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New Representation of Hasimoto Surfaces with the Modified Orthogonal Frame

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Abstract

In this study, we investigate Hasimoto surfaces considering the modified orthogonal frame. Firstly, we recall the relations between the Frenet frame and the modified orthogonal frame, and then we give the evolution equations of the modified orthogonal frame. After that, the first and second fundamental forms, mean curvatures, and Gaussian curvatures of the Hasimoto surfaces are determined with respect to the modified orthogonal frame. We give the definitions and some new theorems about Hasimoto surfaces. Finally, we express the properties of parameter curves of Hasimoto surfaces with a modified orthogonal frame in Euclidean 3-space.

Keywords: *Binormal motion, evolution of curves, Hasimoto surfaces, modified orthogonal frame, smoke ring equations.*

2010 Mathematics Subject Classification: *53A05, 53A04, 37K25*

1. Introduction

In recent years, the theory of surfaces with the connection of the motion of the space curves and nonlinear differential equations is a subject of research attention. At the same time, the applications of the motions of the space curves and the nonlinear differential equations in differential geometry and physics have attracted attention. In 1971, Hasimoto investigated the moving of a vortex filament, and then Hasimoto showed the connection between the vortex filament (smoke ring) equation and the nonlinear Schrodinger equation in 1972 [1, 2]. Moreover, the contributions of these studies to the essential researches in the mathematical physics field are incredible. Therefore, they can be accepted as starting points for new developments. In 3-dimensional Euclidean space E^3, some surfaces can be defined through integrable equations. One of these surfaces to be given as an example is the Hasimoto surface. The geometric properties of Hasimoto surfaces are presented in detail by [3]. Let $r(s,t)$ be a position vector of a moving unit speed space curve for all t on a surface S in E^3. If the position vector $r(s,t)$ satisfies

$$r_t = r_s \times r_{ss}, \tag{1.1}$$

where t is the time parameter, s is the arc-length parameter, and the subscripts denote the partial differential with respect to parameters. Then the surface S is called the Hasimoto surface. Also, this equation is called the vortex filament or smoke ring equation [3]. A lot of researches on Hashimoto surfaces were done using the different frames in Euclidean space [3, 4, 5], Minkowski space [6], and Galilean space [7]. In addition, the parallel surfaces of the Hasimoto surfaces in Euclidean space were investigated by [8]. The Hasimoto surfaces were researched according to three classes of curve evolution with Darboux frame in E^3, and also, the Hasimoto surfaces for two classes of curve evolutions in Minkowski 3-space were given by [9, 10]. Helices, Hasimoto surfaces, and Bäcklund transformations were included in [11]. Bäcklund transformation and vortex filament equation for null Cartan curve and the Bishop frames of pseudo null and null Cartan curves in Minkowski 3-space were studied by Grbović and Nešović [12, 13]. On the other hand, the Serret-Frenet frame is inadequate for studying analytic space curves of which curvatures have discrete zero points since the principal normal and binormal vectors may be discontinuous at zero points of the curvature. For the solution of this problem, Sasai presented an orthogonal frame and obtained a formula, which corresponds to the Frenet-Serret equation [14]. Recently, in Minkowski 3-space, the modified orthogonal frame with non-zero curvature and torsion of a space curve was described by Bukcu and Karacan [15].

In light of these researches, we aim to investigate the geometric properties of Hasimoto surfaces via the modified orthogonal frame. Also, the characterization of the parameter curves of the Hasimoto surface is examined.

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2. Preliminaries

Let r be a space curve with the arc-length s in Euclidean space E^3 defined by the inner product $<,>= dx_1^2+dx_2^2+dx_3^2$ where $x=(x_1,x_2,x_3)\in E^3$. Also, let $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ denote the tangent, principal normal, and unit binormal vectors at point $r(s)$ of r, respectively. Then the orthonormal frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ satisfies the Frenet-Serret equation

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}_s = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where κ and τ are the curvature and the torsion of the curve r, respectively.
Sasai represented the modified orthogonal frame $\{T,N,B\}$ as an alternative to the Frenet orthonormal frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ because the principal normal and binormal vectors of the Frenet frame of a space curve become discontinuous at the points where the curvature is zero.
We assume that the curvature $\kappa(s)$ of r is not zero. Then the modified orthogonal frame $\{T,N,B\}$ is defined as follows:

$$T=\frac{dr}{ds}, N=\frac{dT}{ds}, B=T\times N,$$

where "$\times$" denotes the cross product. The relation between the modified orthogonal frame $\{T,N,B\}$ and the Frenet-Serret frame $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ at non-zero points of κ is given by

$$T=\mathbf{t},\ N=\kappa\mathbf{n},\ B=\kappa\mathbf{b}.$$

Also, the modified orthogonal frame $\{T,N,B\}$ satisfies

$$\langle T,T\rangle = 1,\ \langle N,N\rangle = \langle B,B\rangle = \kappa^2,\ \langle T,N\rangle = \langle T,B\rangle = \langle N,B\rangle = 0.$$

From these last equations, The differentiation formula of the modified orthogonal frame $\{T,N,B\}$ is rewritten in the matrix form:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_s = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa_s}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa_s}{\kappa} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Here κ_s denotes the partial differential of the curvature κ with respect to the arc-length parameter s and $\tau=\frac{\det(r',r'',r''')}{\kappa^2}$ is the torsion of the curve r [14, 15, 16, 17].

3. Properties of Hasimoto Surfaces with Modified Orthogonal Frame

In this section, the Hasimoto surfaces are investigated with the modified orthogonal frame in E^3. At the same time, the parameter curves of the Hasimoto surface are characterized. First, the time evolution equations of a space curve for the modified orthogonal frame are given. Let $r(s,t)$ be a position vector of a moving space curve r which a unit speed curve for all time t in Euclidean 3-spaces. Then, the time evolution of the modified orthogonal frame $\{T,N,B\}$ of the curve r can be written in matrix form as follows:

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}_t = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & \kappa\kappa_t & \gamma \\ -\beta & -\gamma & \kappa\kappa_t \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where α, β and γ are smooth functions, which determine the time evolution of the moving curve for all t, with respect to s and t [16]. Also, the time evolution of the curvature and torsion of the moving curve r are represented by

$$\kappa_t = \frac{1}{\kappa}\left(\alpha_s + \frac{\kappa_s}{\kappa}\alpha - \tau\beta\right) \text{ and } \tau_t = \gamma_s + \beta, \tag{3.1}$$

respectively. Moreover, by using the compatibility conditions, $T_{st}=T_{ts}, N_{st}=N_{ts}$, $B_{st}=B_{ts}$, the smooth functions α, β and γ given in [16], one obtains

$$\alpha=-\tau, \beta=\frac{\kappa_s}{\kappa}, \gamma=\frac{\kappa_{ss}}{\kappa}-\tau^2. \tag{3.2}$$

Now, we give some geometric properties of the Hasimoto surfaces with the modified orthogonal frame and it is defined the following as:

Definition 3.1. *Let $r(s,t)$ be a position vector of a moving unit speed space curve r for all t, on a regular surface in Euclidean 3-space. The generated surface is called Hasimoto surface provided that $r(s,t)$ satisfies the following condition*

$$r_t = r_s \times r_{ss} = B, \tag{3.3}$$

where the vector B is the binormal vector in the modified orthogonal frame of the moving curve r.

Differentiating of the Hasimoto surface $r(s,t)$ defined with the condition (3.3) in terms of s and t, it is obtained that

$$r_s = T \text{ and } r_t = B. \tag{3.4}$$

Then, from the last equations, the unit normal vector field $u(s,t)$ of the Hasimoto surfaces is found as

$$u(s,t) = \frac{r_s \times r_t}{\|r_s \times r_t\|} = \frac{T \times B}{\|T \times B\|} = -N. \tag{3.5}$$

Considering the partial differentiations of $r(s,t)$ with respect to s and t from the equation (3.4), The components E, F and G of the first fundamental form of the Hasimoto surface $r(s,t)$ are obtained as follows:

$$E = \langle r_s, r_s \rangle = 1,\ F = \langle r_s, r_t \rangle = 0,\ G = \langle r_t, r_t \rangle = 1. \tag{3.6}$$

In addition, differentiating r_s and r_t with respect to s and t, we get

$$r_{ss} = N,\ r_{st} = \alpha N + \beta B \text{ and } r_{tt} = -\beta T - \gamma N + \kappa\kappa_t B,$$

where the functions $\alpha = -\tau, \beta = \frac{\kappa_s}{\kappa}$ and $\gamma = \frac{\kappa_{ss}}{\kappa} - \tau^2$ are defined as in (3.2). From the equation (3.5) and the last equations, the components e, f and g of the second fundamental form of the Hasimoto surface $r(s,t)$ are obtained as follows:

$$e = \langle r_{ss}, u \rangle = -1,\ f = \langle r_{st}, u \rangle = -\alpha,\ g = \langle r_{tt}, u \rangle = \gamma. \tag{3.7}$$

Theorem 3.2. *Let $r(s,t)$ be a Hasimoto surface with the modified orthogonal frame $\{T,N,B\}$ in E^3, then the Gaussian curvature K and the mean the curvature H of $r(s,t)$ are*

$$K = -\gamma - \alpha^2 \text{ and } H = \frac{\gamma - 1}{2},$$

respectively. The functions $\alpha = -\tau$ and $\gamma = \frac{\kappa_{ss}}{\kappa} - \tau^2$ are defined as in (3.2).

Proof. Let $r(s,t)$ be a Hasimoto surface with the modified orthogonal frame. Then, from the components (3.6) of the first fundamental form and (3.7) of the second fundamental form, we get that the Gaussian and mean curvatures as

$$K = \frac{eg - f^2}{EG - F^2} = -\gamma - \alpha^2 \quad \text{and} \quad H = \frac{1}{2}\frac{Eg - 2Ff + Ge}{EG - F^2} = \frac{\gamma - 1}{2}, \tag{3.8}$$

respectively. □

Corollary 3.3. *A Hasimoto surface $r(s,t)$ is developable if and only if $\alpha^2 = -\gamma$ or κ_s is constant.*

Corollary 3.4. *A Hasimoto surface $r(s,t)$ minimal if and only if $\gamma = 1$ or $\kappa_{ss} = \kappa\left(\tau^2 + 1\right)$.*

Let us now give the geometric interpretation of the parametric curves of the Hasimoto surfaces and some related theorems.

Theorem 3.5. *Let's assume that $r(s,t)$ is a Hasimoto surface with the modified orthogonal frame $\{T,N,B\}$ in E^3, then s parameter curves of $r(s,t)$ are*

- *geodesic,*
- *not asymptotic.*

Proof. Let's assume that $r(s,t)$ is a Hasimoto surface with the modified orthogonal frame $\{T,N,B\}$ in E^3. Considering the equation $r_{ss} = T_s = N$ and the normal vector field $u(s,t) = -N$ of $r(s,t)$ together, we can easily say that r_{ss} is parallel to the normal vector field $u(s,t)$ of $r(s,t)$. Therefore, s parameter curves of the Hasimoto surfaces are geodesics.
On the other hand, as we know that $r_{ss} = N$, that is, the normal component of r_{ss} cannot be zero. Therefore, s parameter curves of the Hasimoto surfaces cannot be asymptotic. □

Theorem 3.6. *Let's assume that $r(s,t)$ is a Hasimoto surface with the modified orthogonal frame $\{T,N,B\}$ in E^3, then t parameter curves of $r(s,t)$ are*

- *geodesic if and only if $\beta = 0$ and $\kappa_t = 0$,*
- *asymptotic if and only if $\gamma = 0$,*

where $\beta = \frac{\kappa_s}{\kappa}$ and $\gamma = \frac{\kappa_{ss}}{\kappa} - \tau^2$ are defined as in (3.2).

Proof. Let's assume that $r(s,t)$ is a Hasimoto surface with the modified orthogonal frame $\{T,N,B\}$ in E^3. By considering the equation $r_{tt} = B_t = -\beta T - \gamma N + \kappa\kappa_t B$ and the normal vector field $u(s,t) = -N$ of $r(s,t)$ together, we directly see that the vector r_{tt} is parallel to the normal vector field if and only if $\beta = 0$ and $\kappa_t = 0$. Therefore, t parameter curves of the Hasimoto surfaces are geodesics if and only if $\beta = 0$ and $\kappa_t = 0$.
On the other hand, since we know that $r_{tt} = B_t = -\beta T - \gamma N + \kappa\kappa_t B$, we say the normal component of r_{tt} becomes zero if and only if $\gamma = 0$. Therefore, t parameter curves of the Hasimoto surfaces are asymptotic curves if and only if $\gamma = 0$. □

Theorem 3.7. *Let $r(s,t)$ be a Hasimoto surface the modified orthogonal frame $\{T,N,B\}$ in E^3. The s and t parameter curves of the Hasimoto surfaces are lines of curvature if and only if $\alpha = 0$ where $\alpha = -\tau$.*

Proof. The parameter curves of $r(s,t)$ are lines of curvature if and only if the component F of the first fundamental form and the component f of the second fundamental form vanish. This means that

$$F = f = -\alpha = 0. \tag{3.9}$$

Hence, we can easily say that the s and t parameter curves of the Hasimoto surfaces are lines of curvature if and only if $\alpha = 0$. □

4. Conclusion

In this study, we investigated the geometric properties of Hasimoto surfaces with respect to the modified orthogonal frame. Sasai presented the modified orthogonal frame as an alternative to the Frenet frame. Because the principal normal and binormal vectors of the Frenet frame of a space curve become discontinuous at the points where the curvature is zero, However, Hasimoto surfaces have not been examined under these conditions yet. For this reason, this research is a new study in the geometry field. Additionally, in this study, we give the conditions that the parameter curves of the Hasimoto surfaces to be geodesic, asymptotic, or curvature lines.

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