrilik, Möbius eğriliği.

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1. Introduction

The mathematical Bezier curves as known Berstein Polynomial has studied since 1960 by french engineer Pierre Bezier, especially automobile design.

In [3], Marsland and Maclachen investigate of planar shapes and images under the möbius group PSL(2,ℂ) is therefore proportional to the integral of the curvature.

The aim of this study is to observe the Möbius curvature is computed by using curvature of Bezier curve and Bezier curves is therefore curves proportional to the integral of the curvature also correspond to a such as properties of Bezier curves.

2. Preliminaries

2.1. Bezier Curves

Bezier curve is defined as a parametric curve $Q(t)$ that use the Berstein polynomials as a basis. The equation of the general Bezier curve is given by:

$$Q(t)=\sum_{i=0}^{m} P_i^m(t)Q_i$$

where $P_i^m(t)$ is a basis function for Bezier curve Q_i refers to the control points of the curve and they constitute B-spline curve. The function of the $P_i^m(t)$ can be defined as the following:

$$P_i^m(t)=\frac{m!}{(m-1)!i!}(1-t)^{m-i}t^i, \quad i=0,1,2,\ldots,n$$

The curve can be expressed as any degree m with $m+1$ control points [1,2,3,4,5,6,7,8,9,11].

2.2. Frenet Frame of Bezier Curves

Frenet frame of Bezier curves $\{T,N,B,\kappa,\tau\}$ are found firstly by Samanci [9] as follows:

Theorem 2.1.1.

The curvature of a Bezier curve whose control points are $b_0,b_1,b_2,\ldots,b_n$ from $n.$ degree at $t=0$ point

$$\kappa=\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\alpha$$

[11].

Proof. It is obviously seen in [9].

Theorem 2.1.2.

The curvature of a bezier curve whose control points $b_0,b_1,b_2,\ldots,b_n$ from $n.$ degree at $t=1$ point

$$\kappa=\frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|}\sin\alpha$$

Proof. It is obviously seen in [9].

3. Möbius Curvature of Bezier Curves

3.1. Möbius Curvature

Let take a parametrization-invariant Möbius invariant known as the inversive or Möbius curvature [5,9];

$$\kappa_{Möb}=\frac{4\kappa'(\kappa'''-\kappa^2\kappa')-5(\kappa'')^2}{8(\kappa')^3}$$

where $'$ denotes differentiation with respect to arclenght [5,9].

Theorem 3.1.

If take a parametrization-invariant Möbius invariant known as the inversive Bezier curve or Möbius curvature of Bezier curve at $t=0$ point.

Proof. We know from [9] that the curvature of Bezier Curve is as follows:

$$\kappa=\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\alpha$$

and

$$\kappa'=\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\sin\alpha-\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\cos\alpha$$

and

$$\kappa''=\left(\frac{n-1}{n}\right)\frac{\begin{bmatrix}\|\Delta b_1''\|\|\Delta b_0\|+\|\Delta b_0'\|\|\Delta b_0'\| \\ - \\ \|\Delta b_1'\|\|\Delta b_0'\|-\|\Delta b_1\|\|\Delta b_o''\|\end{bmatrix}\|\Delta b_0\|^2-2\left[\|\Delta b_0'\|\|\Delta b_0'\|-\|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0\|\|\Delta b_0'\|}{\|\Delta b_0\|^4}\sin\alpha$$

$$+\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\cos\alpha$$

$$-\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\cos\alpha$$

$$-\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\alpha$$

so

$$\kappa''=\frac{n-1}{n}\frac{\left[\|\Delta b_1''\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_o''\|\right]}{\|\Delta b_0\|^2}\sin\alpha$$

$$-2\frac{n-1}{n}\frac{\left[\|\Delta b_0'\|\|\Delta b_0'\|-\|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|}{\|\Delta b_0\|^3}\sin\alpha$$

$$-\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\alpha$$

Also we must have find the κ''', so we obtain

$$\kappa''' = \frac{n-1}{n}\frac{\begin{bmatrix}\|\Delta b_1'''\|\|\Delta b_0\| + \|\Delta b_1''\|\|\Delta b_0'\| \\ -\|\Delta' b_1\|\|\Delta b_o''\| - \|\Delta b_1\|\|\Delta b_0'''\|\end{bmatrix}\|\Delta b_0\|^2 - 2\left[\|\Delta b_1''\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_o''\|\right]\|\Delta b_0'\|\|\Delta b_0\|}{\|\Delta b_0\|^4}\sin\alpha$$
$$+ \frac{n-1}{n}\frac{\left[\|\Delta b_1''\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_o''\|\right]}{\|\Delta b_0\|^2}\cos\alpha$$
$$-2\frac{n-1}{n}\frac{\begin{bmatrix}\|\Delta b_0''\|\|\Delta b_0'\| + \|\Delta b_1'\|\|\Delta b_0''\| - \|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1\|\|\Delta b_0''\|\end{bmatrix}\|\Delta b_0'\| - \left[\|\Delta b_0'\|\|\Delta b_0'\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0''\|}{\|\Delta b_0\|^6}$$
$$.\sin\alpha$$
$$-2\frac{n-1}{n}\frac{\left[\|\Delta b_0'\|\|\Delta b_0'\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|}{\|\Delta b_0\|^3}\cos\alpha$$
$$-\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\sin\alpha - \frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\cos\alpha$$

After these calculations, if we substitute above differentiations of κ to the formulae of $\kappa_{Möb} = \dfrac{4\kappa'(\kappa''' - \kappa^2\kappa') - 5(\kappa'')^2}{8(\kappa')^3}$, then we have the $\kappa_{Möb}$ of Bezier curves.

Special Case 1: If the angle θ is constant, then we have the values of the diffentials of κ that are κ', κ'' and κ''' as follows

$$\kappa = \frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\alpha,$$

$$\kappa' = \frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\sin\alpha,$$

$$\kappa'' = \left(\frac{n-1}{n}\right)\frac{\begin{bmatrix}\|\Delta b_1''\|\|\Delta b_0\| + \|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\| - \|\Delta b_1\|\|\Delta b_0''\|\end{bmatrix}\|\Delta b_0\|^2 - 2\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|\|\Delta b_0\|}{\|\Delta b_0\|^4}\sin\alpha$$

and we obtain,

$$\kappa'' = \left(\frac{n-1}{n}\right)\frac{\left[\|\Delta b_1''\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0''\|\right]\|\Delta b_0\|^2 - 2\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|\|\Delta b_0\|}{\|\Delta b_0\|^4}\sin\alpha$$

$$\kappa''' = \left(\frac{n-1}{n}\right)\frac{\begin{aligned}&\begin{bmatrix}\begin{bmatrix}\|\Delta b_1'''\|\|\Delta b_0\| + \|\Delta b_1''\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0''\| - \|\Delta b_1\|\|\Delta b_0'''\|\end{bmatrix}\|\Delta b_0\|^2 \\ +2\begin{bmatrix}\begin{bmatrix}\|\Delta b_1''\|\|\Delta b_0\| + \|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\| - \|\Delta b_1\|\|\Delta b_0''\|\end{bmatrix}\end{bmatrix}\|\Delta b_0'\|\|\Delta b_0\|\end{bmatrix}\|\Delta b_0\|^4 \\ &-4\|\Delta b_0\|^3\|\Delta b_0'\|\left[\|\Delta b_1''\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0''\|\right]\|\Delta b_0'\| \\ &-2\begin{bmatrix}\begin{bmatrix}\|\Delta b_1'''\|\|\Delta b_0\| + \|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\| - \|\Delta b_1\|\|\Delta b_0''\|\end{bmatrix}\end{bmatrix}\|\Delta b_0'\|\|\Delta b_0\|\|\Delta b_0\|^4 \\ &-2\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0''\|\|\Delta b_0\|\|\Delta b_0\|^4 \\ &-2\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|^2\|\Delta b_0\|^4 \\ &-8\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|\|\Delta b_0\|\end{aligned}}{\|\Delta b_0\|^8}\sin\alpha$$

If the angle θ is constant, thenwe obtain $\kappa_{Möb}$ as follows:

$$\kappa_{Möb} = \frac{\begin{aligned}&4\left(\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\sin\alpha\right). \\ &\begin{bmatrix}\left(\frac{n-1}{n}\right)\dfrac{\begin{aligned}&\begin{bmatrix}\begin{bmatrix}\|\Delta b_1'''\|\|\Delta b_0\| + \|\Delta b_1''\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0''\| - \|\Delta b_1\|\|\Delta b_0'''\|\end{bmatrix}\|\Delta b_0\|^2 \\ +2\begin{bmatrix}\begin{bmatrix}\|\Delta b_1''\|\|\Delta b_0\| + \|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\| - \|\Delta b_1\|\|\Delta b_0''\|\end{bmatrix}\end{bmatrix}\|\Delta b_0'\|\|\Delta b_0\|\end{bmatrix} \\ &\|\Delta b_0\|^4 \\ &-4\|\Delta b_0\|^3\|\Delta b_0'\|\left[\|\Delta b_1''\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0''\|\right]\|\Delta b_0'\| \\ &-2\begin{bmatrix}\begin{bmatrix}\|\Delta b_1'''\|\|\Delta b_0\| + \|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\| - \|\Delta b_1\|\|\Delta b_0''\|\end{bmatrix}\end{bmatrix} \\ &\|\Delta b_0'\|\|\Delta b_0\|\|\Delta b_0\|^4 \\ &-2\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0''\| \\ &\|\Delta b_0\|\|\Delta b_0\|^4 \\ &-2\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|^2\|\Delta b_0\|^4 \\ &-8\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|\|\Delta b_0\|\end{aligned}}{\|\Delta b_0\|^8}\sin\alpha \\ -\left(\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\alpha\right)^2\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\sin\alpha\end{bmatrix} \\ &-5\left(\left(\frac{n-1}{n}\right)\frac{\begin{bmatrix}\|\Delta b_1''\|\|\Delta b_0\| + \|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\| - \|\Delta b_1\|\|\Delta b_0''\|\end{bmatrix}\|\Delta b_0\|^2 - 2\left[\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|\|\Delta b_0\|}{\|\Delta b_0\|^4}\sin\alpha\right)^2\end{aligned}}{8\left(\dfrac{n-1}{n}\dfrac{\|\Delta b_1'\|\|\Delta b_0\| - \|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\sin\alpha\right)^3}$$

Special Case 2: If the angle $\theta = 90^o$, then $\kappa_{Möb}$ of Bezier curves is obtained as follows:

$$\kappa_{Möb} = \frac{4\left(\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\right) \cdot \left[\begin{array}{l} \left[\begin{array}{l} \left[\begin{array}{l}\|\Delta b_1'''\|\|\Delta b_0\|+\|\Delta b_1''\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0''\|-\|\Delta b_1\|\|\Delta b_0'''\|\end{array}\right]\|\Delta b_0\|^2 \\ +2\left[\left[\begin{array}{l}\|\Delta b_1''\|\|\Delta b_0\|+\|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\|-\|\Delta b_1\|\|\Delta b_0''\|\end{array}\right]\right]\|\Delta b_0'\|\|\Delta b_0\| \end{array}\right] \\ \|\Delta b_0\|^4 \\ -4\|\Delta b_0\|^3\|\Delta b_0'\|\left[\|\Delta b_1''\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0''\|\right]\|\Delta b_0'\| \\ -2\left[\left[\begin{array}{l}\|\Delta b_1'''\|\|\Delta b_0\|+\|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\|-\|\Delta b_1\|\|\Delta b_0''\|\end{array}\right]\right] \\ \|\Delta b_0'\|\|\Delta b_0\|\|\Delta b_0\|^4 \\ -2\left[\left[\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|\right]\right]\|\Delta b_0''\| \\ \|\Delta b_0\|\|\Delta b_0\|^4 \\ -2\left[\left[\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|\right]\right]\|\Delta b_0'\|^2\|\Delta b_0\|^4 \\ \left(\frac{n-1}{n}\right)\frac{-8\left[\left[\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|\|\Delta b_0\|\right]}{\|\Delta b_0\|^8} \\ -\left(\frac{n-1}{n}\frac{\|\Delta b_1\|}{\|\Delta b_0\|}\sin\alpha\right)^2\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2} \end{array}\right] -5\left(\left(\frac{n-1}{n}\right)\frac{\left[\begin{array}{l}\|\Delta b_1''\|\|\Delta b_0\|+\|\Delta b_1'\|\|\Delta b_0'\| \\ -\|\Delta b_1'\|\|\Delta b_0'\|-\|\Delta b_1\|\|\Delta b_0''\|\end{array}\right]\|\Delta b_0\|^2 -2\left[\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|\right]\|\Delta b_0'\|\|\Delta b_0\|}{\|\Delta b_0\|^4}\right)^2}{8\left(\frac{n-1}{n}\frac{\|\Delta b_1'\|\|\Delta b_0\|-\|\Delta b_1\|\|\Delta b_0'\|}{\|\Delta b_0\|^2}\right)^3}$$

Special Case 3: If the angle $\theta = 0^o$, then $\kappa_{Möb} = 0$ of Bezier curves. This means that Bezier curve is lie on the plane. If $\kappa_{Möb} = 0$, then Möbius energy is not computed on the surface.

2020 Mathematics Subject Classification is: 53A04, 53A05

4. Conclusions

In this work Möbius curvature of Bezier curves is computed. These calculations is a step for finding Möbius energy of Bezier curves. Möbius energy is a kind of artificial energy that Möbius energy is found by using curvatures of the Bezier curves in differential geometry.

Also the same operations are calculated for the curvature of a Bezier curve whose control points are $b_0, b_1, b_2, ..., b_n$ from n. degree at $t = 1$ point for below formulae

$$\kappa = \frac{n-1}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|}\sin\alpha$$

5. Acknowledge

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