Journal of Engineering Technology and Applied Sciences, 2022 *e-ISSN:* 2548-0391
Received 1 October 2021, Accepted 13 July 2022 *Vol. 7, No. 2, 69-77*
Published online 20 August 2022, Research Article *doi: 10.30931/jetas.1003445*

Citation: Kaplan, M., "A New Iterative Scheme for Approximating Fixed Points of Suzuki Generalized Multivalued Nonexpansive Mappings". Journal of Engineering Technology and Applied Sciences 7 (2) 2022 : 69-77.

A NEW ITERATIVE SCHEME FOR APPROXIMATING FIXED POINTS OF SUZUKI GENERALIZED MULTIVALUED NONEXPANSIVE MAPPINGS

Makbule Kaplan

Department of Mathematics and Science Education, Sinop University, Turkey
mkaplan@sinop.edu.tr

Abstract

In this paper, we study to approximate fixed points of Suzuki generalized multivalued nonexpansive mappings by using a three-step iterative scheme (1.1) introduced in [17]. We establish some weak and strong convergence results for mappings satisfying condition (C) with the newly proposed iterative scheme in the framework of uniformly convex real Banach spaces.

Keywords: Multivalued mappings, generalized multivalued nonexpansive mappings, strong and weak convergence results, fixed point

1. Introduction and preliminaries

The study of fixed points for multivalued contractive and nonexpansive mappings using the Hausdorff metric was initiated by Markin [2] and Nadler [1]. Since classical fixed point theorems for single-valued nonexpansive mappings are extended to multivalued nonexpansive mappings, fixed point theory for multivalued nonexpansive mappings developed rapidly. The theory of multivalued mappings has applications in control theory, convex optimization, differential equations and economics.

Suzuki [15] established generalized nonexpansive mappings, which satisfy a conditon on mapping called condition (C). Suzuki showed that this condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Suzuki's generalized nonexpansive mappings have been studied some researchers ([3], [4], [5], [8], [9], [10], [11], [12], [13], [16]).

Let D be a nonempty subset of a Banach space E. We denote $CB(D)$ and $C(D)$ the collection of all nonempty closed bounded subsets and nonempty compact subsets of D, respectively. The Hausdorff distance H on $CB(E)$ is defined by

$$H(A,C) := max\{\sup_{a\in A} dist(a,C), \sup_{y\in C} dist(c,A)\} \quad for\ A,C \in CB(E)$$

where $dist(a,C) = inf\{\|a-c\|: c \in C\}$.

A multivalued mapping $T: D \to CB(D)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq \|x-y\|, \qquad \forall x,y \in D$$

A point $x \in D$ is called a fixed point of T if $x \in T(x)$. In this paper, $F(T)$ shows that the set of fixed points of T.

Let $C \neq \emptyset \subset E$ and let $\{y_n\}$ be a bounded sequence in E. For each $y \in E$, the asymptotic radius of $\{y_n\}$ at y is defined by

$$r(y,\{y_n\}) = limsup_{n\to\infty}\|y_n - y\|;$$

the asymptotic radius of $\{y_n\}$ relative to C is a $r > 0$ such that

$$r = r(C,\{y_n\}) = inf\{r(y,\{y_n\}): y \in C\};$$

and the asymptotic center of $\{y_n\}$ relative to C is defined by set A, where

$$A = A(C,\{y_n\}) = \{y \in C: r(y,\{y_n\}) = r\}.$$

If E is a uniformly convex Banach space, $A(C,\{y_n\})$ consists of exactly one point. Unless otherwise stated, in the definitions we will use, E denotes a uniformly convex Banach space and D denotes a nonempty subset of E.

Abkar and Eslamian [3] modified Suzuki's condition to incorporate multivalued mappings. They called these mappings generalized multivalued nonexpansive mappings in the sense of Suzuki or multivalued mappings satisfying the condition (C). Some another generalizations for multivalued mappings are available in literature ([3], [18]). Sadhu and et al. [18] introduced a new class of nonexpansive multivalued mappings, called generalized α-nonexpansive mappings, which properly contains the class of Suzuki-type mapping.

Over the years many researchers have developed several iterative processes for solving fixed point problems for different operators but the researches are still on going in order to develop a faster and more efficient iterative algorithms. Some researchers introduced some different iteration methods for finding the fixed points of a multivalued nonexpansive mappings ([3], [4], [5], [7], [9], [11], [13], [16], [17], [20]). Kaplan and Kopuzlu [17] introduced a new three-step iterative scheme (1.1) to approximate a common fixed point of multivalued nonexpansive mappings in a uniformly convex real Banach space. Mann, Ishikawa, Noor, S- iteration, SP, Abbas, Thakur New and M-iteration processes are few of the most popular methods defined to approximating fixed points of multivalued mappings. Thakur proved in [5] that the Thakur-New iterative process is fast in terms of convergence when compared to Picard, Mann, Ishikawa, Noor, Agarwal and Abbas iteration processes for Suzuki generalized nonexpansive mappings. In 2018, Ullah and Arshad [19] proved that, compared to all the above mentioned iteration processes, M-iteration process have high speeds of convergence for Suzuki generalized nonexpansive mappings.

We now state Suzuki's condition for multivalued mappings as follows.

Definition 1.1 [7] A multivalued mapping $T: E \to CB(E)$ satisfies the condition (C) provided

$$\frac{1}{2}dist(x, Tx) \leq \|x - y\| \Rightarrow H(Tx, Ty) \leq \|x - y\| \quad for\ all\ x, y \in E.$$

Lemma 1.2 [9] Let $T: E \to CB(E)$ be a multivalued nonexpansive mapping, then T satisfies the condition (C).

Lemma 1.3 [9] Let $T: E \to CB(E)$ be a multivalued mapping which satisfies the condition (C) and has a fixed point. Then T is a quasi-nonexpansive mapping.

Lemma 1.4 [7] If $T: D \to P(D)$ satisfies the condition (C), then

$$H(Tx, Ty) \leq 3dist(x, Tx) + \|x - y\|, \quad \forall x, y \in D.$$

Lemma 1.5 [15] Let D be a weakly compact convex subset of E. Let T be a mapping on D. If T satisfies condition (C), so T has a fixed point.

Lemma 1.6 [15] Let T be a mapping on a subset D of a Banach space E with the Opial property. Assume that T satisfies condition (C). If $\{\varkappa_n\} \rightharpoonup z$ and $lim_{n\to\infty}\|\varkappa_n - T\varkappa_n\| = 0$, then $T(z) = z$.

We study with the iteration process (1.1) introduced in [17] as an alternative to all the above mentioned iteration processes to approximate the fixed points of Suzuki generalized multivalued nonexpansive mappings. We prove some weak and strong convergence results using iterative scheme (1.1) for Suzuki-generalized multivalued nonexpansive mappings in uniformly convex Banach space E.

Let D be a nonempty closed convex subset of E and let $T: D \to CB(D)$ be a multivalued mapping,

$$\begin{cases} \varkappa_1 \in E, \\ \varkappa_{n+1} = (1 - a_n)v_n + a_n w_n \\ y_n = (1 - b_n)u_n + b_n v_n \\ z_n = (1 - c_n)\varkappa_n + c_n u_n \end{cases} \tag{1.1}$$

for all $n \geq 1$, where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in (0,1) and $v_n \in T(z_n)$, $u_n \in T(\varkappa_n)$ and $w_n \in T(y_n)$.

Lemma 1.7 [14] Let E be a uniformly convex Banach space and $0 < a \leq k_n \leq b < 1$ for all $n \in N$. Assume that $\{\varkappa_n\}$ and $\{y_n\}$ are sequences of E such that $limsup_{n\to\infty}\|\varkappa_n\| \leq l$, $limsup_{n\to\infty}\|y_n\| \leq l$ and $lim_{n\to\infty}\|k_n\varkappa_n + (1 - k_n)y_n\| = l$ for some $l \geq 0$, then $lim_{n\to\infty}\|\varkappa_n - y_n\| = 0$.

2. Main results

We start with the following lemmas.

Lemma 2.1 Let D be nonempty closed convex subset of a uniformly convex Banach space E. Let $T: D \to CB(D)$ be a mapping satisfying the condition (C) with $F(T) \neq \emptyset$. Let the sequence $\{\varkappa_n\}$ be generated by (1.1). Then $lim_{n\to\infty}\|\varkappa_n - \varkappa^*\|$ exists for all $\varkappa^* \in F(T)$.

Proof: Let $\varkappa^* \in F(T)$ and $z \in D$, as T satisfies condition (C),

$$\frac{1}{2}\|\varkappa^* - T(\varkappa^*)\| = 0 \leq \|\varkappa^* - z\| \text{implies that } H(T(\varkappa^*), T(z)) \leq \|\varkappa^* - z\|.$$

From (1.1), we have

$$\begin{aligned} \|z_n - \varkappa^*\| &= \|(1-c_n)\varkappa_n + c_n u_n - \varkappa^*\| \\ &\leq (1-c_n)\|\varkappa_n - \varkappa^*\| + c_n\|u_n - \varkappa^*\| \\ &\leq (1-c_n)\|\varkappa_n - \varkappa^*\| + c_n dist(u_n, T\varkappa^*) \\ &\leq (1-c_n)\|\varkappa_n - \varkappa^*\| + c_n H\big(T(x_n), T(\varkappa^*)\big) \\ &\leq (1-c_n)\|\varkappa_n - \varkappa^*\| + c_n\|\varkappa_n - \varkappa^*\| \\ &= \|\varkappa_n - \varkappa^*\| \end{aligned} \qquad (2.1)$$

and

$$\begin{aligned} \|y_n - \varkappa^*\| &= \|(1-b_n)u_n + b_n v_n - \varkappa^*\| \\ &\leq (1-b_n)\|u_n - \varkappa^*\| + c_n\|v_n - \varkappa^*\| \\ &\leq (1-b_n)dist(u_n, T(\varkappa^*)) + c_n dist(v_n, T(\varkappa^*)) \\ &\leq (1-b_n)H\big(T(x_n), T(\varkappa^*)\big) + b_n H\big(T(z_n), T(\varkappa^*)\big) \\ &\leq (1-b_n)\|\varkappa_n - \varkappa^*\| + b_n\|z_n - \varkappa^*\| \\ &\leq (1-b_n)\|\varkappa_n - \varkappa^*\| + b_n\|\varkappa_n - \varkappa^*\| \\ &= \|\varkappa_n - \varkappa^*\| \end{aligned} \qquad (2.2)$$

and

$$\begin{aligned} \|x_{n+1} - \varkappa^*\| &= \|(1-a_n)v_n + a_n w_n - \varkappa^*\| \\ &\leq (1-a_n)\|v_n - \varkappa^*\| + c_n\|w_n - \varkappa^*\| \\ &\leq (1-a_n)dist(v_n, T(\varkappa^*)) + a_n dist(w_n, T(\varkappa^*)) \\ &\leq (1-a_n)H\big(T(z_n), T(\varkappa^*)\big) + a_n H\big(T(y_n), T(\varkappa^*)\big) \\ &\leq (1-a_n)\|z_n - \varkappa^*\| + a_n\|y_n - \varkappa^*\| \\ &= \|\varkappa_n - \varkappa^*\|. \end{aligned} \qquad (2.3)$$

Using (2.1), (2.2) and (2.3), we have

$$\|x_{n+1} - \varkappa^*\| \leq (1-a_n)\|z_n - \varkappa^*\| + a_n\|y_n - \varkappa^*\| = \|\varkappa_n - \varkappa^*\|.$$

Thus $\{\|\varkappa_n - \varkappa^*\|\}$ is bounded and nonincreasing sequence, which implies $lim_{n\to\infty}\|\varkappa_n - \varkappa^*\|$ exists for any $\varkappa^* \in F(T)$.

Lemma 2.2 Let D be nonempty closed convex subset of a uniformly convex real Banach space E. Let $T: D \to CB(D)$ be a mapping satisfying the condition (C). Let the sequence $\{\varkappa_n\}$ be generated by (1.1). Assume that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $[a, b] \subset (0,1)$ with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{\varkappa_n\}$ is bounded and $lim_{n\to\infty} dist(\varkappa_n, T(\varkappa_n)) = 0$.

Proof: Let $\varkappa^* \in F(T)$. By Lemma 2.1, $lim_{n\to\infty}\|\varkappa_n - \varkappa^*\|$ exists and $\{\varkappa_n\}$ is bounded. Put for some $c \geq 0$. Let

$$lim_{n\to\infty}\|\varkappa_n - \varkappa^*\| = c. \tag{2.4}$$

From (2.1) and (2.4), we get

$$limsup_{n\to\infty}\|z_n - \varkappa^*\| \leq limsup_{n\to\infty}\|\varkappa_n - \varkappa^*\| = c. \tag{2.5}$$

Similarly, from (2.2) and (2.4), we obtain

$$limsup_{n\to\infty}\|y_n - \varkappa^*\| \leq limsup_{n\to\infty}\|\varkappa_n - \varkappa^*\| = c. \tag{2.6}$$

Also,

$$\begin{aligned} limsup_{n\to\infty}\|u_n - \varkappa^*\| &\leq limsup_{n\to\infty}\|u_n - T(\varkappa^*)\| \\ &\leq limsup_{n\to\infty}H\big(T(\varkappa_n), T(\varkappa^*)\big) \\ &\leq limsup_{n\to\infty}\|\varkappa_n - \varkappa^*\| = c \end{aligned} \tag{2.7}$$

so $limsup_{n\to\infty}\|u_n - \varkappa^*\| \leq c$. By Lemma 1.2, we have

$$limsup_{n\to\infty}\|T(\varkappa_n) - \varkappa^*\| \leq limsup_{n\to\infty}\|\varkappa_n - \varkappa^*\| = c. \tag{2.8}$$

On the other hand,

$$\begin{aligned} \|y_n - \varkappa^*\| &= \|(1 - b_n)u_n + b_n v_n - \varkappa^*\| \\ &\leq (1 - b_n)\|\varkappa_n - \varkappa^*\| + b_n\|z_n - \varkappa^*\| \end{aligned}$$

which implies that

$$\frac{\|y_n - \varkappa^*\| - \|\varkappa_n - \varkappa^*\|}{b_n} \leq \|z_n - \varkappa^*\| - \|\varkappa_n - \varkappa^*\|.$$

Then

$$\|y_n - \varkappa^*\| - \|\varkappa_n - \varkappa^*\| \leq \frac{\|y_n - \varkappa^*\| - \|\varkappa_n - \varkappa^*\|}{b_n} \leq \|z_n - \varkappa^*\| - \|\varkappa_n - \varkappa^*\|.$$

So, we obtain

$$\|y_n - \varkappa^*\| \leq \|z_n - \varkappa^*\|. \tag{2.9}$$

If we apply liminf to inequality (2.9), we get

$$c \leq liminf_{n\to\infty}\|z_n - \varkappa^*\|. \tag{2.10}$$

By (2.5) and (2.10), we get

$$lim_{n\to\infty}\|(1 - c_n)(\varkappa_n - \varkappa^*) + c_n(u_n - \varkappa^*)\| = lim_{n\to\infty}\|z_n - \varkappa^*\| = c. \tag{2.11}$$

From (2.4), (2.7), (2.11) and Lemma 1.7, we obtain that

$$lim_{n\to\infty}\|\varkappa_n - u_n\| = lim_{n\to\infty}dist\big(\varkappa_n, T(\varkappa_n)\big) = 0.$$

Inversely, assume that $\{\varkappa_n\}$ is bounded and $lim_{n\to\infty}dist\big(\varkappa_n, T(\varkappa_n)\big) = 0$. Let $\varkappa^* \in A(C,\{\varkappa_n\})$. By Lemma 1.4, we have

$$\begin{aligned} r(T(\varkappa^*),\{\varkappa_n\}) &= limsup_{n\to\infty}\|\varkappa_n - T(\varkappa^*)\| \\ &\leq\ limsup_{n\to\infty}3dist(T(\varkappa_n),\varkappa_n)) + \|\varkappa_n - \varkappa^*\|) \\ &=\ limsup_{n\to\infty}\|\varkappa_n - \varkappa^*\| \\ &=\ r(\varkappa^*,\{\varkappa_n\}). \end{aligned}$$

It follows that $T(\varkappa^*) \in A(C,\{\varkappa_n\})$. Due to the uniformly and convex properties of Banach space E, therefore $A(C,\{\varkappa_n\})$ is singleton. That is, $T(\varkappa^*) = \varkappa^*$. It means $F(T) \neq \emptyset$.

Definiton 2.3 A Banach space E satisfies Opial property [6] if for each sequence $\{\varkappa_n\}$ in E which weakly converges to $\varkappa \in E$

$$limsup_{n\to\infty}\|\varkappa_n - \varkappa\| \leq limsup_{n\to\infty}\|\varkappa_n - y\|$$

holds, for all $y \in E$ with $y \neq \varkappa$.

Theorem 2.4 Let be a uniformly convex Banach space E with Opial property. Let D, T and $\{\varkappa_n\}$ be as in Lemma 2.2. Then $\{\varkappa_n\}$ converges weakly to a fixed point of T.

Proof: Lemma 2.2 guarantees that $\{\varkappa_n\}$ is bounded and $lim_{n\to\infty}dist\big(\varkappa_n, T(\varkappa_n)\big) = 0$. Since E uniformly convex Banach space, we can assume that $\varkappa_n \to a$ weakly as n→∞ for some $a \in D$. We prove that $a \in F(T)$. Since $T(a)$ is compact, for every $n \geq 1$, we can select $y_n \in T(a)$ such that $\|\varkappa_n - y_n\| = dist(\varkappa_n, T(a))$. As $T(a)$ is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}$ with $lim_{k\to\infty}y_{n_k} = w \in T(a)$. By Lemma 1.4, we have

$$\begin{aligned} dist(\varkappa_{n_k}, T(a)) &\leq\ dist(\varkappa_{n_k}, T(\varkappa_{n_k})) + H(T(\varkappa_{n_k}), T(a)) \\ &\leq\ 3dist(\varkappa_{n_k}, T(\varkappa_{n_k})) + \|\varkappa_{n_k} - a\|. \end{aligned}$$

Note that

$$\begin{aligned} \|\varkappa_{n_k} - w\| &\leq\ \|\varkappa_{n_k} - y_{n_k}\| + \|y_{n_k} - w\| \\ &\leq\ 3dist(\varkappa_{n_k}, T(\varkappa_{n_k})) + \|\varkappa_{n_k} - a\| + \|y_{n_k} - w\|, \end{aligned}$$

which implies that

$$limsup_{n\to\infty}\big\|\varkappa_{n_k} - w\big\| \leq limsup_{n\to\infty}\big\|\varkappa_{n_k} - a\big\|.$$

As E has Opial property, we can write that $w = a \in T(a)$. Consequently $a \in F(T)$. Now, we prove that $\{\varkappa_n\}$ has a unique weak subsequentail limit in $F(T)$. To prove this, let w and v weak limits of the subsequence$\{\varkappa_{n_k}\}$ and$\{\varkappa_{n_m}\}$ of $\{\varkappa_n\}$, respectively and $v \neq w$. Firstly, we claim that $w = v$. On the contrary, we suppose that $v \neq w$, then from Lemma 2.2 and using Opial's property, we obtain

$$\begin{aligned} lim_{n\to\infty}\|\varkappa_n - w\| &= lim_{n_k\to\infty}\|\varkappa_{n_k} - w\| \\ &< lim_{n_k\to\infty}\|\varkappa_{n_k} - v\| \\ &= lim_{n\to\infty}\|\varkappa_n - v\| \\ &< lim_{n_m\to\infty}\|\varkappa_{n_m} - v\| \\ &< lim_{n_m\to\infty}\|\varkappa_{n_m} - w\| \\ &= lim_{n\to\infty}\|\varkappa_n - w\| \end{aligned}$$

which is a contradiction. Hence, $w = v$. Therefore, $\{\varkappa_n\}$ converges weakly to a fixed point of T.

Theorem 2.5 Let D be a nonempty compact convex subset of E. Let E, T and $\{\varkappa_n\}$ the same as in the Lemma 2.2. Then $\{\varkappa_n\}$ converges strongly to a fixed point of T if and only if $liminf_{n\to\infty} dist(\varkappa_n, F(T)) = 0$.

Proof: By Lemma 1.5, $F(T) \neq \emptyset$, so by Lemma 2.2, we get $lim_{n\to\infty} dist(\varkappa_n, T(\varkappa_n)) = 0$. Since D is compact, there exists a subsequence $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ such that $\{\varkappa_{n_k}\}$ converges strongly to $\varkappa \in D$. By Lemma 1.4, we have

$$\|\varkappa_{n_k} - T(\varkappa)\| \leq 3dist(\varkappa_{n_k}, T(\varkappa_{n_k})) + \|\varkappa_{n_k} - \varkappa\|$$

letting $k \to \infty$, we get that $\{\varkappa_{n_k}\}$ converges strongly to $T(\varkappa)$. This implies that $T(\varkappa) = \varkappa$. That is, $\varkappa \in F(T)$. By Lemma 1.4, $lim_{n\to\infty}\|\varkappa_n - \varkappa\|$ exists and hence $\varkappa$ is the limit of $\{\varkappa_n\}$.

Definition 2.6 [9] Let D be a subset of a Banach space E. A mapping $T: D \to CB(D)$ satisfies condition (I) if there exists a nondecreasing function $f: (0,\infty) \to 0,\infty)$ with $f(0) = 0$, $f(k) > 0$ for all $k \in (0,\infty)$ such that $d(\varkappa, T\varkappa) \geq f(d(\varkappa, F(T))$ for all $\varkappa \in D$.

Theorem 2.7 Let D be nonempty closed convex subset of a uniformly convex Banach space E. Let $T: D \to CB(D)$ be a mapping satisfying the condition (C) with $F(T) \neq \emptyset$. Let the sequence $\{\varkappa_n\}$ be generated by (1.1). Then $\{\varkappa_n\}$ converges strongly to a fixed point of T if and only if $liminf_{n\to\infty} d(\varkappa_n, F(T)) = 0$.

Proof: The necessity is obvious. Inversely, suppose that $liminf_{n\to\infty} d(\varkappa_n, F(T)) = 0$. From Lemma 2.1, $lim_{n\to\infty}\|\varkappa_n - \varkappa\|$ exists for all $\varkappa \in F(T)$ and by hypothesis $liminf_{n\to\infty} d(\varkappa_n, F(T)) = 0$, so $lim_{n\to\infty} d(\varkappa_n, F(T)) = 0$. Seeing that $lim_{n\to\infty} d(\varkappa_n, F(T)) = 0$, for given $\xi > 0$, assume that for all $n \geq n_0$

$$\begin{aligned} d(\varkappa_n, F(T)) &< \xi/2 \\ &\Rightarrow inf\{\|\varkappa_n - \varkappa\| : \varkappa \in F(T)\} < \xi/2. \end{aligned}$$

In especially, $inf\{\|\varkappa_{n_0} - \varkappa\| : \varkappa \in F(T)\} < \xi/2$; hence there exists a $\varkappa^* \in F(T)$ such that

$$\|\varkappa_{n_0} - \varkappa^*\| < \xi/2.$$

For $m, n \geq n_0$,

$$\begin{aligned} \|\varkappa_{n+m} - \varkappa_n\| &\leq \|\varkappa_{n+m} - \varkappa^*\| + \|\varkappa_n - \varkappa^*\| \\ &\leq 2\|\varkappa_{n_0} - \varkappa^*\| \\ &< 2(\xi/2) = \xi. \end{aligned}$$

Therefore, $\{\varkappa_n\}$ is a Cauchy sequence in D. Hence, there exists a point $q^* \in D$ such that $lim_{n\to\infty}\varkappa_n = q^*$. Also, $lim_{n\to\infty} d(\varkappa_n, F(T)) = 0$ gives that

$$\begin{aligned} dist(q^*, T(q^*)) &\leq \|\varkappa_n - q^*\| + dist(\varkappa_n, T(\varkappa_n)) + H(T(\varkappa_n), T(q^*)) \\ &\leq \|\varkappa_n - q^*\| + \|\varkappa_n\} - u_n\| + \|\varkappa_n - q^*\| \\ &\to 0 \ as\ n \to \infty. \end{aligned}$$

Theorem 2.8 Let E, D, T and $\{\varkappa_n\}$ be as in Theorem 2.7. If T satisfy conditon (I), then $\{\varkappa_n\}$ converges strongly to a fixed point of T.

Proof: By Lemma 2.1 $lim_{n\to\infty}\|\varkappa_n - \varkappa\|$ exists for each $\varkappa \in F(T)$, so $lim_{n\to\infty} d(\varkappa_n, F(T))$ exists. Assume that $lim_{n\to\infty}\|\varkappa_n - \varkappa\| = c$ for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$, from the hypothesis and condition (I), we have

$$f(d(\varkappa_n, F(T))) \leq d(\varkappa_n, T(\varkappa_n)). \quad (2.12)$$

Since $F(T) \neq \emptyset$, so by Lemma 2.2, we have $lim_{n\to\infty} d(\varkappa_n, F(T)) = 0$. So (2.12) implies that

$$lim_{n\to\infty} f(d(\varkappa_n, F(T))) = 0. \quad (2.13)$$

Since f is a decreasing function and $f(0) = 0$, so from (2.13) we obtain

$$lim_{n\to\infty} d(\varkappa_n, F(T)) = 0.$$

Since all the conditions of Theorem 2.7 are satisfied, the conclusion follows from the proof of Theorem 2.7.

References

[1] Nadler, S.B., "Multivalued contraction mappings", Pacific J. Math. 30(2) (1969) : 475-488.

[2] Markin, J.T., "A fixed point theorem for set valued mappings", Bull. Amer. Math. Soc. 74(4) (1968) : 545-547.

[3] Abkar, A., Eslamian, M., "Fixed point theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces", Fixed Point Theory and Applications 2010 (2010) : 1-10.

[4] Yildirim, I., "On convergence of an implicit algorithm for multivalued mappings in Banach spaces", Miskolc Mathematical Notes 15(2) (2014) : 771-780.

[5] Thakur, B.S., Thakur, D., Postolache, M., "A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings", Applied Mathematics and Computation 275 (2016) : 147-155.

[6] Opial,Z., "Weak convergence of the sequence of successive approximations for nonexpansive mappings", Bull Amer. Math. Soc. 73 (1967) : 591-597.

[7] Eslamian, M., Abkar, A., "One-step iterative process for a finite family of multivalued mappings", Mathematical and Computer Modelling, 54 (2011) : 105-111.

[8] Garcia-Falset, J., Lorens-Fusters, E., Suzuki, T., "Fixed point theory for a class of generalized nonexpansive mappings", J.Math. Anal. Appl. 375 (2011) : 185-195.

[9] Kaewchareon, A., Panyanak, B., "Fixed point theorems for some generalized multivalued nonexpansive mappings", Nonlinear Analysis, 74 (2011) : 5578-5584.

[10] Ali, J., Ali, F., Kumar, P., "Approximation of fixed points for Suzuki's generalized nonexpansive mappings", Mathematics 7(6) (2019) : 522.

[11] Yildirim, I., "Strong and weak convergence of an iterative process for a finite family of multivalued mappings satisfying the condition (C)", Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 66 (2017) : 51-65.

[12] Phuengrattana, W., "Approximating fixed points of Suzuki-generalized nonexpansive mappings", Nonlinear Anal. Hybrid Syst. 5(3) (2011) : 583-590.

[13] Akkasriworn, N., Sokhuma, K., Chuikamwong, K., "Ishikawa iterative process for a pair of Suzuki generalized nonexpansive single valued and multivalued mappings in Banach spaces , Int. J. Math. Anal. 6(19) (2012) : 923-932.

[14] Schu, J., "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings", Bull. Aust. Math. Soc. 43 (1991) : 153-159.

[15] Suzuki, T., "Fixed point theorems and convergence theorems for some generalized nonexpansive mappings", J. Math. Anal. Appl. 340(2) (2008) : 1088-1095.

[16] Abkar, A., Eslamian, M., "Convergence theorems for a finite family of generalized nonexpansive multivalued mappings in CAT (0) spaces" Nonlinear Analysis: Theory, Methods & Applications 75(4) (2012) : 1895-1903.

[17] Kaplan, M., Kopuzlu, A., "Three-step iterative scheme for Approximating fixed points of multivalued nonexpansive mappings", Advances in Fixed Point Theory 3(2) (2013) : 273-285.

[18] Sadhu, R., Majee, P., Nahak, C., "Fixed point theorems on generalized α-nonexpansive multivalued mappings", The Journal of Analysis 29 (2021) : 1165-1190.

[19] Ullah, K., Arshad, M., "Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process", *Filomat*, *32*(1) (2018) : 187-196.

[20] Khan, S. H., Yildirim, I., "Fixed points of multivalued nonexpansive mappings in Banach spaces", Fixed Point Theory Appl. (2012) : 73.