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New Weyl-Type Inequalities by Multiplicative Injective and Surjective s-Numbers of Operators in Reflexive Banach Spaces

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Abstract

In this work, two problems are investigated. In general, Weyl-type inequalities of operators in complex reflexive Banach spaces are discussed. First, we obtained the Weyl-type inequalities using arbitrary multiplicative surjective and injective s-numbers that are dual of each other. Second, we introduced the Weyl-type inequalities by multiplicative injective and surjective s-numbers under certain conditions for S and S' operators in complex reflexive Banach space. So, new Weyl-type inequalities are investigated for both dual s-number sequences and dual operators.

Keywords: Dual s- numbers; Dual operators; Multiplicative injective and surjective s-numbers; s-numbers; Weyl-Type inequalities.

Yansımalı Banach Uzaylarda Operatörlerin Çarpımsal İnjektiv ve Surjektiv s-Sayıları ile Yeni Weyl-Tipi Eşitsizlikleri

Öz

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Bu çalışmada iki problem incelenmiştir. Genel olarak, kompleks yansımalı Banach uzaylarında operatörlerin Weyl-tipi eşitsizlikleri üzerinde durulmuştur. İlk olarak, birbirinin duali olan keyfi çarpımsal surjektif ve injektif s-sayılarını kullanarak Weyl-tipi eşitsizlikler elde edilmiştir. İkinci olarak, kompleks yansımalı Banach uzayındaki S ve S' operatörleri için belirli koşullar altında çarpımsal injektif ve surjektif s-sayıları ile Weyl-tipi eşitsizlikler ifade edilmiştir. Böylece hem dual s-sayı dizileri hem de dual operatörler için yeni Weyl-tipi eşitsizlikleri araştırılmıştır.

Anahtar Kelimeler: Dual s-sayıları; Dual operatörler; Çarpımsal injektif ve surjektif s-sayıları; s-sayıları; Weyl-Tipi eşitsizlikler.

1. Introduction

The definition of s-number (or singular numbers) was firstly used by E. Schmidt in the theory of non- selfadjoint integral equation. The axiomatic structure of the original s-numbers in Banach spaces was developed by A. Pietsch [1].

Let us first give the theorem which expresses the classical Weyl inequality in Hilbert spaces [2]. Let H be a Hilbert space and $S \in C_\infty(H)$ a compact operator. Then

$$\prod_{k=1}^{n} |\lambda_k(S)| \leq \prod_{k=1}^{n} s_k(S)$$

for $n = 1,2,\ldots$

This inequality is an important tool to prove the correlation between eigenvalues and s-numbers. Thus, an important contribution is made to the investigation of the optimum asymptotic behavior of the eigenvalues. A. Pietsch developed the Weyl inequality for operators in Banach spaces [3].

$$\left(\prod_{j=1}^{n} |\lambda_j(S)|\right)^{\frac{1}{n}} \leq \left(\frac{n}{k}\right)^{\frac{n-k}{2n}} n^{\frac{k}{2n}} \|S\|^{1-\frac{k}{n}} \left(\prod_{j=n-k+1}^{n} h_j(S)\right)^{\frac{1}{n}}.$$

This inequality applies to any s-number sequence, because the Hilbert numbers are the smallest s-numbers in Banach spaces. We can also look at [4] for better constants.

For these inequalities, the Weyl numbers are considered to be suitable s-numbers. This fact has been confirmed as a result of extensive studies on the eigenvalues about of integral operators moving in function spaces. Researchers obtained similar inequalities by taking different s-

numbers instead of Weyl numbers. These inequalities are generally referred to as Weyl-type inequalities in the literature. We can see several Weyl-type inequalities in [2, 5, 6]. However, various Weyl-type inequalities were obtained for different operators (Riesz operator, Compact operator, etc.) in Banach space [5-7]. We can see some Weyl-type inequalities by injective and surjective s-numbers in [8, 9]. In our study, we will use the multiplicative injective and surjective s-numbers.

In the studies done in the ever-evolving literature, it has been concluded that many problems of the theory of multi-point differential operators can be easily solved on the direct sum of Banach spaces [10, 11]. In this context, some s-number functions of the direct sum of operator defined on the direct sum of Banach spaces, which can contribute to the field, and the s-number functions of the same type of coordinate operators have been investigated [12, 13]. In addition, s-numbers have a very important place for studies related to Lorentz-Schatten sequence classes [17-24].

We denote by B_X the closed unit ball of X. In what follows X, Y, Z, e.t.c . always denote complex Banach spaces. Then $L(X,Y)$ and $C_\infty(X,Y)$ respectively are denote the set of bounded linear operators and compact operators from X into Y. Also, if $X = Y$, it is denoted by $L(X) = L(X,X)$ and $C_\infty(X) = C_\infty(X,X)$. Moreover, S' is a dual operator of S.

2. *s*-Numbers and basic results

Definition 1. Let $S \in L(X)$. If $S^n \in C_\infty(X,Y)$ for $n \in \mathbb{N}$ then S is called power compact [5-7].

Let's give the definition of an s-number sequence [14].

Definition 2. A rule $s_n(S) : L \to [0,\infty]$ assigning to every operator $S \in L$ a non-negative scalar sequence $s_n(S)_{n\in\mathbb{N}}$ is called an s-number sequence if the following conditions are satisfied:

(i) Monotonicity:

$\|S\| = s_1(S) \geq s_2(S) \geq \cdots \geq 0$ for $S \in L(X,Y)$,

(ii) Additivity:

$s_n(S+T) \leq s_n(S) + \|T\|$ for $S, T \in L(X,Y)$ and $n, m = 1,2, \ldots$,

(iii) Ideal-Property:

$s_n(RST) \leq \|R\| s_n(S) \|T\|$ for $R \in L(X_0, X), S \in L(X,Y)$ and $T \in L(Y, Y_0)$

(iv) Rank-Property:

$s_n(S) = 0$ for $S \in L(X,Y)$ with $rank(S) < n$

(v) Norming Property:

$s_n(I_n) = 1$ for the identity maps $I_n: l_2^n \to l_2^n$ on l_2.

Let's give important s-number definitions. For $S \in L(X,Y)$ and $n = 1,2,\dots$, the n-th approximation number is defined by

$$a_n(S) = inf\{\|S - A\|: A \in L(E,F),\ rank\ (A) < n\},$$

the n-th Gelfand number by

$$c_n(S) := inf\{\|SJ_M\| : M \subset X,\ codim(M) < n\},$$

where $J_M: M \to X$ is the natural embedding from a subspace M of X into X, and the n-th Kolmogorov number by

$$d_n(S) := inf\{\|Q_N S\| : N \subset Y,\ dim(N) < n\},$$

where $Q_N: Y \to Y/N$ defines the canonical quotient map from Y into the quotient space Y/N, and the n-th Weyl number by

$$x_n(S) = sup\{a_n(SA): \ \|A: l_2 \to X\| \leq 1\},$$

and the n-th Hilbert number by

$$h_n(S) := sup\{a_n(BSA): \|A: l_2 \to X\| \leq 1, \|B: Y \to l_2\| \leq 1\}.$$

Remark 1. The following inequality exists for s-numbers in Banach spaces

$$h_n(S) \leq s_n(S) \leq a_n(S),$$

where $s_n(S)_{n\in\mathbb{N}}$ is an arbitrary s-number [5, 14].

Now let us express the relation between s-numbers in Hilbert spaces [5, 6].

Let us first give a brief description of the s-number in the classical Hilbert spaces. Suppose H is a Hilbert space and let $S \in C_\infty(H)$. Then $s_n(S) = \lambda_n\left((S^*S)^{\frac{1}{2}}\right)$ are called the singular numbers of S. We will show the sequence of eigenvalues of the S transformation with $\lambda_n(S)_{n\in\mathbb{N}}$. This sequence is ordered in decreasing absolute value and counted according to their multiplicity,

$$|\lambda_1(S)| \geq \cdots \geq |\lambda_n(S)| \geq \cdots \geq 0.$$

If S possesses less than n eigenvalues λ with $\lambda \neq 0$ we put $\lambda_n(S) = \lambda_{n+1}(S) = \cdots = 0$. It is well known that the sequence of eigenvalues form a null sequence.

Lemma 1. Let X, Y be Hilbert spaces and $S \in C_\infty(X, Y)$. Then we have

$$a_n(S) = c_n(S) = x_n(S) = d_n(S) = h_n(S) = \lambda_n(S^*S)^{\frac{1}{2}},$$

where S^* is the Hilbert adjoint of S.

In addition to these, let's give the following definitions [5, 6].

a. A s-number sequence $s = (s_n)$ is called injective if, given any metric injection $J \in L(Y, \tilde{Y})$ i.e. $\|Jy\| = \|y\|$ for $y \in Y, s_n(S) = s_n(JS)$ for all $S \in L(X, Y)$ and all Banach spaces X.

b. A s-number sequence $s = (s_n)$ is called surjective if, given any metric surjection $Q \in L(\tilde{X}, X)$ i.e. $Q(B_{\tilde{X}}) = B_{\tilde{X}}, s_n(S) = s_n(SQ)$ for all $S \in L(X, Y)$ and all Banach spaces Y.

c. If a s-number sequence satisfies (a) and (b) then it is called injective and surjective.

Moreover, we have

$$c_n(S) = a_n(J_\infty S) \text{ and } d_n(S) = a_n(SQ_1),$$

where $J_\infty: Y \to l_\infty(B_{Y'})$ is the metric surjection defined by $J_\infty y = (\langle y, a \rangle)_{a\in B_{Y'}}$ and with values in the space $l_\infty(B_{Y'})$ of bounded sequences and where $Q_1: l_1(B_X) \to X$ is the metric surjection from $l_1(B_X)$ onto X, defined by $Q_1\big((\xi_x)\big) \coloneqq \sum_{x\in B_X} \xi_x\, x$.

On the other hand, a s-numbers sequence (s_n) is called multiplicative if

$$s_{n+m-1}(TS) \leq s_n(T)\, s_m(S)$$

for $S \in L(X, Y), T \in L(Y, Z)$and $m, n \in 1,2, \ldots$.

Now, we recall useful mixing multiplicativity property for an arbitrary s-number sequence $s = (s_n)$ from [4]. For $S \in L(X,Y)$ and $T \in L(Y,Z)$,

a. $s_{n+m-1}(TS) \leq s_n(T)a_m(S)$ and $s_{n+m-1}(TS) \leq a_n(T)s_m(S)$.
b. If $s = (s_n)$ is injective, then $s_{n+m-1}(TS) \leq c_n(T)s_m(S)$.
c. If $s = (s_n)$ is surjective, then $s_{n+m-1}\,(TS) \leq s_n\,(T)d_m(S)$.

The following result can be easily deduced from the above inequalities. If $s = (s_n)$ is an injective and surjective s-number sequence, then

$$s_{n+m+l-1}(TSR) \leq c_n(T)s_m(S)d_l(R)$$

for $R \in L(X_0, X), S \in L(X,Y)$ and $T \in L(Y, Y_0)$.

Let us now state a lemma that we frequently refer to in our proofs [15].

Lemma 2. Let $I \in L(X,Y)$ identity map. Then,

$$x_k(I: l_\infty^n \to l_2^n) = \left(\frac{n}{k}\right)^{\frac{1}{2}},$$

for $1 \leq k \leq n$.

A. Pietsch's principle of related operators in the context of operators on X factorizing through Y. And we give the following lemma which is quite useful as a result of the principle of related operators [5-7].

Definition 3. Let $S \in L(X)$ and $T \in L(Y)$. If there are maps $P \in L(X,Y)$ and $R \in L(Y,X)$ such that $S = RP$ and $T = PR$, then S and T are called related.

Lemma 3. Let $S \in L(X)$ and $T \in L(Y)$ be related operators and S power compact. Then T is power compact and

$$\sigma(S)\backslash\{0\} = \sigma(T)\backslash\{0\}, \;\; m(S,\lambda) = m(T,\lambda) \text{ for all } 0 \neq \lambda \in \sigma(S).$$

Hence we have $\lambda_n(S) = \lambda_n(T)$ for all $n \in \mathbb{N}$.

We need another fact about the 2-summing norms due to Garling-Gordon [15].

Lemma 4. Let X_n be any n-dimensional Banach space. Then

$$\pi_2(I_{X_n}) = \sqrt{n} \;.$$

The following basic fact is quite important to prove the main result [5].

Lemma 5. Let $S \in C_\infty(X)$ be a compact operator and $\lambda_n(S) \neq 0$. Then there is a n-dimensional subspace X_n of X, invariant under S, such that the operator $S_n \in L(X_n)$ induced by S has exactly the eigenvalues $\lambda_1(S), \ldots, \lambda_n(S)$.

The following theorems give the relationship between operators and their duals for some s-numbers, see [1] for similar relations.

Theorem 1. Let $S \in L(X,Y)$; then

$$c_n(S) = d_n(S'),$$

where S' is dual operator of S.

Theorem 2. Let $S \in C_\infty(X,Y)$; then

$$d_n(S) = c_n(S'),$$

where S' is dual operator of S.

3. Main results

In the first section, we have recalled that Weyl-type inequalities are optimal for estimating eigenvalues of operators in Banach spaces. We can see several Weyl-type inequalities in [5, 6, 16].

Let us give the Weyl-type inequalities expressing the relation between the eigenvalues and Weyl numbers established by Pietsch in [3]:

$$\left(\prod_{k=1}^{2n-1} |\lambda_k(S)|\right)^{\frac{1}{2n-1}} \leq \sqrt{2e}\left(\prod_{k=1}^{n} x_k(S)\right)^{\frac{1}{n}}$$

and

$$\left(\sum_{k=1}^{n} |\lambda_k(S)|^p\right)^{\frac{1}{p}} \leq c_p\left(\sum_{k=1}^{n} x_k^p(S)\right)^{\frac{1}{p}}$$

for $S \in C_\infty(X,Y)$.

3.1. Weyl-Type inequality by dual s-numbers of power compact operators

Power compact operators are a classical subject in the context of integral operators to relate the properties (e.i. kernel properties) of an operator to the decay of its eigenvalues [6]. So, we firstly obtained Weyl-type inequalities of power compact operators in Banach spaces through multiplicative injective and surjective s-numbers. Here $[x]$ denotes the integer part of x for $1 \leq x < \infty$ and if $0 < x \leq 1$ we put $[x] := 1$.

Theorem 3. Let $S \in L(X)$ is a power compact operator such that all complex Banach space X and $s = (s_n)$ be a multiplicative injective s-number sequence for $n \in \mathbb{N}$. Then,

$$\left(\prod_{k=1}^{n} |\lambda_k(S)|\right)^{\frac{1}{n}} \leq C(\delta)\sqrt{n}\sqrt{e}\left(\prod_{k=1}^{m} s_k(S)\right)^{\frac{1}{m}},$$

where $m := \left[\frac{n}{1+\delta}\right]$ and $0 < \delta \leq 1$, $C(\delta) = 2\left(\frac{1+\delta}{\delta}\right)^{\frac{1}{2}}$ for $n, m \in \mathbb{N}$.

Proof. Since $S \in L(X)$ is a power compact we can find a n −dimensional subspace X_n of X invariant under S such that the restriction of S to X_n, $S_n = S_{|X_n}$ has precisely $\lambda_1(S), \ldots, \lambda_n(S)$ as its eigenvalues. By Lemma 4, $\pi_2(I_{|X_n}) = \sqrt{n}$. Hence by the Grothendieck-Pietsch factorization $B \in L(H, X_n)$ with $BA = I_{|X_n}$ and $\|A\| = \pi_2(A) = \sqrt{n}$, $\|B\| = 1$. We may assume that the Hilbert space H is n −dimensional (by restriction), $H = l_2^n$ so that $B = A^{-1}$. Define $T_n = AS_nA^{-1} \in L(l_2^n)$; T_n has the same eigenvalues $\left(\lambda_j(S)\right)_{j=1}^{n}$ as S_n with the principle of related operators. Using Weyl' s inequality in Hilbert space and the multiplicative of s-number sequence, we obtain

$$\left(\prod_{k=1}^{n} |\lambda_k(S)|\right)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} |\lambda_k(S_n)|\right)^{\frac{1}{n}}$$

$$= \left(\prod_{k=1}^{n} |\lambda_k(T_n)|\right)^{\frac{1}{n}}$$

$$\leq \left(\prod_{k=1}^{n} s_k(T_n)\right)^{\frac{1}{n}}. \quad (1)$$

Moreover, $0 < \delta \le 1$ and a non-increasing sequence of positive numbers $(s_k)_{k\in\mathbb{N}}$ we use the estimate

$$\left(\prod_{k=1}^{n} s_k\right)^{\frac{1}{n}} \le \left(\prod_{k=1}^{m} s_{[\delta k]+k-1}\right)^{\frac{1}{m}},$$

where $m:=\left[\frac{n}{1+\delta}\right]$ and $n, m \in \mathbb{N}$. Thus,

$$\left(\prod_{k=1}^{n} s_k(T_n)\right)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} s_k(AS_nA^{-1})\right)^{\frac{1}{n}}$$

$$\le \left(\prod_{k=1}^{m} s_{[\delta k]+k-1}(AS_nA^{-1})\right)^{\frac{1}{m}}. \tag{2}$$

The mixing multiplicative (b) property of an injective s-number sequence yields the estimate for the single s-numbers

$$\left(\prod_{k=1}^{m} s_{[\delta k]+k-1}(AS_nA^{-1})\right)^{\frac{1}{m}} \le \left(\prod_{k=1}^{m} s_k(AS_n)c_{[\delta k]}(A^{-1})\right)^{\frac{1}{m}}. \tag{3}$$

Also using property (iii) of Definition 2, $\|A\| = \pi_2(A) = \sqrt{n}$,$\|B\| = 1$ by the Grothendieck-Pietsch factorization and Lemma 2 we have

$$s_k(AS_n)c_{[\delta k]}(A^{-1}) \le \|A\|\, s_k(S)\, c_{[\delta k]}(I_nB)$$

$$\le \sqrt{n}\, s_k(S)\, \|B\|\, c_{[\delta k]}(I_n: l_2^n \to l_2^n)$$

$$= \sqrt{n}\, s_k(S)\, \|B\|\, x_{[\delta k]}(I_n: l_2^n \to l_2^n)$$

$$\le \sqrt{n}\, s_k(S) \sqrt{\frac{n}{[\delta k]}} \quad , \tag{4}$$

where $I_n: l_2^n \to l_2^n$ identity maps for $n \in \mathbb{N}$. From $m := \left[\frac{n}{1+\delta}\right] \ge \frac{n}{2(1+\delta)}$ and $[\delta k] \ge \frac{\delta k}{2}$ and Stirling's formula $e^m \ge \frac{m^m}{m!}$ we can combine Eqns. (1)-(4)

$$\left(\prod_{k=1}^{n}|\lambda_k(S)|\right)^{\frac{1}{n}} \le \sqrt{n}\left(\prod_{k=1}^{m} s_k(S)\left(\frac{n}{[\delta k]}\right)^{\frac{1}{2}}\right)^{\frac{1}{m}}$$

$$\le \sqrt{n}\left(\prod_{k=1}^{m}\left(\frac{n}{[\delta k]}\right)^{\frac{1}{2}}\right)^{\frac{1}{m}}\left(\prod_{k=1}^{m} s_k(S)\right)^{\frac{1}{m}}$$

$$\le \sqrt{n}\left(\prod_{k=1}^{m}\left(\frac{2n}{\delta k}\right)^{\frac{1}{2}}\right)^{\frac{1}{m}}\left(\prod_{k=1}^{m} s_k(S)\right)^{\frac{1}{m}}$$

$$\le \sqrt{n}\left(\frac{2}{\delta}\right)^{\frac{1}{2}}\left(\frac{n^m}{m!}\right)^{\frac{1}{2m}}\left(\prod_{k=1}^{m} s_k(S)\right)^{\frac{1}{m}}$$

$$\le \sqrt{n}\left(\frac{2}{\delta}\right)^{\frac{1}{2}}(e^m)^{\frac{1}{2m}}\left(2(1+\delta)\right)^{\frac{1}{2}}\left(\prod_{k=1}^{m} s_k(S)\right)^{\frac{1}{m}}$$

$$\le 2\left(\frac{1+\delta}{\delta}\right)^{\frac{1}{2}}\sqrt{n}\sqrt{e}\left(\prod_{k=1}^{m} s_k(S)\right)^{\frac{1}{m}}.$$

Finally, we obtain following inequality

$$\left(\prod_{k=1}^{n}|\lambda_k(S)|\right)^{\frac{1}{n}} \le C(\delta)\sqrt{n}\sqrt{e}\left(\prod_{k=1}^{m} s_k(S)\right)^{\frac{1}{m}},$$

where $0 < \delta \le 1$, $C(\delta) = 2\left(\frac{1+\delta}{\delta}\right)^{\frac{1}{2}}$ and $n, m \in \mathbb{N}$.

Remark 2. If we put $\delta = 1$, $m = \left[\frac{n}{2}\right] = n - 1$ replace (s_n) by (x_n) then we have Pietsch's Weyl inequality

$$\left(\prod_{k=1}^{n}|\lambda_k(S)|\right)^{\frac{1}{n}} \le 2\sqrt{2e}n^{\frac{1}{2}}\left(\prod_{k=1}^{n-1} x_k(S)\right)^{\frac{1}{n-1}},$$

where $m, n \in \mathbb{N}$.

Theorem 4. Let $S \in L(X)$ be a power compact operator such that all complex reflexive Banach space X and $s = (s_n)$ be a multiplicative surjective s-number sequence for $n \in \mathbb{N}$. Then,

$$\left(\prod_{k=1}^{n} |\lambda_k(S)|\right)^{\frac{1}{n}} \leq C(\delta)\sqrt{n}\sqrt{e}\left(\prod_{k=1}^{m} s_k(S)\right)^{\frac{1}{m}},$$

where $m := \left[\frac{n}{1+\delta}\right]$ and $0 < \delta \leq 1$, $C(\delta) = 2\left(\frac{1+\delta}{\delta}\right)^{\frac{1}{2}}$ for $n, m \in \mathbb{N}$.

Proof. If S is a compact power operator, the dual operator S' is also a compact power operator. Furthermore, the eigenvalues sequences of S and S' can be arrenged in such a way that $\lambda_n(S') = \lambda_n(S)$ for all $n \in \mathbb{N}$, see e.g. [5]. For $n \in \mathbb{N}$ and any operator S, define $\tilde{s}_n(S) = s_n(S')$. $\tilde{s} = (\tilde{s})$ is known to be a sequence of s-numbers [5]. Also, since the dual of a metric injection is a metric surjection [14] , $\tilde{s}$ is an injective s-number sequence. Then we obtain from Theorem 3 applied to S' that

$$\left(\prod_{k=1}^{n} |\lambda_k(S)|\right)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} |\lambda_k(S')|\right)^{\frac{1}{n}}$$

$$\leq C(\delta)\sqrt{n}\sqrt{e}\left(\prod_{k=1}^{m} \tilde{s}_k(S')\right)^{\frac{1}{m}}$$

$$\leq C(\delta)\sqrt{n}\sqrt{e}\left(\prod_{k=1}^{m} s_k(S'')\right)^{\frac{1}{m}}.$$

We have $S'' = S$ since X is reflexive. Thus, the alleged inequality is proved.

3.2. Weyl-Type inequality of dual operators by arbitrary multiplicative injective and surjective s-numbers

We mentioned that Weyl numbers with minimum s-numbers are considered the best s-numbers for working with Weyl-type inequalities. Nevertheless, whether there exists a minimal multiplicative s-number sequence another from Weyl numbers for was investigated by [4].

We will obtain an important inequality between arbitrary multiplicative injective and surjective s-numbers (s_n) and (r_n) with the property that $s_n(S) \leq r_n(S)$ for all $S \in C_\infty(X)$ compact operators. We will also investigate the relation that $s = (s_n)$ s-numbers which is an arbitrary multiplicative injective and surjective s-number sequence of S is compact operator and it's S' is dual operator in complex reflexive Banach space.

Theorem 5. Let (s_n) and (r_n) be multiplicative injective and surjective s-numbers sequence with the property that $s_n(S) \leq r_n(S)$ for all $S \in C_\infty(X)$ compact operators such that complex Banach space X. Then,

$$r_{2n-1}(S) \leq \sqrt{e}\left(\prod_{k=1}^{n} s_k(S)\right)^{\frac{1}{n}},$$

for $n = 1, 2, \ldots$

Proof. Since (r_n) is an injective and surjective s-number sequence, we have that operator $Q_1 \in L(l_1^n, X)$ and $J_\infty \in L(X, l_\infty^n)$.

$$r_{2n-1}(S) = r_{2n-1}(J_\infty S Q_1) \leq \left(\prod_{k=1}^{n} r_{2k-1}(J_\infty S Q_1)\right)^{\frac{1}{n}} \tag{5}$$

We easily see that for Q_1, an operator acting in Hilbert space, and (s_n), a multiplicative injective and surjective s-number sequence, the following inequality holds.

$$\left(\prod_{k=1}^{n} r_{2k-1}(J_\infty S Q_1)\right)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} s_{2k-1}(J_\infty S Q_1)\right)^{\frac{1}{n}}$$

$$\leq \left(\prod_{k=1}^{n} c_{\frac{k}{2}}(J_\infty) s_k(S) d_{\frac{k}{2}}(Q_1)\right)^{\frac{1}{n}}. \tag{6}$$

Thus using property (iii) of Definition 2, J_∞ is an operator acting in Hilbert space, $\|J_\infty\| \leq 1$ and $\|Q_1\| \leq 1$ we obtain

$$\left(\prod_{k=1}^{n} c_{\frac{k}{2}}(J_\infty) s_k(S) d_{\frac{k}{2}}(Q_1)\right)^{\frac{1}{n}} \leq \left(\prod_{k=1}^{n} c_{\frac{k}{2}}(I_n: l_\infty^n \to l_\infty^n) \|J_\infty\| s_k(S) \|Q_1\| d_{\frac{k}{2}}(I_n: l_1^n \to l_1^n)\right)^{\frac{1}{n}}$$

$$\leq \left(\prod_{k=1}^{n} c_{\frac{k}{2}}(I_n: l_\infty^n \to l_\infty^n) s_k(S) d_{\frac{k}{2}}(I_n: l_1^n \to l_1^n)\right)^{\frac{1}{n}}, \tag{7}$$

where I_n is a identity map. Since I_n is an operator acting in Hilbert space and combining Eqns. (5)-(7) we arrive

$$r_{2n-1}(S) \leq \left(\prod_{k=1}^{n} x_{\frac{k}{2}}(I_n: l_\infty^n \to l_\infty^n) s_k(S) x_{\frac{k}{2}}(I_n: l_1^n \to l_1^n) \right)^{\frac{1}{n}}.$$

We easily get the following estimates by using the identity maps I_n for $n = 1,2,...$ with Lemma 2, from the last equation and known inequality $\frac{n^n}{n!} \leq e^n$,

$$r_{2n-1}(S) \leq \left(\prod_{k=1}^{n} \left(\frac{2n}{k} \right) \right)^{\frac{1}{n}} \left(\prod_{k=1}^{n} s_k(S) \right)^{\frac{1}{n}}$$

$$\leq 2 \left(\frac{n^n}{n!} \right)^{\frac{1}{n}} \left(\prod_{k=1}^{n} s_k(S) \right)^{\frac{1}{n}}$$

$$\leq 2e \left(\prod_{k=1}^{n} s_k(S) \right)^{\frac{1}{n}}.$$

Finally, we get that

$$r_{2n-1}(S) \leq 2e \left(\prod_{k=1}^{n} s_k(S) \right)^{\frac{1}{n}},$$

for $n = 1, 2, ...$

Let's express the relation for dual operators under the conditions of Theorem 5.

Theorem 6. Let (s_n) and (r_n) be be multiplicative injective and surjective s-numbers sequence with the property that $s_n(S) \leq r_n(S)$ for all $S \in C_\infty(X)$ compact operators such that complex reflexive Banach space X. Then,

$$r_{2n-1}(S') \leq 2e \left(\prod_{k=1}^{n} s_k(S') \right)^{\frac{1}{n}}, \quad n \in \mathbb{N}$$

where S' is dual operator of S .

Proof. Since (r_n) is an injective and surjective s-number sequence, for operator $Q_1' \in L(X', l_\infty^n)$ and $J_\infty' \in L(l_1^n, X')$ we have

$$r_{2n-1}(S') = r_{2n-1}(Q_1'S'J_\infty') \leq \left(\prod_{k=1}^{n} r_{2k-1}(Q_1'S'J_\infty')\right)^{\frac{1}{n}}. \tag{8}$$

We easily see that for J_∞' , an operator acting in Hilbert space, and (s_n), a multiplicative injective and surjective s-number sequence, we can write

$$\left(\prod_{k=1}^{n} r_{2k-1}(Q_1'S'J_\infty')\right)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} s_{2k-1}(Q_1'S'J_\infty')\right)^{\frac{1}{n}}$$

$$\leq \left(\prod_{k=1}^{n} c_{\frac{k}{2}}(Q_1')s_k(S')d_{\frac{k}{2}}(J_\infty')\right)^{\frac{1}{n}}. \tag{9}$$

Moreover we have from Theorem 1 and Theorem 2,

$$\left(\prod_{k=1}^{n} c_{\frac{k}{2}}(Q_1')s_k(S')d_{\frac{k}{2}}(J_\infty')\right)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} d_{\frac{k}{2}}(Q_1)s_k(S')c_{\frac{k}{2}}(J_\infty)\right)^{\frac{1}{n}}. \tag{10}$$

Thus using property (iii) of Definition 2, J_∞ is an operator acting in Hilbert space, $\|J_\infty\| \leq 1$ and $\|Q_1\| \leq 1$ we obtain

$$\left(\prod_{k=1}^{n} d_{\frac{k}{2}}(Q_1)s_k(S')c_{\frac{k}{2}}(J_\infty)\right)^{\frac{1}{n}} \leq \left(\prod_{k=1}^{n} \|Q_1\| d_{\frac{k}{2}}(I_n: l_1^n \to l_1^n)s_k(S')c_{\frac{k}{2}}(I_n: l_\infty^n \to l_\infty^n)\|J_\infty\|\right)^{\frac{1}{n}}$$

$$\leq \left(\prod_{k=1}^{n} d_{\frac{k}{2}}(I_n: l_1^n \to l_1^n)s_k(S')c_{\frac{k}{2}}(I_n: l_\infty^n \to l_\infty^n)\right)^{\frac{1}{n}}, \tag{11}$$

where I_n is a identity map. Since I_n is an operator acting in Hilbert space and combining Eqns. (8)-(11) we arrive

$$r_{2n-1}(S) \leq \left(\prod_{k=1}^{n} x_{\frac{k}{2}}(I_n: l_1^n \to l_1^n)s_k(S')x_{\frac{k}{2}}(I_n: l_\infty^n \to l_\infty^n)\right)^{\frac{1}{n}}.$$

We get the following estimates by using the identity maps I_n for $n = 1,2, \ldots$ with Lemma 2, from the last equation and known inequality $\frac{n^n}{n!} \leq e^n$,

$$r_{2n-1}(S') \leq 2e\left(\prod_{k=1}^{n} s_k(S')\right)^{\frac{1}{n}}.$$

4. Conclusion

In this study, the role and importance of s-numbers in the literature were investigated. First of all, the development of s-numbers in Banach spaces in the literature was given. Moreover, Weyl-type inequalities, which have an important place in applied mathematics, were presented. Information about the optimality of these Weyl-type inequalities was given. New Weyl-type inequalities were obtained by using multiplicative injective and surjective s-numbers and dual s-numbers in complex reflexive Banach spaces. In addition, important relations for dual operators were expressed.

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