RATE of EQUI-CONVERGENCE of SOME SERIES ASSOCIATED WITH FOURIER SERIES and CERTAIN INTEGRALS

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Received: 08.04.2010 Accepted: 25.06.2010

ABSTRACT

Salem and Zygmund have studied the convergence problem of factored Fourier series and conjugate series the factor being $n^{\gamma},\ 0<\gamma<1$. In the present work we study the rate of convergence of the some series for functions belonging to $Lip(w,p),\ p\geq 1$ class.

AMS subject Classification (2000) : 42A24, 41A25

Keywords: Equi-convergence, monotonic type, $Lip(w,p)$.

FOURİER SERİLERİ VE BELİRLİ İNTEGRALLER İLE ALAKALI BAZI SERİLERİN EŞ -YAKINSAMA ORANLARI

ÖZET

Salem ve Zygmund, carpanlı Fourier serilerinin ve çarpanı $n^{\gamma},\ 0<\gamma<1$ olan eşlenik serilerin yakınsaklık problemini çalışmışlardır. Şimdiki çalışmada, biz $Lip(w,p),\ p\geq 1$ sınıfına ait fonksiyonlar için bazı serilerin yakınsaklık oranlarını inceleyeceğiz.

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1. DEFINITIONS

Let f be a 2π- periodic and Lebesgue integrable function over (-π, π). The Fourier series of f at x *is given*

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x) \tag{1.1}$$

The series conjugate to (1.1) is given by

$$\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)=\sum_{n=0}^{\infty}B_n(x). \tag{1.2}$$

We write

$$\phi(x,t)=f(x+t)+f(x-t)-2f(x)$$

$$\psi(x,t)=f(x+t)-f(x-t)$$

$$\tilde{f}(x)=-\frac{1}{\pi}\int_0^{\pi}\psi(x,t)\frac{1}{2}\cot\frac{1}{2}t\,dt$$

whenever the integral exists in Cauchy sense at the origin. Let w be a modulus of continuity. If

$$\|f(.+t)-f(.)\|_p=\begin{cases}O(w(|t|))\\ 0(w(|t|))\end{cases}$$

then respectively we say that [1] $f\in Lip(w,p),\ p\geq 1,\ \ f\in lip(w,p),\ p\geq 1.$

If $w(t)=t^{\alpha}, 0<\alpha\leq 1$ then $Lip(w,p),\ p\geq 1$ reduces to familiar $Lip(\alpha,p),\ p\geq 1$ class.

A function f is called of monotonic type $([2],p.33)$ if for some constant C the function $f(x)+Cx$ is either monotonic increasing or monotonic decreasing.

For $0<\gamma<1$ and $0<\in<\infty$, we write

$$I_{\gamma}(x;\in)=-\frac{1}{\pi}\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}\int_{\in}^{\infty}\frac{\psi(x,t)}{t^{1+\gamma}}dt$$

$$J_{\gamma}(x;\in)=-\frac{1}{\pi}\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}\int_{\in}^{\infty}\frac{\phi(x,t)}{t^{1+\gamma}}dt.$$

We define

$$I_{\gamma}(x)=\lim_{\varepsilon\to 0+}I_{\gamma}(x;\varepsilon)\quad\text{and}\quad J_{\gamma}(x)=\lim_{\varepsilon\to 0+}J_{\gamma}(x;\varepsilon)$$

whenever the limits exist. Let $C_{2\pi}$ denote the class of 2π-periodic continuous functions.

2. INTRODUCTION

Results on equi-convergence of conjugate series $\sum_{n=1}^{\infty} B_n(x)$ and the conjugate integral $\tilde{f}(x)$ can be found in Salem and Zygmund $([2], p.33)$ when they have studied the equi-convergence of the series $\sum n^{\gamma} B_n(x)$ and $\sum n^{\gamma} A_n(x), 0<\gamma<1$ respectively with the integrals $I_{\gamma}(x)$ and $J_{\gamma}(x)$. Their results read as follows.

Theorem A ([2] p.33). Suppose that $f \in C_{2\pi}$. Let *f* be a function of bounded variation and of monotonic type. If $f \in lip\gamma,\ 0<\gamma<1$, then the difference

$$\alpha_n(x) = I_{\gamma}\left(x;\frac{1}{n}\right) - \sum_{k=1}^{n} k^{\gamma} B_k(x) \tag{2.1}$$

tends to zero uniformly in x as $n \to \infty$. If $f \in lip\gamma,\ 0<\gamma<1$ the difference is uniformly bounded.

Theorem B ([2], p.34). Suppose that $f \in C_{2\pi}$. Let f be function of bounded variation and of monotonic type. If $f \in lip\gamma,\ 0<\gamma<1$ then the difference

$$\beta_n(x) = J_{\gamma}\left(x;\frac{1}{n}\right) - \sum_{k=1}^{n} k^{\gamma} A_k(x) \tag{2.2}$$

tends to zero uniformly in x as $n \to \infty$. If $f \in Lip\,\gamma,\ 0<\gamma<1,$ the difference is uniformly bounded.

In this present work we study the rate of equi-convergence of the sequence $\alpha_n(x)$ and $\beta_n(x)$.

3. MAIN RESULTS

We prove the following theorems:

Theorem 1. Suppose that $f \in C_{2\pi} \cap Lip(w,p),\ p \geq 1$. If further f is of monotonic type, then for $0<\gamma<1$.

$$\left\|\alpha_n(.)\right\|_p = O(1)\frac{1}{n^{1-\gamma}}\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du \tag{3.1}$$

where $\alpha_n(x)$ is defined in (2.1).

Theorem 2. Suppose that $f \in C_{2\pi} \cap Lip(w, p), p \geq 1$. If further f is of monotonic type, then for $0 < \gamma < 1$.

$$\left\|\beta_n(.)\right\|_p = O(1)\frac{1}{n^{1-\gamma}}\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du \tag{3.2}$$

where $\beta_n(x)$ is defined in (2.2).

Remark. Das, Nath and Ray [1] have obtained the following result on the degree of approximation of functions in $Lip(w, p)$ class by their Fourier series.

Theorem C. If $f \in Lip(w, p), p \geq 1$ and is of monotonic type, then

$$\left\|s_n(f,x) - f\right\|_p = O(1)\frac{1}{n}\int_{1/n}^{\pi}\frac{w(t)}{t^2}dt$$

where $s_n(f,x)$ is the nth partial sum of (1.1) and w is a modulus of continuity.

4. LEMMAS.

For the proof of Theorem 1 we need the following notations and lemmas.

$$g(x,t,u) = \psi(x,t+u) + \psi(x,t-u) - 2\psi(x,t) \tag{4.1}$$

$$m = \left[\frac{\log n}{\log 2}\right], \eta = \frac{1}{2^m} \tag{4.2}$$

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n}\cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin(t/2)} \tag{4.3}$$

We know $([3], \text{vol. I, p.50})$ that

$$\psi(x,t) \sim -2\sum_{k=1}^{\infty}B_k(x)\sin kt \tag{4.4}$$

$$S_n(x,t) = -\sum_{k=1}^{n} 2\sin kt B_k(x) \cdot \qquad (4.5)$$

Lemma1. Let $\psi(x,t)$ and $g(x,t,u)$ be defined as above. If $f \in C_{2\pi} \cap Lip(w,p),\ p \geq 1,$
then

(i) $\left\| g(x,t,u) \right\|_p = O(w(u))$

(ii) $\left\| \psi(x,t+u) - \psi(x,t-u) \right\|_p = O(w(u))$

Proof of (i). By Minkowski's inequality

$$\begin{aligned}\left\| g(x,t,u) \right\|_p &= \left\| \psi(x,t+u) + \psi(x,t-u) - 2\psi(x,t) \right\|_p \\ &\leq \left\| f(x+t+u) - f(x+t) \right\|_p + \left\| f(x-t) - f(x-t-u) \right\|_p \\ &+ \left\| f(x+t-u) - f(x+t) \right\|_p + \left\| f(x-t) - f(x-t+u) \right\|_p \\ &= O(w(u))\end{aligned}$$

which proves Lemma 1 (i).

Proof of (ii). By Minkowski's inequality

$$\begin{aligned}\left\| \psi(x,t+u) - \psi(x,t-u) \right\|_p &\leq \left\| f(x+t+u) - f(x+t) \right\|_p + \left\| f(x-t) - f(x-t+u) \right\|_p \\ &+ \left\| f(x+t) - f(x+t-u) \right\|_p + \left\| f(x-t+u) - f(x-t) \right\|_p \\ &= O(w(u))\end{aligned}$$

which proves Lemma 1 (ii).

Lemma 2. If the hypotheses of Theorem 1 holds, then

$$\left\| S_n(.;t) - \psi(.;t) \right\|_p = O(1) \frac{1}{n} \int_{\pi/n}^{\pi} \frac{w(u)}{u^2} du \,.$$

Proof. Now after simplification

$$\begin{aligned}S_n(x,t) &= -\sum_{k=1}^{n} 2\sin kt \left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x,u) \sin ku du \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x,u) \sum_{k=1}^{n} \left[\cos k(u-t) - \cos k(u+t) \right] du\end{aligned}$$

$$=\frac{1}{\pi}\int_0^{\pi}\left[\psi(x,t+u)+\psi(x,t-u)\right]D_n(u)\,du$$
$$=\frac{1}{\pi}\int_0^{\pi}g(x,t,u)\,D_n(u)\,du+\psi(x,t)$$

from which it follows that

$$S_n(x,t)-\psi(x,t)=\frac{1}{\pi}\left(\int_0^{n}+\int_{\eta}^{\pi}\right)g(x,t,u)\,D_n(u)\,du$$
$$=\frac{1}{\pi}\left[I(x,t)+J(x,t)\right] \tag{4.6}$$

For the first integral using Lemma 1 (i), we get

$$\left\|I(.,t)\right\|_p=\int_0^{\eta}O\left(w(u)\right)\left|D_n(u)\right|du$$
$$=O(1)\,w(\eta)\,n\eta=O(1)\,w\left(\frac{1}{n}\right). \tag{4.7}$$

Since the function f is of monotonic type there exists a constant C so that $F(x)=f(x)+Cx$ is either never decreasing or never increasing in $(-\infty,\infty)$. Without loss of generality, we may assume that F is increasing as in the case when F is decreasing the problem can be dealt with in a similar manner.

Replacing $f(x)$ by $F(x)-Cx,$ we get

$$g(x,t,u)=\left[F(x+t+u)-F(x+t)\right]-\left[F(x-t+u)-F(x-t)\right]$$
$$+\left[F(x-t)-F(x-t-u)\right]-\left[F(x+t)-F(x+t-u)\right]. \tag{4.8}$$

Using (4.8), we get

$$J(x,t)=\int_{\eta}^{\pi}\left[F(x+t+u)-F(x+t)\right]D_n(u)\,du+\int_{\eta}^{\pi}\left[F(x-t)-F(x-t-u)\right]D_n(u)\,du$$
$$-\int_{\eta}^{\pi}\left[F(x-t+u)-F(x-t)\right]D_n(u)\,du$$
$$-\int_{\eta}^{\pi}\left[F(x+t)-F(x+t-u)\right]D_n(u)\,du$$
$$=J_1(x,t)+J_2(x,t)-J_3(x,t)-J_4(x,t),\ \text{(say)} \tag{4.9}$$

We write

$$J_1(x,t)=\sum_{j=1}^{m}\int_{\pi/2^j}^{\pi/2^{j-1}}\left[F(x+t+u)-F(x+t)\right]D_n(u)\,du$$
$$=\sum_{j=1}^{m}Q_j(x,t) \tag{4.10}$$

where

$$Q_j(x,t)=\int_{\pi/2^j}^{\pi/2^{j-1}}\left[F(x+t+u)-F(x+t)\right]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin u/2}\,du\,. \tag{4.11}$$

As $F(x+t+u)-F(x+t)$ is increasing in u and $\frac{1}{2\sin(u/2)}$ is decreasing in u, by repeated application of Mean–Value Theorem, we get

$$Q_j(x,t)=\frac{1}{2\sin\frac{1}{2^{j+1}}}\int_{1/2^j}^{\xi}\left[F(x+t+u)-F(x+t)\right]\sin\left(n+\frac{1}{2}\right)udu\left(\frac{1}{2^j}<\zeta<\frac{1}{2^{j-1}}\right)$$

$$=\frac{1}{2\sin\frac{1}{2^{j+1}}}\left[F(x+t+\zeta)-F(x+t)\right]\int_{\zeta_1}^{\zeta}\sin\left(n+\frac{1}{2}\right)udu\left(\frac{1}{2^j}<\zeta_1<\zeta\right)$$

$$=\frac{F(x+t+\zeta)-F(x+t)}{2\sin\frac{1}{2^{j+1}}}\left[\frac{\cos\left(n+\frac{1}{2}\right)\zeta_1-\cos\left(n+\frac{1}{2}\right)\zeta}{\left(n+\frac{1}{2}\right)}\right],$$

from which it follows that for $j = 1,2,...,m$

$$\left\|Q_j(.,t)\right\|_p\le\frac{2}{\left(n+\frac{1}{2}\right)}\frac{\left\|F(.+t+\zeta)-F(.+t)\right\|_p}{2.\frac{2}{\pi}\frac{1}{2^{j+1}}}\left(\sin u\ge\frac{2u}{\pi}\right)$$

$$=O(1)\frac{2^j}{n}w(\zeta)=O(1)2^jw\left(\frac{1}{2^{j-1}}\right).$$

Therefore

$$\left\|J_1(.;t)\right\|_p\le\sum_{j=1}^{m}\left\|Q_j(.,t)\right\|_p$$

$$=O(1)\frac{1}{n}\sum_{j=1}^{m}2^jw\left(\frac{1}{2^{j-1}}\right)$$

$$=O(1)\frac{1}{n}\int_{\pi/n}^{\pi}\frac{w(u)du}{u^2}. \tag{4.12}$$

Similarly we prove that

$$\left\|J_k(.,t)\right\|_p=O(1)\frac{1}{n}\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du \quad \text{for } k=2,3,4 \tag{4.13}$$

Using (4.12) and (4.13), we obtain

$$\left\|J(.,t)\right\|_p\le\sum_{k=1}^{4}\left\|J_k(.,t)\right\|_p=O(1)\frac{1}{n}\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du \tag{4.14}$$

Using (4.3) and (4.14) in (4.6), we get

$$\left\|S_n(.,t)-\psi(.,t)\right\|_p=O(1)w\left(\frac{1}{n}\right)+O(1)\int_{\pi/n}^{\pi}\frac{w(u)du}{u^2}$$

$$= O(1)\frac{1}{n}\int_{\pi/n}^{\pi}\frac{w(u)\,du}{u^2}$$

since by the monotonicity of w,

$$\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du \geq w(\pi/n)\int_{\pi/n}^{\pi}\frac{du}{u^2} = w(\pi/n)\left(\frac{n-1}{\pi}\right)\cdot$$

Lemma 3. Let the hypotheses of Theorem 1 hold. Then

$$\left\|S_n^{'}(.,t)\right\|_p = \left\|\frac{\partial}{\partial t}\left(S_n(.,t)\right)\right\|_p = O(1)\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du\cdot$$

Proof. We have

$$\frac{\partial}{\partial t}\left(S_n(x,t)\right) = -\sum_{k=1}^{n} 2k\cos kt\, B_k(x)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(x,u)\sum_{k=1}^{n}k\left[\sin k(u+t) - \sin k(u-t)\right]du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(x,u)\frac{\partial}{\partial u}D_n(u+t)\,du - \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(x,u)\frac{\partial}{\partial u}D_n(u-t)\,du$$

$$= -\frac{1}{\pi}\left[\int_0^{\eta} + \int_{\eta}^{\pi}\right]\left[\psi(x,t+u) + \psi(x,t-u)\right]D_n^{'}(u)\,du$$

$$= -\frac{1}{\pi}\left[L(x,t) + K(x,t)\right]. \tag{4.15}$$

Using Lemma 1 (ii), we get

$$\left\|L(.,t)\right\|_p \leq \int_0^{\eta}\left\|\psi(x,t+u) - \psi(x,t-u)\right\|_p \left|D'_n(u)\right|du$$

$$= O(1)\,w(u)\int_0^{\eta}n^2du = O(1)\,w(\eta)\,n^2\eta$$

$$= O(1)\,nw\left(\frac{1}{n}\right) \tag{4.16}$$

Replacing $f(x)$ by $Fx - Cx,$ we get, after rearrangement,

$$K(x,t) = \int_{\eta}^{\pi}\left[F(x+t+u) - F(x-t-u) + F(x-t+u) - F(x+t-u) - 4Cu\right]D'_n(u)\,du$$

$$= \int_{\eta}^{\pi}\left[F(x+t+u) - F(x+t)\right]D'_n(u)\,du$$

$$+\int_{\eta}^{\pi}\left[F(x-t+u) - F(x-t)\right]D'_n(u)\,du + \int_{\eta}^{\pi}\left[F(x-t) - F(x-t-u)\right]D'_n(u)\,du$$

$$+\int_{\eta}^{\pi}\left[F(x+t) - F(x+t-u)\right]D'_n(u)\,du - 4C\int_{\eta}^{\pi}uD'_n(u)\,du$$

$$= \sum_{i=1}^{4}K_i(x,t) - 4C\int_{\eta}^{\pi}uD'_n(u)\,du\cdot \tag{4.17}$$

We write

$$K_1(x,t)=\sum_{j=1}^{m}P_j(x,t)$$

where

$$P_j(x,t)=\int_{\pi/2^j}^{\pi/2^{j-1}}\left[F(x+t+u)-F(x+t)\right]D'_n(u)\,du\,. \tag{4.18}$$

As $F(x+t+u)-F(x+t)$ is increasing in u, by using Mean–Value theorem, we have, for $\pi/2^j<\zeta<\pi/2^{j-1}$

$$P_j(x,t)=\int_{\pi/2^j}^{\zeta}\left[F(x+t+u)-F(x+t)\right]D'_n(u)\,du$$

$$=\left[F\left(x+t+\frac{1}{2^{j-1}}\right)-F(x+t)\right]\left(D_n(\zeta)-D_n\left(\frac{\pi}{2^j}\right)\right)$$

which ensures that

$$\left\|P_j(.,t)\right\|_p=O(1)\,w\left(\frac{1}{2^{j-1}}\right)\left|D_n(\zeta)-D_n\left(\frac{\pi}{2^j}\right)\right|$$

$$=O(1)\,2^j\,w\left(\frac{1}{2^{j-1}}\right).$$

Therefore

$$\left\|K_1(.,t)\right\|_p\le\sum_{j=1}^{m}\left\|P_j(.,t)\right\|_p=O(1)\sum_{j=1}^{m}2^j\,w\left(\frac{1}{2^{j-1}}\right)$$

$$=O(1)\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du \tag{4.19}$$

Similarly we can prove that , for *i*= 2,3,4

$$\left\|K_i(.,t)\right\|_p=O(1)\int_{\pi/n}^{\pi}\frac{w(u)\,du}{u^2}\,. \tag{4.20}$$

Integrating by parts, we get

$$4C\int_{\eta}^{\pi}uD'_n(u)\,du=4C\left[\pi\left(D_n(\pi)-\eta D_n(\eta)\right)\right]-\int_{\eta}^{\pi}D_n(u)\,du$$

$$=O(1) \tag{4.21}$$

Using (4.20) and (4.21) in (4.17), we get

$$\left\|K(.,t)\right\|_p=O(1)\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du \tag{4.22}$$

Using (4.16) and (4.22) in (4.15), we get

$$\left\|\frac{\partial}{\partial t}S_n(.,t)\right\|_p=O(1)\int_{\pi/n}^{\pi}\frac{w(u)}{u^2}du$$

which proves Lemma 3.

Lemma 4. $([2],p.31)$.

$$\int_0^\infty \frac{\sin kt}{t^{1+\alpha}} dt = \frac{\pi k^\alpha}{2\cos(\pi\alpha/2)\Gamma(\alpha+1)}$$

5. PROOF OF THEOREM 1.

We have from (4.5) and Lemma 4,

$$\int_0^\infty \frac{S_n(x,t)}{t^{1+\gamma}} dt = -\sum_{k=1}^n 2B_k(x)\int_0^\infty \frac{\sin kt}{t^{1+\gamma}} dt$$
$$= -\frac{\pi}{\cos\frac{\pi\gamma}{2}\Gamma(\gamma+1)}\sum_{k=1}^n k^\gamma B_k(x) \tag{5.1}$$

Now from (2.1) and (5.1)

$$I_\gamma\left(x,\frac{1}{n}\right)-\sum_{k=1}^n k^\gamma B_k(x) = \frac{-\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}}{\pi}\left[\int_{1/n}^\infty \frac{\psi(x,t)}{t^{1+\gamma}}dt - \int_0^\infty \frac{S_n(x,t)}{t^{1+\gamma}}\right]dt$$
$$= \frac{\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}}{\pi}\left[\int_{1/n}^\pi \frac{S_n(x,t)-\psi(x,t)}{t^{1+\gamma}}dt + \int_0^{1/n}\frac{S_n(x,t)}{t^{1+\gamma}}dt\right] \tag{5.2}$$

Using Lemma 2, we get

$$\left\|\int_{1/n}^\infty \frac{S_n(.,t)-\psi(.,t)}{t^{1+\gamma}}dt\right\|_p \le \int_{1/n}^\infty \frac{\|S_n(.,t)-\psi(.,t)\|}{t^{1+\gamma}}dt$$
$$= O(1)\frac{1}{n}\int_{\pi/n}^\pi \frac{w(u)\,du}{u^2}\int_{1/n}^\infty \frac{dt}{t^{1+\gamma}}$$
$$= O(1)\frac{1}{n^{1-\gamma}}\int_{\pi/n}^\pi \frac{w(u)\,du}{u^2}. \tag{5.3}$$

As $S_n(x,0)=0$, we get $S_n(x,t)=S_n(x,t)-S_n(x,0)=tS'_n(x,\theta), 0<\theta<t$.
Therefore, by Lemma 3,

$$\left\|\int_0^{1/n} \frac{S_n(.,t)}{t^{1+\gamma}}dt\right\|_p \le \int_0^{1/n} \frac{\|S_n(.,t)\|_p}{t^{1+\gamma}}dt$$
$$= O(1)\left(\int_{\pi/n}^\pi \frac{w(u)}{u^2}du\right)\int_0^{1/n}\frac{dt}{t^\gamma}$$
$$= O(1)\frac{1}{n^{1-\gamma}}\int_{\pi/n}^\pi \frac{w(u)}{u^2}du. \tag{5.4}$$

Using (5.3) and (5.4) in (5.2), we get

$$I_\gamma\left(x,\frac{1}{n}\right)-\sum_{k=1}^n k^\gamma B_k(x) = O(1)\frac{1}{n^{1-\gamma}}\int_{\pi/n}^\pi \frac{w(u)}{u^2}du$$

which completes the proof of Theorem 1.
By taking $w(t)=t^{\alpha}, 0<\alpha\leq 1,$ in Theorem 1, we obtain the following corollaries:

Corollary 1. If $f\in C_{2\pi}\cap Lip(\alpha,p), p\geq 1, 0<\alpha\leq 1$ and f is of monotonic type then

$$\left\|\alpha_n(.)\right\|_p = O(1) \begin{cases} \frac{1}{n^{\alpha-\gamma}}, 0<\gamma\leq\alpha<1 \\ \frac{\log n}{n^{1-\gamma}}, 0<\gamma<\alpha=1 \end{cases}$$

Putting $p=\infty$, in Corollary 1, we get

Corollary 2. If $f\in C_{2\pi}\cap Lip\alpha, 0<\alpha\leq 1$ and f is of monotonic type then

$$\left\|\alpha_n(.)\right\|_c = O(1) \begin{cases} \frac{1}{n^{\alpha-\gamma,}} 0<\gamma\leq\alpha<1 \\ \frac{\log n}{n^{1-\gamma}}, 0<\gamma<\alpha=1 \end{cases}$$

Note: In Corollary 2 in the special case when $\gamma=\alpha$ the result reduces to second part of Theorem A due to Salem and Zygmund.

6. ADDITIONAL NOTATIONS AND LEMMAS FOR THEOREM 2

We Write

$$\phi^*(x,t)=f(x+t)+f(x-t)$$
$$\phi(x,t)=\phi^*(x,t)-2f(x)$$
$$h(x,t,u)=\phi(x,t+u)+\phi(x,t-u)-2\phi(x,t)$$
$$m=\left[\frac{\log n}{\log 2}\right], \eta=\frac{1}{2^m}$$

We know ([4] Vol. I, p. 50) that

$$\phi^*(x,t)\sim 2\sum_{n=1}^{\infty}A_n(x)\cos nt \tag{6.1}$$

$$\tilde{S}_n(x,t)=-2\sum_{k=1}^{n}(1-\cos kt)A_k(x). \tag{6.2}$$

We need the following additional lemmas for the proof of Theorem 2.

Lemma 5. Let $\phi(x,t)$ and $h(x,t,u)$ be defined as above, If

$$f \in C_{2\pi} \cap Lip(w, p),\ p \geq 1$$

then

(i) $\left\| h(x,t,u) \right\|_p = O(w(u))$

(ii) $\left\| \phi(x,t+u) - \phi(x,t-u) \right\|_p = O(w(u))$.

Proof:

(i). By Minikowski's inequality

$$\left\| h(x,t,u) \right\|_p = \left\| \phi(x,t+u) + \phi(x,t-u) - 2\phi(x,t) \right\|_p = O(w(u))$$

proceeding as in Lemma 1 (i)

(ii). By Minkowski's inequality

$$\left\| \phi(x,t+u) + \phi(x,t-u) \right\|_p = \left\| f(x+t+u) + f(x-t-u) - 2f(x) - f(x+t-u) - f(x-t+u) + 2f(x) \right\|_p$$
$$= O(w(u)) \cdot$$

proceeding as in Lemma 1 (ii)

Lemma 6. Under the hypotheses of Theorem 2

$$\left\| \tilde{S}_n(.,t) - \phi(.,t) \right\|_p = O(1)\,\omega(\pi/n)$$

where $\tilde{S}_n(x,t)$ is defined in (6.2).

Proof. We have

$$\tilde{S}_n(x,t) = -2\sum_{k=1}^{n}(1-\cos kt)\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^*(x,u)\cos ku du$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi^*(x,u)\sum_{k=1}^{n}\left\{\cos k(u-t) + \cos k(u+t) - 2\cos ku\right\}du$$
$$= \frac{1}{\pi}\int_0^{\pi}\left[\phi^*(x,t+u) + \phi^*(x,t-u) - 2\phi^*(x,u)\right]D_n(u)\,du$$
$$= \frac{1}{\pi}\int_0^{\pi}h(x,t,u)D_n(u)\,du - \frac{2}{\pi}\int_0^{\pi}\phi(x,u)D_n(u)\,du + \frac{2}{\pi}\phi(x,t)\int_0^{\pi}D_n(u)\,du \quad (6.3)$$

As

$$\frac{2}{\pi}\int_0^{\pi}D_n(u)\,du = 1$$

from (6,3), it follows that

$$\tilde{S}_n(x,t) - \phi(x,t) = \frac{1}{\pi}\int_0^{\pi}h(x,t,u)D_n(u)\,du - \frac{2}{\pi}\int_0^{\pi}\phi(x,u)D_n(u)\,du\,.$$

Proceeding in the lines of proof adopted in the proof of Lemma 1, $h(x,t,u)$ takes the place of $g(x,t,u)$, it can be shown that

$$\left\| \frac{1}{\pi} \int_0^{\pi} h(x,t,u) D_n(u)\, du \right\|_p = O(1)\,\omega(\pi/n)\cdot$$

As $\left\| \phi(x,u) \right\|_p$ and $\left\| g(x,t,u) \right\|_p$ have the same estimate, adopting the argument used in the proof of Lemma 1 it can be shown that

$$\left\| \frac{2}{\pi} \int_0^{\pi} \phi(x,u) D_n(u)\, du \right\|_p = O\omega(\pi/n)\cdot$$

This completes the proof of Lemma 6.

Lemma 7. Under the hypotheses of Theorem 2

$$\left\| \frac{\partial}{\partial t}\left(\tilde{S}_n(x,t) \right) \right\|_p = O(1) n w(\pi/n).$$

Proof. We have

$$\begin{aligned}
\frac{\partial}{\partial t} \tilde{S}_n(x,t) &= -\sum_{k=1}^{n} 2k \sin kt A_k(x) \\
&= -\sum_{k=1}^{n} 2k \sin kt \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x,u) \cos ku du \right) \\
&= -\frac{1}{2\pi} \sum_{k=1}^{n} k \int_{-\pi}^{\pi} \phi(x,u) \left[\sin k(u+t) - \sin k(u-t) \right] du \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\phi(x,t-u) - \phi(x,t+u) \right] D'_n(u)\, du \\
&= \frac{1}{\pi} \left(\int_0^{\eta} + \int_{\eta}^{\pi} \right) \left[\phi(x,t-u) - \phi(x,t+u) \right] D'_n(u)\, du. \qquad (6.4)
\end{aligned}$$

Using Lemma 5(ii), we get

$$\begin{aligned}
&\left\| \int_0^{\eta} \left[\phi(x,t-u) - \phi(x,t+u) \right] D'_n(u)\, du \right\|_p \\
&\leq \int_0^{\eta} \left\| \phi(.,t-u) - \phi(.,t+u) \right\|_p \left| D'_n(u) \right| du \\
&= O(n^2) \int_0^{\eta} \omega(u)\, du = O(1)\, n\omega(\pi/n) \qquad (6.5)
\end{aligned}$$

Writing $F(x) - Cx$ in place of $f(x)$, we get

$$\int_{\eta}^{\pi} [\phi(x,t-u) - \phi(x,t+u)] D'_n(u) du$$

$$= -\int_{\eta}^{\pi} \left[F(x+t+u) - F(x+t-u) \right] D'_n(u)\, du + \int_{\eta}^{\pi} \left\{ F(x-t+u) - F(x-t-u) \right\} D'_n(u)\, du.$$

As the expressions in the curly bracket are increasing functions of u, proceeding as in Lemma 3, it can be shown that

$$\left\| \int_{\eta}^{\pi} \left[\phi(x,t-u) - \phi(x,t+u) \right] D'_n(u)\,du \right\|_p = O(1) \int_{\pi/n}^{\pi} \frac{\omega(u)}{u^2}\,du \cdot \qquad (6.6)$$

This completes the proof of Lemma 7.

Lemma 8. ([2], p.32)

$$\int_0^{\infty} \frac{\sin^2 kt/2}{t^{1+\alpha}} dt = \frac{\pi k^{\alpha}}{4\sin(\pi\alpha/2)\Gamma(\alpha+1)}.$$

7. PROOF of THEOREM 2

Now for $0<\gamma<1$

$$\int_0^{\infty} \frac{\tilde{S}_n(x,t)}{t^{1+\gamma}} dt = -4\int_0^{\infty} \frac{\sum_{k=1}^{n} \sin^2 \frac{kt}{2} A_k(x)}{t^{1+\gamma}} dt$$

$$= -4\sum_{k=1}^{n} A_k(x) \int_0^{\infty} \frac{\sin^2 \frac{kt}{2}\, dt}{t^{1+\gamma}}. \qquad (7.1)$$

Using (7.1) and Lemma 8 we get

$$\int_0^{\infty} \frac{\tilde{S}_n(x,t)}{t^{1+\gamma}} dt = \frac{-\pi}{\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}} \sum_{k=1}^{n} k^{\gamma} A_k(x). \qquad (7.2)$$

From (7.2) and (2.2) we get

$$\beta_n(x) \equiv J_{\gamma}\left(x, \frac{1}{n}\right) - \sum_{k=1}^{n} k^{\gamma} A_k(x)$$

$$= \frac{\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}}{\pi} \left[-\int_{1/n}^{\infty} \frac{\phi(x,t)}{t^{1+\gamma}} dt + \int_0^{\infty} \frac{\tilde{S}_n(x,t)\,dt}{t^{1+\gamma}} \right]$$

$$= \frac{\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}}{2} \left[\int_0^{1/n} \frac{\tilde{S}_n(x,t)}{t^{1+\gamma}} dt + \int_{1/n}^{\infty} \frac{\tilde{S}_n(x,t) - \phi(x,t)}{t^{1+\gamma}} dt \right]$$

$$= \frac{\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}}{2} \left[\Delta_1(x) + \Delta_2(x) \right]. \qquad (7.3)$$

By Lemma 6

$$\left\|\Delta_2(x)\right\|_p \le \int_{1/n}^{\pi} \frac{\left\|\tilde{S}_n(x,t)-\phi(x,t)\right\|_p}{t^{1+\gamma}}dt \tag{7.4}$$

As $\tilde{S}_n(x,0)=0$, we have

$$\tilde{S}_n(x,t)=\tilde{S}_n(x,t)-\tilde{S}_n(x,0)=t\left[\frac{\partial}{\partial t}\tilde{S}_n(x,t)\right]_{t=\theta},\ 0<\theta<t.$$

Hence by Lemma 7,

$$\begin{aligned}\left\|\tilde{S}_n(.,t)\right\|_p &= O(t)\left\|\left[\frac{\partial}{\partial t}\tilde{S}_n(.,t)\right]_{t=\theta}\right\|_p\\ &= O(1)\,t\,n\omega(\pi/n).\end{aligned} \tag{7.5}$$

Using (7.5) , we get

$$\begin{aligned}\left\|\Delta_1(x)\right\|_p &= O(1)\int_0^{1/n}\frac{\left\|\tilde{S}_n(x,t)\right\|_p}{t^{1+\gamma}}dt\\ &= O(1)n\omega(\pi/n)\int_0^{1/n}t^{-\gamma}dt\\ &= O(1)n^{\gamma}\omega(\pi/n).\end{aligned} \tag{7.6}$$

Using (7.5) , (7.6) in (7.3) , we get

$$\left\|\beta_n(x)\right\|_p = O(1)n^{\gamma}\omega(\pi/n).$$

and this completes the proof of Theorem 2.

By taking $\omega(t)=t^{\alpha}, 0<\alpha\le 1$, in Theorem 2, we obtain the following corollaries:

Corollary 1. If $f\in C_{2\pi}\cap Lip(\alpha,p),\ p\ge 1, 0<\alpha\le 1$ and f is of monotonic type then

$$\left\|\beta_n(.)\right\|_p = O(1)\begin{cases}\dfrac{1}{n^{\alpha-\gamma}}, 0<\gamma\le\alpha<1\\[2mm] \dfrac{\log n}{n^{1-\gamma}}, 0<\gamma<\alpha=1\end{cases}$$

Putting $p=\infty$, in Corollary 1, we get

Corollary 2. If $f\in C_{2\pi}\cap Lip(\alpha), 0<\alpha\le 1$ and f is of monotonic type then

$$\left\|\beta_n(.)\right\|_c = O(1)\begin{cases} \dfrac{1}{n^{\alpha-\gamma}}, & 0<\gamma<\alpha<1 \\ \dfrac{\log n}{n^{1-\gamma}}, & 0<\gamma<\alpha=1 \end{cases}$$

Note: In Corollary 2 in the special case when $\gamma=\alpha$ the result reduces to second part of Theorem B due to Salem and Zygmund.

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