Ikonion Journal of Mathematics 4(1) (2022) 9-16

Ikonion Journal of Mathematics

https://dergipark.org.tr/tr/pub/ikjm

Research Article

Open Access

https://doi.org/ 10.54286/ikjm.1005765

ISSN: 2687-6531

The Gershgorin type theorem on localization of the eigenvalues of infinite matrices and zeros of entire functions

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Keywords
Infinite matrix, Compact operator, Localization of spectrum, Zeros of entire functions

Abstract — Let $1 < p < \infty$, $1/p + 1/p' = 1$ and $A = (a_{jk})_{j,k=1}^{\infty}$ be a p-Hille-Tamarkin infinite matrix, i.e.

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{jk}|^{p'}\right)^{p/p'} < \infty.$$

It is proved that the spectrum of A lies in the union of the discs

$$\left\{z \in \mathbb{C} : |a_{jj} - z| \le \left[\sum_{j=1}^{\infty}\Big(\sum_{k=1,k\neq j}^{\infty}|a_{jk}|^{p'}\Big)^{p/p'}\right]^{1/p}\right\} \quad (j = 1,2,...).$$

In addition, an application of that result to finite order entire functions is discussed. An illustrative example is also presented.

Subject Classification (2020): 47A10; 47A11; 30B10.

1. Introduction and statement of the main result

Let $1 < p < \infty$, $1/p + 1/p' = 1$ and $A = (a_{jk})_{j,k=1}^{\infty}$ be a p-Hille-Tamarkin infinite matrix, i.e.

$$c_p(A) := \left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{jk}|^{p'}\right)^{p/p'}\right]^{1/p} < \infty.$$

The paper is devoted to the localization of the eigenvalues of such matrices.

The literature on the localization of the eigenvalues of finite and infinite matrices is very rich, cf. [1, 3, 5, 9, 14, 15, 18, 19] and the references which are given therein. At the same time, to the best of our knowledge, the location of the eigenvalues of Hille-Tamarkin matrices has not been considered in the available literature.

As is well-known, Hille-Tamarkin matrices represent numerous integral operators, arising in various applications, cf. [17]. About properties of Hille-Tamarkin matrices, see for instance, [17], [6],[7, Section 18]. In particular, in the well-known book [17], the convergence of the powers of the eigenvalues of these matrices is investigated. The works [6, 7] deal with infinite matrices, whose upper-triangular parts are Hille-Tamarkin matrices. Besides, the invertibility and positive invertibility conditions are explored, as well as

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Article History: Received: 10.10.2021 - Accepted: 03.03.2022 - Published: 10.03.2022

upper bounds for the spectra have been derived.

Denote

$$\tau_p(A) := \left[\sum_{j=1}^{\infty}\left(\sum_{k=1,k\neq j}^{\infty}|a_{jk}|^{p'}\right)^{p/p'}\right]^{1/p}.$$

Throughout the paper $\lambda_k(A)$ $(k=1,2,...)$ are the eigenvalues of A taken with their multiplicities and enumerated in the non-increasing way of the absolute values: $|\lambda_{k+1}(A)| \le |\lambda_k(A)|$ $(k=1,2,...)$, and $\sigma(A)$ is the spectrum of A as the operator in l^p. Recall that l^p is the Banach space of the sequences $x=(x_k)_{k=1}^{\infty}$ of complex numbers with the finite norm

$$\|x\|_{l^p} = \sum_{k=1}^{\infty}(|x_k|^p)^{1/p}.$$

The following theorem is the main result of this paper.

Theorem 1.1. Let $c_p(A) < \infty$ for a finite $p > 1$. Then with the notation

$$U_{j,p}(A) := \{z \in \mathbb{C} : |a_{jj} - z| \le \tau_p(A)\} \ \ (j = 1,2,...),$$

one has

$$\sigma(A) \subset \cup_{j=1}^{\infty} U_{j,p}(A).$$

The proof of this theorem is presented in the next section.

For a positive integer n, let $\mathbb{C}^{n\times n}$ be the set of $n \times n$-matrices and $A_n = (a_{jk})_{j,k=1}^{n} \in \mathbb{C}^{n\times n}$. Recall that by the Gershgorin theorem [15],

$$\sigma(A_n) \subset \cup_{j=1}^{n} \hat{U}_j(A_n),$$

where

$$\hat{U}_j(A_n) := \{z \in \mathbb{C} : |a_{jj} - z| \le \sum_{k=1,k\neq j}^{n} |a_{jk}|\} \ \ (j = 1,...,n).$$

This result can be easily extended to the infinite dimensional case, provided

$$\sup_j \sum_{k=1}^{\infty} |a_{jk}| < \infty.$$

Thus, Theorem 1.1 can be considered as an extending of the Gershgorin theorem to a finite $p > 1$. Furthermore, the quantity $\hat{s}(A) := \sup_{j,k}|\lambda_j(A) - \lambda_k(A)|$ is called the spread of A. In the finite dimensional case the spread plays an essential role, cf. [15, Section III.4]. Since

$$|\lambda_j(A) - \lambda_k(A)| \le |\lambda_j(A) - a_{jj}| + |\lambda_k(A) - a_{kk}| + |a_{jj} - a_{kk}|,$$

for a p-Hille-Tamarkin matrix A Theorem 1.1 gives us the inequality

$$\hat{s}(A) \le \sup_{j,k}|a_{jj} - a_{kk}| + 2\tau_p(A). \tag{1.1}$$

Similarly, let $A = (a_{jk})$ and $B = (b_{jk})$ be two p-Hille-Tamarkin matrices. Then the quantity $s(A,B) := \sup_{j,k}|\lambda_j(A)-$

$\lambda_k(B)|$ will be called the mutual spread of A and B. Since

$$|\lambda_j(A)-\lambda_k(B)| \le |\lambda_j(A)-a_{jj}| + |\lambda_k(B)-b_{kk}| + |a_{jj}-b_{kk}|,$$

Theorem 1.1 gives us the inequality

$$s(A,B) \le \sup_{j,k} |a_{jj}-b_{kk}| + \tau_p(A) + \tau_p(B). \tag{1.2}$$

Let $r_s(A)$ denote the spectral radius of A. Then clearly, $|r_s(A)-r_s(B)| \le s(A,B)$. Now we can apply inequality (1.2).

2. Proof of Theorem 1.1

Let $A_n = (a_{jk})_{j,k=1}^n \in \mathbb{C}^{n\times n}$, $\mu \in \sigma(A_n)$ and $x=(x_j)$ be the eigenvector of A_n corresponding to μ:

$$\sum_{k=1}^n a_{jk}x_k = \mu x_j \ \ (j=1,...n).$$

Then

$$(\mu - a_{jj})x_j = \sum_{k=1,k\ne j}^n a_{jk}x_k$$

and

$$|a_{jj}-\mu||x_j| \le \sum_{k=1,k\ne j}^n |a_{jk}||x_k| \ \ (j=1,...,n).$$

So by the Hőlder inequality,

$$|a_{jj}-\mu|^p|x_j|^p \le (\sum_{k=1,k\ne j}^n |a_{jk}|^{p'})^{p/p'} \sum_{i=1}^n |x_i|^p$$

and

$$\sum_{j=1}^n |a_{jj}-\mu|^p|x_j|^p \le \tau_p^p(A_n) \sum_{i=1}^n |x_i|^p.$$

Here

$$\tau_p(A_n) := \left[\sum_{j=1}^n \left(\sum_{k=1,k\ne j}^n |a_{jk}|^{p'} \right)^{p/p'} \right]^{1/p}.$$

Consequently,

$$\min_j |a_{jj}-\mu| \le \tau_p(A_n).$$

In other words, for any eigenvalue μ of A_n, there is an integer $m \le n$, such that $|a_{mm}-\mu| \le \tau_p(A_n)$. We thus have proved the following result.

Lemma 2.1. Let $A_n \in \mathbb{C}^{n\times n}$. Then for any finite $p>1$ we have

$$\sigma(A_n) \subset \cup_{j=1}^n U_{j,p}(A_n),$$

where

$$U_{j,p}(A_n) = \{z \in \mathbb{C} : |a_{jj}-z| \le \tau_p(A_n)\}.$$

Proof of Theorem 1.1: By the Hőlder inequality we have

$$\|A\|_{l^p} \le c_p(A), \tag{2.1}$$

where $\|A\|_{l^p}$ means the operator norm of A in l^p. Since $\tau_p(A_n) \to \tau_p(A)$ as $n \to \infty$, according to (2.1), $A_n \to A$ in the operator norm and therefore, by the upper semicontinuity of spectra [11, Theorem IV.3.1], for any finite k we have $\lambda_k(A_n) \to \lambda_k(A)$ as $n \to \infty$. Now Lemma 2.1 implies the required result. □

3. Applications to entire functions

Let us consider the entire function

$$h(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{(k!)^\alpha} \ \ (0 < \alpha \le 1,\ z \in \mathbb{C},\ a_0 = 1, a_k \in \mathbb{C}, k \ge 1). \tag{3.1}$$

Denote the zeros of h with the multiplicities in non-decreasing order of their absolute values by $z_k(h)$: $|z_k(h| \le |z_{k+1}(h)|$ $(k = 1,2,...)$ and assume that for a $p > 1/\alpha$,

$$\theta_p(h) := \sum_{k=2}^{\infty} |a_k|^{p'} < \infty \ \ (1/p + 1/p' = 1). \tag{3.2}$$

Introduce the quantity

$$b_p(h) := [\theta_p^{p/p'}(h) + \sum_{j=2}^{\infty} \frac{1}{j^{\alpha p}}]^{1/p} = [\theta_p^{p/p'}(h) + \zeta(\alpha p) - 1]^{1/p},$$

where

$$\zeta(z) = \sum_{k=1}^{\infty} k^{-z} \ \ (\text{Re}\, z > 1)$$

is the Riemann zeta function. Our aim in this section is to prove the following theorem.

Theorem 3.1. Let h be defined by (3.1) and condition (3.2) hold. Then for any zero $z(h)$ of h we have either $|a_1 - \frac{1}{z(h)}| \le b_p(h)$ or $|z(h)| \ge \frac{1}{b_p(h)}$.

To prove this theorem introduce the polynomial

$$f_n(z) = \sum_{k=0}^{n} \frac{a_k z^{n-k}}{(k!)^\alpha}$$

and $n \times n$-matrix

$$F_n = \begin{pmatrix} -a_1 & -a_2 & ... & -a_{n-1} & -a_n \\ 1/(2^\alpha) & 0 & ... & 0 & 0 \\ 0 & 1/(3^\alpha) & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & 1/(n^\alpha) & 0 \end{pmatrix}.$$

Let $z_k(f_n)$ $(k = 1,...,n)$ be the zeros of f_n with their multiplicities enumerated in non-increasing order of their absolute values, and $\lambda_k(F_n)$ be the eigenvalues of F_n taken with the multiplicities enumerated in the non-increasing order of their absolute values.

Lemma 3.2. One has $\lambda_k(F_n) = z_k(f_n)$ $(k = 1, ..., n)$.

Proof.

Clearly, f_n is the characteristic polynomial of the matrix

$$B = \begin{pmatrix} -a_1 & -\frac{a_2}{2^\alpha} & \dots & -\frac{a_{n-1}}{((n-1)!)^\alpha} & -\frac{a_n}{(n!)^\alpha} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Following [8, Lemma 5.2.1, p. 117], put

$$m_k = \frac{1}{k^\alpha} \text{ and } \psi_k = \frac{1}{(k!)^\alpha} = m_1 m_2 \cdots m_k \ \ (k = 1, ..., n).$$

Then

$$F_n = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ m_2 & 0 & \dots & 0 & 0 \\ 0 & m_3 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & m_n & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -a_1 & -a_2\psi_2 & \dots & -a_{n-1}\psi_{n-1} & -a_n\psi_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let μ be an eigenvalue of B, i.e. for the eigenvector $(x_k)_{k=1}^n \in \mathbb{C}^n$, we have

$$-a_1 x_1 - a_2\psi_2 x_2 - \dots - a_{n-1}\psi_{n-1} x_{n-1} - a_n \psi_n x_n = \mu x_1,$$

$$x_k = \mu x_{k+1} \ \ (k = 1, ..., n-1).$$

Since $\psi_1 = 1$, substituting $x_k = y_k/\psi_k$, we obtain

$$-a_1 y_1 - a_2 y_2 \dots - a_{n-1} y_{n-1} - a_n y_n = \mu y_1$$

and

$$\frac{y_k}{\psi_k} = \mu \frac{y_{k+1}}{\psi_{k+1}} \ \ (k = 1, ..., n-1).$$

Or

$$m_{k+1} y_k = \frac{y_k \psi_{k+1}}{\psi_k} = \mu y_{k+1} \ \ (k = 1, ..., n-1).$$

These equalities are equivalent to the equality $F_n y = \mu y$ with $y = (y_k)$. In other words $TBT^{-1} = F_n$, where

$T = \operatorname{diag}(1, \psi_2, ..., \psi_n)$ and therefore

$$T^{-1} = \operatorname{diag}(1, \frac{1}{\psi_2}, ..., \frac{1}{\psi_n}).$$

This proves the lemma. □

The simple calculations show that

$$\tau_p(F_n) = [(\sum_{k=2}^{n} |a_k|^{p'})^{p/p'} + \sum_{j=2}^{n} \frac{1}{j^{\alpha p}}]^{1/p}.$$

Due to Lemmas 2.1 and 3.2, for any zero $z(f_n)$ of f_n either

$$|a_1 - z(f_n)| \le \tau_p(F_n), \text{ or } |z(f_n)| \le \tau_p(F_n). \tag{3.3}$$

With

$$h_n(z) = z^n f_n(1/z) = \sum_{k=0}^{n} \frac{a_k z^k}{(k!)^\alpha},$$

we obtain

$$z_k(h_n) = \frac{1}{z_k(f_n)} = \frac{1}{\lambda_k(F_n)} \quad (k = 1, ..., n).$$

Here $z_k(h_n)$ are the zeros of h_n with their multiplicities enumerated in non-decreasing order of their absolute values. According to (3.3) for any zero $z(h_n)$ of h_n either

$$|a_1 - \frac{1}{z(h_n)}| \le \tau_p(F_n), \text{ or } |z(h_n)| \ge \tau_p(F_n). \tag{3.4}$$

Proof of Theorem 3.1: Clearly, $\tau_p(F_n) \to b_p(h)$ as $n \to \infty$. In each compact domain $\Omega \in \mathbb{C}$, we have $h_n(z) \to h(z)$ $(n \to \infty)$ uniformly in Ω. Due to the Hurwitz theorem [16, p. 5] $z_k(h_n) \to z_k(h)$ for $z_k(h) \in \Omega$. Now (3.4) prove the theorem. □

From Theorem 3.1 it follows

$$\inf |z_j(h)| \ge \frac{1}{|a_1| + b_p(h)}$$

So the disc $|z| < \frac{1}{|a_1| + b_p(h)}$ is a zero-free domain.

Note that the classical results on the zeros of entire functions are presented in [13]; the recent investigations on localization of the zeros of entire functions can be found, for instance, in the works [2, 4, 8, 10, 12] and the references which are given therein.

4. Example

The following example characterizes the sharpness of Theorem 1.1.

Let $A = \operatorname{diag}(B_j)_{j=1}^{\infty}$, where

$$B_j = \begin{pmatrix} \frac{1}{j} & \frac{\sqrt{3}}{2j} \\ \frac{\sqrt{3}}{2j} & \frac{1}{j} \end{pmatrix} \quad (j = 1, 2, ...).$$

Under consideration we have

$$\tau_2(A) = [2\sum_{j=1}^{\infty}\frac{3}{4j^2}]^{1/2} = \sqrt{3\zeta(2)/2} \approx \sqrt{3\cdot 1.645/2} \le 1.570.$$

By Theorem 1.1

$$\sigma(A) \subseteq \cup_{j=1}^{\infty}\{z \in \mathbb{C} : |z - \frac{1}{j}| \le 1.570\}.$$

Simple calculations show that $\lambda_1(B_j) = \frac{3}{2j}$, $\lambda_2(B_j) = \frac{1}{2j}$ $(j = 1, 2, ...)$.

Author Contributions

The author read and approved the final version of the manuscript.

Conflicts of Interest

The author declares no conflict of interest.

Acknowledgement

The author is very grateful to the referee of this paper for his (her) helpful remarks.

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