e-ISSN: 2564-7954 CUJSE 19(02): 062-069 (2022) Research Article

Çankaya University

Journal of Science and Engineering

CUJSE

https://dergipark.org.tr/cankujse

Non-selfadjoint Finite System of Discrete Sturm-Liouville Operators with Hyperbolic Eigenparameter

Nimet Coskun[1]

[1] Department of Mathematics, Karamanoglu Mehmetbey University, Turkey

Keywords

Discrete Equations, Spectral Analysis, Eigenvalues, Spectral Singularities.

Abstract

In this paper, spectrum and spectral properties of the operator generated by the finite system of Sturm-Liouville discrete equations with hyperbolic eigenparameter have been taken under investigation. The transformation choosen for the eigenparameter affects drastically the representation of Jost solution and analicity region of the Jost function. Besides obtaining resolvent operator of the problem, finiteness of the eigenvalues and spectral singularities have been proved by using the analicity of the Jost solution on the complex left half-plane. Hence, generalizing the recent results, this paper lays the groundwork for future research questions in different branches of science like inverse scattering theory, quantum physics, applied mathematics and etc.

1. Introduction

Many branches of natural sciences make use of differential and discrete equations for modelling and solving natural phenomenas. For this reason, spectral analysis of differential and discrete equations has attracted intensive attention from both mathematicians and applied sciences. Discretization of continuous problems serves the purpose of solving complex problems by the aid of computers. Interested reader may refer to the books $[1-3]$ and the references therein to understand theory and applications of the spectral theory of operators. Also, $[4]$ presents a usefull background for the theory and solutions of discrete equations.

Naimark was the first to investigate the spectrum of the non-selfadjoint singular differential operator [5,6]. He elobarated on the boundary value problem (BVP)

$$-y'' + q(x)y - \lambda^2 y = 0, x \in \mathbb{R}_+, \tag{1}$$

$$y'(0) - hy(0) = 0, \tag{2}$$

where $h \in \mathbb{C}$ and q is a complex valued function. Note that the operator generated by the BVP (1)-(2) is called singular since the definition set of equation (1) is infinite and non-selfadjoint for $h \in \mathbb{C}$ and q is a complex valued function. He proved that the spectrum of the BVP (1)-(2) is comprised of eigenvalues, spectral singularities and continuous spectrum. It may be remarked that the set of spectral singularities is usually instrict to the non-selfadjoint operators. Naimark has also investigated the quantitative properties of the operator. Denote the bounded solution of (1) holding the asymptotic

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, Im\lambda \geq 0\},$$

by $e(x,\lambda)$. $e(x,\lambda)$ is introduced as the Jost solution of (1).

Under the condition

* Corresponding Author: cannimet@kmu.edu.tr
Received: September 20, 2021, Accepted: April 12, 2022

$$\int_0^\infty x|q(x)|dx < \infty,$$

the Jost solution can be represented by the Volterra type integral equation

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt, \tag{3}$$

where the function $K(x,t)$ is defined by q $[6]$. Also, Wronskian of regular and irregular solutions of (1) is defined as Jost function of the problem. To determine the spectrum of a given operator, it is necessary to obtain the resolvent of the corresponding operator. It is also well-known fact that the expression of a resolvent operator includes Jost function as a denominator term by definition. Hence, the spectrum of an operator directly depends on Jost function. Interested reader may consult to $[6,\ \text{Part II, Pages: } 292-331]$ for the detailed proofs and definitions.

Let us take into consideration the boundary value problem with the finite system of Sturm-Liouville type differential equations

$$y_j{}'' + \lambda^2 y_j = \sum_{k=1}^{n} v_{jk}(x)y_k, \qquad 0 < x < \infty \ (j = 1,2,\ldots,n),$$

and the boundary condition

$$y_j(0) = 0,$$

where $y_j \in L_2(\mathbb{R}_+)$, $(j = 1,2,\ldots,n)$ and $V(x) = \left[v_{jk}(x)\right]_{n\times n}$ is an $n \times n$ Hermitian matrix-valued function called the potential matrix. Agranovich and Marchenko have investigated the inverse scattering theory of this boundary value problem in detail in their book [7]. Note that there is one-to-one correspondence between n vector solutions $y = (y_1, y_2, \ldots, y_n)$ of above system and the $n \times n$ matrix solutions $Y(x,\lambda)$ of the matrix equation

$$Y'' + \lambda^2 Y = V(x)Y, 0 < x < \infty.$$

That is to say, both representations are equivalent and every finite sytem of Sturm-Liouville type equations can be conceived as matrix valued equation and vice versa.

Pursuing the notions $[5-7]$, the spectral characteristics of the non-selfadjoint singular boundary value problems have been the object of many more studies $[8-21]$. In particular, matrix-valued non-selfadjoint problems were studied in $[8-10]$. Quantitative spectral properties of the discrete analogue of Sturm-Liouville type operators have been seriously attacked in papers $[12-16,18,19]$. Finiteness of the eigenvalues and spectral singularities of the non-selfadjoint Dirac type operators have been searched in depth in studies $[12,20,21]$.

In this paper, we will investigate the spectral properties of L generated in the Hilbert space $l_2(\mathbb{N},\mathbb{C}^N)$ by the finite system of discrete Sturm-Liouville type equations

$$a_{n-1}^{(v)}y_{n-1}^{(v)} + b_n^{(v)}y_n^{(v)} + a_n^{(v)}y_{n+1}^{(v)} = \lambda y_n^{(v)}, n \in \mathbb{N} = \{1,2,\ldots\}, v = 1,2,\ldots,N, \tag{4}$$

and the boundary condition

$$y_0^{(v)} = 0, v = 1,2,\ldots,N, \tag{5}$$

where $\left\{a_n^{(v)}\right\}_{n=1}^{\infty}$ and $\left\{b_n^{(v)}\right\}_{n=1}^{\infty}$ are complex sequences such that $a_0^{(v)} = 1$ for $v = 1,2,\ldots,N$. Take notice of that we can write the equation (4) in the matrix form

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = \lambda Y_n, n \in \mathbb{N},$$

where $A_n = diag\left(a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(N)}\right)$, $B_n = diag\left(b_n^{(1)}, b_n^{(2)}, \ldots, b_n^{(N)}\right)$ are $N \times N$ diagonal matrices and $Y_n = \left(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(N)}\right) \in l_2(\mathbb{N},\mathbb{C}^N)$.

Concerning the non-selfadjoint discrete boundary value problems, let us mention some different approaches. For instance, in paper [21], the eigenparameter of the non-selfadjoint boundary value problem was taken as

$$\lambda = (iz) - (iz)^{-1}, |z| \leq 1.$$

As a result of this transformation, Jost solution obtained the polynomial type representation which is analytic in unit disc. Also, in [22], the spectrum of discrete analogue of Sturm-Liouville equation has been investigated for

$$\lambda = \frac{1}{2}(z^{-1} + z), |z| \geq 1.$$

A non-standard representation for Jost solution has been obtained under this eigenparameter transformation, too. Therefore, it is clear that there is a gap in the literature investigating the problem of under what transformations of the eigenparameter one can obtain solvable systems for Sturm-Liouville type discrete equations (which is also known as infinite Jacobi matrices).

Differently from other studies, Sturm-Liouville type difference operators with hyperbolic eigenparameter was taken into consideration in recent papers [18,19]. Note that hyperbolic eigenparameter shifts the analycity region of the Jost fuction from upper half-plane to left half-plane. As a consequence of this shift, analitic continuity regions of the Jost solution of the operator L differs.

Along with a diagonal complex valued potential, the eigenparameter of the non-selfadjoint Sturm-Liouville problem with an hyperbolic transformation has been considered. Thus, the calculations for the establishment of the Naimark's and Pavlov's conditions for the potential require a new point of view with respect to trigonometric parameter cases.

The paper is arranged as follows. Jost solution and Jost function of the operator L are have been presented in the next section. Section 3 is devoted to investigation of the quantitative properties of the eigenvalues and spectral singularities of the operator L.

2. Jost solution and Jost function of L

Suppose that complex sequences $(\boldsymbol{a_n^{(v)}})$ and $(\boldsymbol{b_n^{(v)}})$ satisfy

$$\sum_{n\in\mathbb{N}} n\left(\left|1 - a_n^{(v)}\right| + \left|b_n^{(v)}\right|\right) < \infty, v = 1,2,\ldots,N. \tag{6}$$

The matrix solution $E_n(z)$ of (4) satisfying the asymptotic $\lim_{n\to\infty} E_n(z)e^{-nz} = I$ for I is $N \times N$ unit matrix, $z \in \overline{\mathbb{C}}_{left} := \{z: z \in \mathbb{C}, Rez \leq 0\}$ and $\lambda = 2\cosh z$ is introduced as the Jost solution. It was proved with complete analogy with what have been referred [10,15,18] in previous section that, under this assumption, there exists a solution (Jost solution) represented by

$$E_n(z) = \left\{e_n^{(v)}(z)\right\}_{n\in\mathbb{N}} = \left\{\begin{bmatrix} e_n^{(1)}(z) & 0 & . & . & . & 0 \\ 0 & e_n^{(2)}(z) & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & e_n^{(N)}(z) \end{bmatrix}_{N\times N}\right\}_{n\in\mathbb{N}},$$

where

$$e_n^{(v)}(z) = \alpha_n^{(v)} e^{nz}\left(1 + \sum_{m=1}^{\infty} K_{n,m}^{(v)} e^{mz}\right), n \in \mathbb{N} \cup \{0\}, v = 1,2,\ldots,N, \tag{7}$$

for $\lambda = 2\cosh z$, where $z \in \overline{\mathbb{C}}_{left}$. Moreover, we have the inequality for the kernel $K_{n,m}^{(v)}$

$$\left|K_{n,m}^{(v)}\right| \leq C \sum_{r=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(\left|1 - a_r^{(v)}\right| + \left|b_r^{(v)}\right|\right), n \in \mathbb{N} \cup \{0\}, v = 1,2,\ldots,N, \tag{8}$$

for $\lambda = 2\cosh z$, where $z \in \overline{\mathbb{C}}_{left}$ and $K_{n,m}^{(v)}$, α_n are expressed in terms of $(a_n^{(v)})$ and $(b_n^{(v)})$. Hence, $e_n^{(v)}(z)$ is analytic with respect to z in $\mathbb{C}_{left} := \{z : z \in \mathbb{C}, Rez < 0\}$ and continuous in $Rez = 0$ and they also satisfy

$$e_n^{(v)}(z) = \alpha_n^{(v)} e^{nz}[1 + o(1)], n \in \mathbb{N}, v = 1,2,\dots,N, z = \xi + i\tau, \xi \to -\infty.$$

Analogous to $E_n(z)$, let $\hat{\varphi}(\lambda) = \{\hat{\varphi}_n(\lambda)\} = \varphi_n^{(v)}(z)$, $n \in \mathbb{N} \cup \{0\}$ be the solution of (4) holding the initial conditions

$$\varphi_0^{(v)}(z) = 0, \quad \varphi_1^{(v)}(z) = 1, v = 1,2,\dots,N.$$

If we define

$$\varphi(z) = \hat{\varphi}(2\cosh z) = \{\hat{\varphi}_n(2\cosh z)\}, \quad n \in \mathbb{N} \cup \{0\},$$

then φ is an entire function and

$$\varphi(z) = \varphi(z + 2\pi i).$$

Let us define the semi-strips $P_0 := \left\{z : z \in \mathbb{C}, z = \xi + i\tau, -\frac{\pi}{2} \le \tau \le \frac{3\pi}{2}, \xi < 0\right\}$ and

$$P := P_0 \cup \left\{z : z \in \mathbb{C}, z = \xi + i\tau, -\frac{\pi}{2} \le \tau \le \frac{3\pi}{2}, \xi = 0\right\}.$$

The Wronskian of the solutions of $y_n^{(v)}(z)$ and $u_n^{(v)}(z)$ of the equation (4) is defined as usual

$$W\left[y_n^{(v)}, u_n^{(v)}\right] = a_n^{(v)}\left[y_n^{(v)} u_{n+1}^{(v)} - y_{n+1}^{(v)} u_n^{(v)}\right], v = 1,2,\dots,N.$$

Hence, we get

$$W\left[e_n^{(v)}(z), \varphi_n^{(v)}(z)\right] = e_0^{(v)}(z) = E_0(z), v = 1,2,\dots,N.$$

For all $z \in P$ and $e_0^{(v)}(z) \ne 0$, the Green's function of the BVP (4), (5) is obtained by standard computations as

$$G_{nk}(z) = \begin{cases} \varphi_n(z) E_k(z) H^{-1}(z), & k \le n, \\ \varphi_k(z) E_n(z) H^{-1}(z), & k > n, \end{cases} \tag{9}$$

where

$$H(z) := \det E_n(z) = \prod_{v=1}^{N} \left\{e_n^{(v)}(z)\right\}, z \in \overline{\mathbb{C}}_{left}.$$

It is clear that, for $g = (g_k) \in l_2(\mathbb{N})$ and $k \in \mathbb{N} \cup \{0\}$,

$$(Rg)_n := \sum_{k=0}^{\infty} G_{nk}(z) g_k(z) \quad , n \in \mathbb{N} \cup \{0\}, \tag{10}$$

is the resolvent of the BVP (4), (5).

3. Eigenvalues and spectral singularities of the BVP (4)-(5)

We symbolize the set of all eigenvalues and spectral singularities of the operator L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From (9) and (10) and the definition of eigenvalues and spectral singularities, we have [12]

$$\sigma_d(L) = \{\lambda : \lambda = 2\cosh z, z \in P_0, H(z) = 0\}, \tag{11}$$

$$\sigma_{ss}(L) = \left\{\lambda : \lambda = 2\cosh z, z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], H(z) = 0\right\} \setminus \{0\}. \tag{12}$$

Definition 3.1. The multiplicity of a zero of H in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (4), (5).

Let us introduce the sets

$$R_1 := \{z: z \in P_0, H(z) = 0\},$$
$$R_2 := \left\{z: z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], H(z) = 0\right\},$$

and R_3, R_4 as the sets of limit points of the sets R_1 and R_2, respectively, and R_5 as the set of zeros in P_0 of the function $H(z)$ with infinite multiplicity. Clearly, the following relations hold

$$R_3 \subset R_2, R_4 \subset R_2, R_5 \subset R_2, R_1 \cap R_5 = \emptyset, \tag{13}$$

and the linear Lebesgue measures of R_2, R_3, R_4 and R_5 are zero. Due to the continuity of all derivatives of $H(z)$ up to the imaginary axis, it can be written that

$$R_3 \subset R_5, R_4 \subset R_5.$$

Obviously, the sets of eigenvalues and spectral singularities can be expressed as

$$\sigma_d(L) = \{\lambda: \lambda = 2\cosh z, z \in R_1\},$$
$$\sigma_{ss}(L) = \{\lambda: \lambda = 2\cosh z, z \in R_2\}.$$

Theorem 3.1. Assume that (6) holds. It follows that:

i) $\sigma_d(L)$ is bounded, countable and its limit points can lie only in $[-2,2]$.

ii) The set of spectral singularities of L is subset of $[-2,2]$, $\mu(\sigma_{ss}(L)) = 0$ where μ denotes the linear Lebesgue measure and $\sigma_{ss}(L) = \overline{\sigma_{ss}(L)}$.

Proof. It is known that $H(z)$ is analytic in the left-plane and continuous up to the imaginary axis. Moreover, the following asymptotic holds

$$H(z) = \alpha_0^{(v)}[1 + o(1)], v = 1,2,\ldots,N, z \in P_0, Rez \to -\infty. \tag{14}$$

From this point, it is easy to prove i) and ii) by using (11), (12) and (14) and uniqueness theorems of analytic functions [24]. ■

So far, Jost solution, resolvent operator and the sets of eigenvalues and spectral singularities of the operator L have been discussed under the condition (6). Now, we will investigate the impact of strickter conditions on the potential which are known as Naimark's and Pavlov's conditions.

We shall assume

$$\sum_{n=1}^{\infty} e^{\varepsilon n^{\beta}} \left(\left|1 - a_n^{(v)}\right| + \left|b_n^{(v)}\right|\right) < \infty, \varepsilon > 0, v = 1,2,\ldots,N, \frac{1}{2} \le \beta \le 1. \tag{15}$$

For $\beta = 1$, (15) reduces to Naimark's condition:

$$\sum_{n=1}^{\infty} e^{\varepsilon n} \left(\left|1 - a_n^{(v)}\right| + \left|b_n^{(v)}\right|\right) < \infty, \varepsilon > 0, v = 1,2,\ldots,N. \tag{16}$$

Theorem 3.2. Let (16) is satisfied. Then the operator L has a finitely many number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof. Taking into account (8) and (16), the following inequality

$$\left|K_{n,m}^{(v)}\right| \le C\exp\left(\frac{-\varepsilon}{2}(n+m)\right), \tag{17}$$

is satisfied for all $C > 0$ constant, $n = 0,1,2,\ldots$ and $m = 1,2,\ldots$. Using (7), (16) and (17) and after some algebra, one writes

$$|H(z)| \le \sum_{m=1}^{\infty} e^{-m\left(\frac{\varepsilon}{4} - Rez\right)}. \tag{18}$$

(18) implies that $H(z)$ continues analytically from real axis to the left half-plane $Rez < \frac{\varepsilon}{4}$. In addition to this, $H(z)$ is a $2\pi i$ periodic function. Hence, the limit points of its zeros in the region P can not be in the interval

$$\left\{z \in \mathbb{C}: z = \xi + i\tau, \xi = 0, \tau \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}.$$

As a result of these facts, the finiteness of eigenvalues and spectral singularities of L is achieved by Theorem 3.1. ■

Clearly, (16) guarantees the analytic continuity of $H(z)$ from the real axis to the left half-plane. Pay regard to the condition (15) for $\frac{1}{2} \leq \beta < 1$,

$$\sum_{n=1}^{\infty} e^{\varepsilon n^{\beta}} \left(\left| 1 - a_n^{(v)} \right| + \left| b_n^{(v)} \right| \right) < \infty, \varepsilon > 0, v = 1,2,\ldots,N. \tag{19}$$

Analicity in the left half-plane and infinite differentiability on the imaginary axis of $H(z)$ are clear. However, under the condition (19), $H(z)$ does not continue analitically from the real axis to the lower half-plane. This leads to requirement of a new technique for the investigation of the finiteness of the eigenvalues and spectral singularities of L. To handle this problem, we will make use of the following lemma.

Lemma 3.3. ([3]) Assume the 2π periodic function ξ is anaytic in the open half-plane, all of its derivatives are continuous in the closed upper half-plane and

$$\sup_{z \in P} \left| \xi^{(k)}(z) \right| \leq \eta_k, k \in \mathbb{N} \cup \{0\}. \tag{20}$$

If the set G with linear Lebesgue measure zero is the set of all zeros of the function ξ with infinite multiplicity in P, and

$$\int_0^{\omega} \ln t(s) d\mu(G_s) > -\infty, \tag{21}$$

where $\mu(G_s)$ is the Lebesgue measure of the s-neighborhood of G, $t(s) = \inf_k \frac{\eta_k s^k}{k!}$, $k \in \mathbb{N} \cup \{0\}$, and $\omega \in (0,2\pi)$ is an arbitrary constant, then $\xi \equiv 0$.

Theorem 3.4. If (19) holds, then $R_5 = \emptyset$.

Proof. The following inequality for the k. th derivative of $H(z)$ can be obtained from (19), (7) and (8) and after some algebra

$$\left| H^{(k)}(z) \right| \leq \eta_k, k \in \mathbb{N} \cup \{0\},$$

where

$$\eta_k = 2^k C \sum_{m=1}^{\infty} m^k \exp\left(-\varepsilon m^{\beta}\right), \tag{22}$$

and $C > 0$ is a constant. As a next step, one obtains the inequality for η_k using the classical inequalities in the literature

$$\eta_k \leq 2^k C \int_0^{\infty} x^k e^{-\varepsilon x^{\beta}} dx \leq D d^k k! \, k^{k\frac{1-\beta}{\beta}}$$

where D and d are constants depending C, ε and β.

Now, we adopt the previous lemma to our problem. Taking into account $t(s) = \inf_k \frac{\eta_k s^k}{k!}$, $k \in \mathbb{N} \cup \{0\}$, $\mu(R_{5,s})$ is the Lebesgue measure of the s-neighborhood of R_5 and η_k is introduced by (22), the following inequality is clear

$$\int_0^{\omega} \ln t(s) d\mu(R_{5,s}) > -\infty. \tag{23}$$

We get

$$t(s) \leq D \exp\left\{ -\frac{1-\beta}{\beta} e^{-1} d^{-\frac{\beta}{1-\beta}} s^{-\frac{\beta}{1-\beta}} \right\}, \tag{24}$$

by (22). (23) and (24) yield,

$$\int_0^\omega s^{-\frac{\beta}{1-\beta}} d\mu(R_{5,s}) < \infty. \tag{25}$$

Because of $\frac{\beta}{1-\beta} \geq 1,$ the inequality (24) is true for arbitrary s if and only if $\mu(R_{5,s}) = 0$ or $R_5 = \emptyset$. ■

Theorem 3.5. If (19) holds to be true, then the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.

Proof. We are supposed to prove that $H(z)$ has a finite number of zeros with finite multiplicities in the region P. From Theorem 3.4 and (13), we have $R_3 = R_4 = \emptyset$. Therefore, the accummulation points of the bounded sets R_1 and R_2 do not exist. Due to the these reasons, $H(z)$ must have only finite number of zeros in the region P. Because of $R_5 = \emptyset$, these zeros must be of finite multiplicity. ■

4. Conclusion

To sum up, there are many beneficial aspects of investigating discrete analogues of Sturm-Liouville type problems. While it allows to use of computers in calculations, it is also adaptable to the problems arising from some events in nature. For this reasons, this paper contributes to the literature in many different ways. First, we consider the non-selfadjoint operator which has different spectral structure compared to the self-adjoint operators. Secondly, we investigate the hyperbolic type spectral parameter and as a result of this case analicity region of the Jost solution shifts. Finally, quantitative spectral properties of the problem have been obtained for the Naimark's and Pavlov's conditions.

Declaration of Competing Interest

The author declares that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Nimet Coşkun: Conceptualization, Methodology, Writing- Original draft preparation, Reviewing and Editing.

References

[1] K. Chadan and P. C. Sabatier, "Inverse problems in Quantum Scattering Theory," Springer-Verlag, New York Inc., 1977.

[2] C. R. Oliveira, "Intermediate Spectral Theory and Quantum Dynamics," Progress in Mathematical Physics, vol. 54, Birkhauser, 2009.

[3] V. A. Marchenko, "Sturm-Liouville Operators and Applications," Operator Theory: Advances and Applications, Birkhauser Verlag, Basel, 1986.

[4] R. P. Agarwal, Difference Equations and Inequalities Theory, Methods and Applications, Second Edition, Marcel Dekker Inc, New York - Basel, 2000.

[5] M. A. Naimark, "Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint operator of second order on a semi-axis," AMS Transl. 2, pp. 103-193, 1960.

[6] M. A. Naimark, Linear Differential Operators I, II. Ungar, New York; 1968.

[7] Z. S. Agranovich and V. A. Marchenko, The inverse problem of scattering theory. Gordon and Breach, New York,

1967.

[8] C. Coskun and M. Olgun, “Principal functions of non-selfadjoint matrix Sturm–Liouville equations,” Journal of Computational and Applied Mathematics, vol. 235, no. 16, pp. 4834-4838, 2011.

[9] N. Yokus and N. Coskun, “A note on the matrix Sturm-Liouville operators with principal functions,” Mathematical Methods in the Applied Sciences, vol. 42, no. 16, pp. 5362-5370, 2019.

[10] E. Kir, “Spectral properties of a finite system of Sturm-Liouville difference operators,” Journal of Difference Equations and Applications, vol. 17, no. 3, pp. 255-266, 2011.

[11] G. Mutlu and E. Kir Arpat, “Spectral properties of non-selfadjoint Sturm-Liouville operator equation on the real axis,” Hacettepe Journal of Mathematics and Statistics, pp. 1-9, 2020.

[12] M. Adivar and E. Bairamov, “Difference equations of second order with spectral singularities,” Math. Anal. Appl., vol. 277, pp. 714-721, 2003.

[13] T. Koprubasi and N. Yokus, “Quadratic eigenparameter dependent discrete Sturm–Liouville equations with spectral singularities,” Applied Mathematics and Computation, vol. 244, pp. 57-62, 2014.

[15] M. Adıvar and E. Bairamov, “Spectral properties of non-selfadjoint difference operators,” Journal of Mathematical Analysis and Applications, vol. 261, no. 2, pp. 461–478, 2001.

[16] E. Bairamov, A. M. Krall and O. Cakar, “Non-selfadjoint difference operators and Jacobi matrices with spectral singularities,” Math. Nachr., vol. 229, pp. 5-14, 2001.

[17] G. Mutlu, “Associated functions of non-selfadjoint Sturm-Liouville operator with operator coefficient,” TWMS Journal of Applied and Engineering Mathematics, vol. 11, no. 1, pp. 113-121, 2021.

[18] N. Yokus and N. Coskun, “Jost Solution and the spectrum of the discrete Sturm-Liouville equations with hyperbolic eigenparameter,” Neural, Parallel, and Scientific Computations, vol. 24, pp. 419-430, 2016.

[19] N. Yokus and N. Coskun, “Principal functions of discrete Sturm-Liouville equations with hyperbolic eigenparameter,” Creat. Math. Inform., vol. 26, no. 3, pp. 353-359, 2017.

[20] T. Koprubasi, “A study of impulsive discrete Dirac system with hyperbolic eigenparameter,” Turk. J. Math, vol. 45, no. 1, pp. 540-548, 2021.

[21] E. Bairamov, Y. Aygar and M. Olgun, “Jost solution and the spectrum of the discrete Dirac systems,” Boundary Value Problems, pp. 1-11, 2010.

[22] E. Bairamov, S. Cebesoy and I. Erdal. “Properties of eigenvalues and spectral singularities for impulsive quadratic pencil of difference operators,” Journal of Applied Analysis & Computation, vol. 9, pp. 1454-1469, 2019.

[23] V. E. Lyantse, “The spectrum and resolvent of a non-selfadjoint difference operator,” Ukrainian Mathematical Journal, vol. 20, no. 4, pp. 422-434, 1968.

[24] E. P. Dolzhenko, “Boundary-value uniqueness theorems for analytic functions,” Mathematical Notes of the Academy of Sciences of the USSR, vol. 25, no. 6, pp. 437-442, 1979.