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Modified Finite Difference Method for Solution of Two-interval Boundary Value Problems with Transition Conditions

Semih Çavuşoğlu[1,*], Oktay Sh. Mukhtarov[2,3]

[1] *Institute of Graduate Studies, Tokat Gaziosmanpaşa University, Tokat, Turkey.*
[2] *Department of Mathematics, Faculty of Science, Tokat Gaziosmanpaşa University, Tokat, Turkey.*
[3] *Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan.*

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Abstract. In this study, we have proposed a new modification of classical Finite Difference Method (FDM) for the solution of boundary value problems which are defined on two disjoint intervals and involved additional transition conditions at an common end of these intervals. The proposed modification of FDM differs from the classical FDM in calculating the iterative terms of numerical solutions. To illustrate the efficiency and reliability of the proposed modification of FDM some examples are solved. The obtained results are compared with those obtained by the standart FDM and by the analytical method. Corresponding graphical illustration are also presented.

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1. Introduction

A lot of mechanical and physical processes are modeled by linear or nonlinear differential equations, whose exact solutions are impossible to find by using analytical methods. Many researchers have tried to do this in various semi-analytical, numerical and approximate methods, such as the Finite Element Methods, the Adomian Decomposition Methods, the Differential Transform Method, the Explicit Euler Method, the Taylor's Expansion Method etc. One of them is the Finite Difference Method, which can be applied to wide class of problems appearing in mathematical physics and engineering. Many important theoretical and numerical results have been obtained during the last seven decades regarding the stability, accuracy and convergence of the FDM for different type initial and/or BVPs (see, [1, 3–6, 9] and references cited therein).

The standard FDM is intended for solving one-interval initial and/or boundary value problems without jump conditions.

Note that, finite difference methods are numerical methods for pproximating the solution to various type differential equations. The idea is to replace ordinary or partial derivatives appearing in the boundary-value problem by finite differences that approximate them [2, 8, 9, 12, 13].

Note that the finite difference methods deal without interior singular point and corresponding transmission conditions. It is our main goal here to develop finite difference method to deal with an additional transmission conditions

*Corresponding Author
Email addresses: semihcavusoglu@gmail.com (S. Çavuşoğlu), omukhtarov@yahoo.com (O.Sh. Mukhtarov)

at the interior singular point. Based on FDM, we have developed a new technique for solving two-interval Sturm-Liouville problems (SLPs), that included additional transition conditions across the common endpoint of these intervals. We note that, some important theoretical aspects of Sturm-Liouville problems with transition conditions were studied in [7, 10, 11].

2. Analysis of the Method

Let y(x) represent a function of one variable that, unless otherwise stated, will always be assumed to be smooth, meaning that we can differentiate the function several times and each derivative is a well-defined bounded function over an interval containing a particular point of interest x.

Let us consider a linear boundary-value problem for two-interval Sturm-Liouville equation

$$y'' + p(x)y' + q(x)y = f(x), \quad x \in [a, c) \cup (c, b] \tag{2.1}$$

together with the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \tag{2.2}$$

where $p(x)$, $q(x)$ and $f(x)$ are continious functions on $[a, c) \cup (c, b]$ having finite limit values $p(c \pm 0), q(c \pm 0)$ and $f(c \pm 0)$, respectively, α, β are real numbers. To discretize the problem (2.1), (2.2) the definition range $[a, b]$ is divided into N equal ranges $[x_0, x_1], [x_1, x_2], ..., [x_{N-1}, x_N]$ that is,

$$x_i = a + ih, \quad h = \frac{b-a}{N}, \quad i = 0, 1, 2, ..., N.$$

By using the Taylor expansion

$$y(x_i + h) \approx y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \dots,$$

we can express the first derivative in the ordinary differential equation using one of the following apprroximate expression

$$D_+y(x) \approx \frac{y(x+h) - y(x)}{h},$$

$$D_-y(x) \approx \frac{y(x) - y(x-h)}{h},$$

or

$$D_0y(x) \approx \frac{y(x+h) - y(x-h)}{2h},$$

where $D_+y(x)$, $D_-y(x)$ and $D_0y(x)$ denotes the forward finite difference, backward finite difference and centered finite difference of the unknown solution $y(x)$, respectively.

The first and second derivative expressions in the boundary value problem can be expressed in the same way, as

$$y'(x) \approx \frac{D_+y(x) + D_-y(x)}{2} \tag{2.3}$$

and

$$y''(x) \approx \frac{D_+y(x) - D_-y(x)}{h}. \tag{2.4}$$

Let us define the finite difference solution for $y(x)$ at all grid points $x_0, x_1, \cdots, x_N$ by $y_i = y(x_i)$. Substituting (2.3) and (2.4) in the boundary value problem (2.1)-(2.2), we have the following linear system of algebraic equations

$$\left(1 - \tfrac{1}{2}hp_i\right) y_{i-1} + \left(-2 + h^2 q_i\right) y_i + \left(1 + \tfrac{1}{2}hp_i\right) y_{i+1} = h^2 f(x_i)$$

$$1 \le i \le N-1, \qquad i = 1, 2, 3, ..., N-1,$$

where

$$y_0 = \alpha, \quad y_N = \beta.$$

Note that each equation of this system involves solution values at three nodal points x_{i-1} , x_i and x_{i+1}. The linear system of algebraic equations can be written in the matrix and vector form

$$My = B, \tag{2.5}$$

where M is a tridioganal matrix of size $(N-1)\times(N-1)$, given by

$$M = \begin{pmatrix} -2+h^2q_1 & 1+\frac{1}{2}hp_1 & 0 & \cdots & 0 & 0 & 0 \\ 1-\frac{1}{2}hp_2 & -2+h^2q_2 & 1+\frac{1}{2}hp_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & -2+h^2q_{N-2} & 1+\frac{1}{2}hp_{N-2} & 0 \\ 0 & 0 & 0 & \cdot & 0 & 1-\frac{1}{2}hp_{N-1} & -2+h^2q_{N-1} \end{pmatrix}_{(N-1)\times(N-1)}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix}_{(N-1)\times 1} \quad \text{and} \quad B = \begin{pmatrix} h^2 f(x_1) - \left(1-\frac{1}{2}hp_1\right)\alpha \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{N-2}) \\ h^2 f(x_{N-1}) - \left(1-\frac{1}{2}hp_1\right)\beta \end{pmatrix}_{(N-1)\times 1}.$$

This is the tridiagonal linear system of algebraic equations (2.5) and therefore can be solved efficiently by the Crout or Cholesky algoritm [3].

3. Convergence and Error Estimates of Finite Difference Method

When the FDM is applied to solve a boundary value problem, it is very important to know how accurate the numerical solution is compared to the exact solution.

3.1. **Global Error.** Let $\tilde{Y} = (y_1, y_2, \ldots, y_n)$ denote the finite difference solution and $\tilde{y} = (y(x_1), y(x_2), \ldots, y(x_n))$ is the exact solution at the grid points $x_1, x_2, \ldots, x_n$. Then, the vector

$$\tilde{E} = (y_1 - y(x_1), y_2 - y(x_2), \ldots, y_n - y(x_n)) = \tilde{Y} - \tilde{y}$$

is said to be the global error vector. You usually want to find an admissible upper bound for this error with respect to the infinite norm (so-called maximum norm), defined by

$$\| \tilde{E} \| = \max_{1\leqslant i\leqslant n} | y_i - y(x_i) |$$

or p-norm ($p \geq 1$)

$$\| \tilde{E} \|_p = \left(\sum_{i=1}^{n} | y_{x_i} - y(i) |^p \, (x_{i+1} - x_i) \right)^{1/2}.$$

Denote

$$h_i := \max_{1\leqslant i\leqslant n} (x_{i+1} - x_i).$$

If $\| \tilde{E} \|_p$ converges to zero as $h \to 0$, then a finite difference method is called convergent. Moreover, if there is $c \geq 0$ such that

$$\| \tilde{E} \|_p \leq Ch^q, q > 0,$$

the FDM is called q-th order accurate.

Definition 3.1. A FDM is called convergent, if

$$\lim_{h\to 0} \| \tilde{E} \| = 0.$$

4. Local Truncation Errors

We shall show that the FDM solution converges to the exact solution of the BVP (2.1)- (2.2) when h converges to zero. Using formulas (2.3) and (2.4), one can show that the exact solution $\tilde{y} = (y(x_1), y(x_2), \ldots, y(x_n))$ satisfies the following linear system of equation

$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} - \frac{h^2}{12}u^{(4)}(\xi_i) + p_i\frac{u(x_{i+1}) - u(x_{i-1})}{2h} - \frac{h^2}{6}u^{(3)}(\eta_i) + q_i u(x_i) = f(x_i), \quad 1 \leqslant i \leqslant n,$$

for same $\xi_i \in [a, b]$.
On the other hand the FDM solution $\tilde{Y} = (y_1, y_2, \ldots, y_n)$ satisfies the linear system of equation

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + p_i\frac{u_{i+1} - u_{i-1}}{2h} + q_i u_i = f_i, \quad 1 \leqslant i \leqslant n.$$

Substracting these equation one from the other, we get

$$\frac{e_{i+1} - 2e_i + e_{i-1}}{h^2} + p_i\frac{e_{i+1} - e_{i-1}}{2h} + q_i e_i = h^2 f_i, \quad 1 \leqslant i \leqslant n, \tag{4.1}$$

where e_i is the global error $e_i := u(x_i) - u_i$ and $h^2 f_i$ is the local truncation error at the grid point $x = x_i$ and

$$f_i = \frac{1}{12}u^{(4)}(\xi_i) - \frac{1}{6}u^{(3)}(\eta_i).$$

After multiplying both sides of (4.1) by h^2 and then collecting the corresponding terms, we have

$$\left(1 - \frac{h}{2}p_i\right)e_{i-1} + \left(-2 + h^2 q_i\right)e_i + \left(1 + \frac{h}{2}p_i\right)e_{i+1} = h^4 f_i. \tag{4.2}$$

To estimate the magnitude of the error vector $\tilde{e}$, it is necessary to use an infinite norm $\| \tilde{e} \|_p$, $p \geq 1$ for some spesific value p.
We will apply the infinite norm $\| \tilde{e} \|_\infty$, because it is used to measure grid functions and is easily estimated.
The equation (4.2) can be written as

$$(2 + h^2 q_i)e_i = (1 - \frac{h}{2}p_i)e_{i+1} - (1 + \frac{h}{2}p_i)e_i + h^4 f_i.$$

Consequently,

$$\begin{aligned} | 2 + h^2 q_i \| e_i | \leq & | 1 - \frac{h}{2}p_i \| e_{i+1} | + | 1 + \frac{h}{2}p_i \| e_i | + h^4 | f_i | \\ \leq & | 1 - \frac{h}{2}p_i \|| \tilde{e} \|_\infty + | 1 + \frac{h}{2}p_i \|| \tilde{e} \|_\infty + h^4 \| \tilde{f} \|_\infty, \end{aligned}$$

where $\| \tilde{f} \|_\infty = \max\limits_{1\leqslant i\leqslant n} | f_i |$.
From this inequality, it follows immediately that

$$| 2 + h^2 q_i \|| \tilde{e} \|_\infty \leq \left(| 1 - \frac{h}{2}p_i | + | 1 + \frac{h}{2}p_i |\right) \| \tilde{e} \|_\infty + h^4 \| \tilde{f} \|_\infty . \tag{4.3}$$

Since $q(x) < 0$, one can choose $h > 0$ small enough to satisfy

$$| 1 - \frac{h}{2}p_i | + | 1 + \frac{h}{2}p_i | = 2 \quad \text{and} \quad | 2 + h^2 q_i | = 2 + h^2 | q_i |,$$

for all $i = 1, 2, \ldots, n$.
Consequently, for sufficiently small $h > 0$ we have from (4.3) that

$$| q_i \|| \tilde{e} \|_\infty \leq h^2 \| \tilde{f} \|_\infty .$$

Denoting $C = \frac{\|\tilde{f}\|_\infty}{\min\limits_{1\leqslant i\leqslant n}|q_i|}$, we obtain

$$\| \tilde{e} \|_\infty \leq Ch^2.$$

Hence, the FDM is convergent and 2-order accurate.

5. Numerical Example

Consider the following two-interval SLP, consisting of the differential equation

$$xy^{''} + 2y^{'} - xy = e^x, \quad x \in [1, 2) \cup (2, 3], \tag{5.1}$$

subject to the boundary conditions at the endpoints $x = 1$ and $x = 3$, given by

$$y(1) = 2, \quad y(3) = -1 \tag{5.2}$$

together with transition conditions across the common endpoint $x = 2$, given by

$$y(2^-) = 3y(2^+), \quad 2y'(2^-) = y'(2^+). \tag{5.3}$$

At first, we consider the problem (5.1)-(5.3) without jump conditions (5.3). It is easy to verify that the function

$$y = \frac{e^{x+4}(x-3) - 4e^{x-1} - 6e^{x+1} - e^x(x-1) + 2e^4(e^3 + 2e^2 + 3)}{2(e^4 - 1)x} \tag{5.4}$$

satisfies the equation (5.1) on whole $[1, 2) \cup (2, 3]$ and both boundary conditions (5.2). For simplicity, we will use the uniform cartesian grid

$$x_i = 1 + ih, \quad i = 0, 1, \ldots, 50,$$

for $h = 0,06$. In particular, we have $x_0 = 2, x_{50} = -1$.
The central finite difference (CFD) approximation of the derivatives $y^{'}$ and $y^{''}$ are defined by

$$y'(x) \approx \frac{1}{2}\left(D_+y(x) + D_-y(x)\right)$$

and

$$y''(x) \approx \frac{1}{h}\left(D_+y(x) - D_-y(x)\right),$$

where $D_+y(x)$ and $D_-y(x)$ denotes the forward finite difference and backward finite difference of $y(x)$.
By applying the CFD to the differential equation (5.1) at a typical grid point $x = x_i$ and denoting $y_i = y(x_i)$, we have the following finite difference equations

$$(2 - x_i)y_{i-1} - (4 + 2h^2)x_iy_i + 2(h + x_i)y_{i+1} = 2h^2e^{x_i}, \quad i = 1, 2, \ldots, 49. \tag{5.5}$$

That is, we have the linear algebraic system of equations with respect to the variables $y_1, y_2, \ldots, y_{49}$. The system of linear algebraic equations (5.5) can be written in a tridiagonal matrix-vector form

$$Ay = b,$$

where

$$A = \begin{pmatrix} -(4+2h^2)x_1 & 2+2x_1 & 0 & \cdots & 0 & 0 & 0 \\ 2x_2 - 2h & -(4+2h^2)x_2 & 2+2x_2 & \cdots & 0 & 0 & 0 \\ 0 & 2x_3 - 2h & -(4+2h^2)x_3 & 2+2x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x_{48} - 2h & -(4+2h^2)x_{48} & 2+2x_{48} \\ 0 & 0 & 0 & \cdots & 0 & 2x_{49} - 2h & -(4+2h^2)x_{49} \end{pmatrix},$$

$$y = \begin{pmatrix} y_1 \\ \\ y_2 \\ \\ y_3 \\ \vdots \\ y_{48} \\ \\ y_{49} \end{pmatrix}, \qquad b = \begin{pmatrix} 2h^2 e^{x_1} - 4(x_1 - h) \\ \\ 2h^2 e^{x_2} \\ \\ 2h^2 e^{x_3} \\ \vdots \\ 0 \\ \\ 2h^2 e^{x_{49}} + 2(x_{49} + h) \end{pmatrix}.$$

The solution of this system can be found by using MATLAB-Octave. The obtained numerical FDM solutions are graphically compared with the exact solution (5.4) (see, Figures 1, 2, 3 and 4).

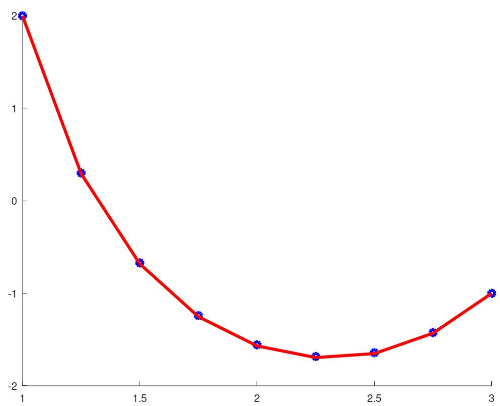

FIGURE 1. The FDM-solution and exact solution for the problem (5.1)-(5.2) where N=8

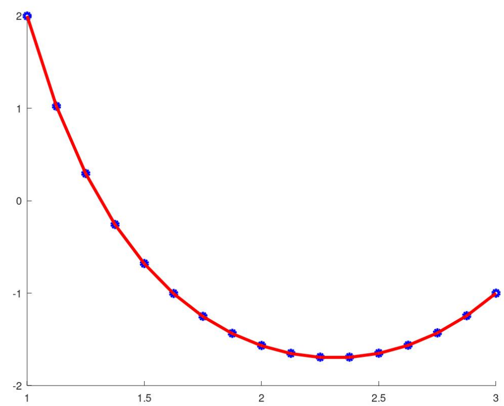

FIGURE 2. The FDM-solution and exact solution for the problem (5.1)-(5.2) where N=16

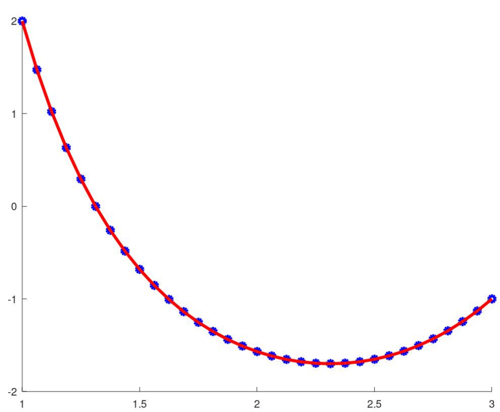

FIGURE 3. The FDM-solution and exact solution for the problem (5.1)-(5.2) where N=32

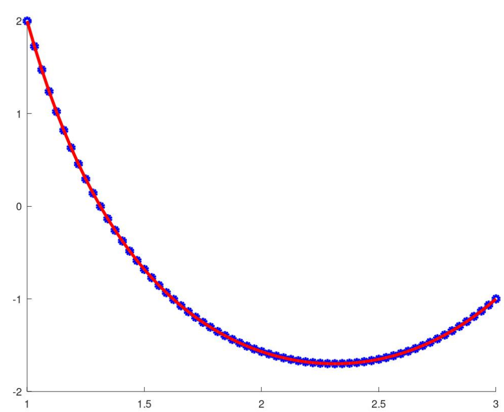

FIGURE 4. The FDM-solution and exact solution for the problem (5.1)-(5.2) where N=64

TABLE 1. Maximum absolute error (MAE) for the problem (5.1)-(5.2)

N	h	$\|\| e \|\|_\infty$	N	h	$\|\| e \|\|_\infty$
4	$\frac{3}{4}$	0.046856	128	$\frac{3}{128}$	0.000047501
8	$\frac{3}{8}$	0.012013	256	$\frac{3}{256}$	0.000011876
16	$\frac{3}{16}$	0.0030277	512	$\frac{3}{512}$	0.0000029690
32	$\frac{3}{32}$	0.00075960	1024	$\frac{3}{1024}$	0.00000074225
64	$\frac{3}{64}$	0.00018998	2048	$\frac{3}{2048}$	0.00000018556

5.1. **Remark.** In figures 1,2,3 and 4 the exact solution (5.4) is compared with the numerical FDM solutions for $N = 8, 16, 32, 64$, respectively. It can be seen from these graphical illustrations that, the error between the FDM solutions and the exact solution decreases as the number of grid points N increases.

6. Solution of Transition Problem

Now, we will investigate the problem (5.1)-(5.3). If we select $N = 32$ and apply the transition conditions (5.3), then we have two additional algebraic equations

$$y_{16} - 3\tilde{y}_0 = 0 \tag{6.1}$$

and

$$2y_{14} - 2y_{16} - \tilde{y}_0 + \tilde{y}_2 = 0. \tag{6.2}$$

Note that, each equation of this system involves solution values at three nodal points x_{i-1} , x_i and x_{i+1}.
By adding equations (5.5) to the system of equations (6.1) and (6.2), a linear equation system is obtained, in the form

$$My = B,$$

where the matrix M is not tridiagonal. The solution of this linear system of algebraic equations can be found by using MATLAB/Octave.

In the Figures 5, 6, 7 and 8 the finite difference solution of the problem (5.1)-(5.3) is graphically compared with the exact solution.

TABLE 2. Maximum absolute error (MAE) for transition problem

N	h	$\|\| e \|\|_\infty$
8	$\frac{2}{8}$	0.69141
16	$\frac{2}{16}$	0.33075
32	$\frac{2}{32}$	0.16166
64	$\frac{2}{64}$	0.078916

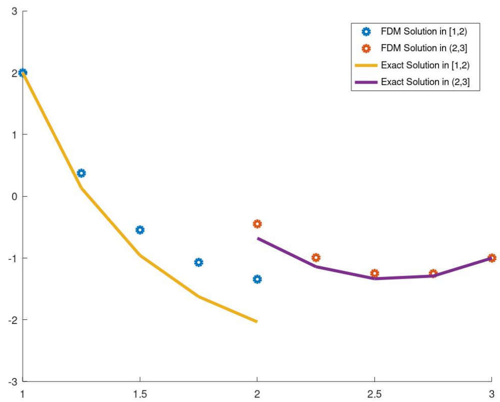

FIGURE 5. The FDM-solution and exact solution for the problem (5.1)-(5.3) where N=8

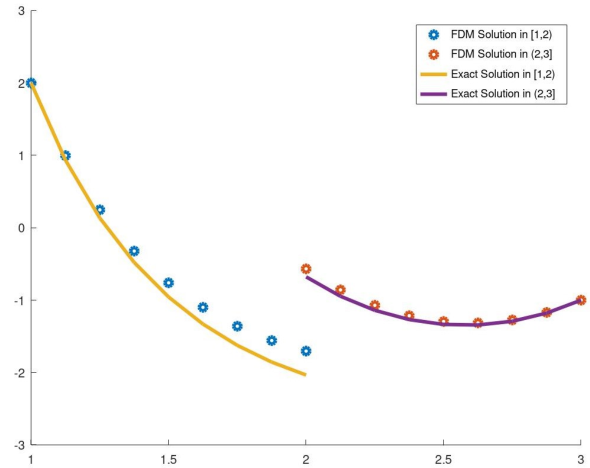

FIGURE 6. The FDM-solution and exact solution for the problem (5.1)-(5.3) where N=16

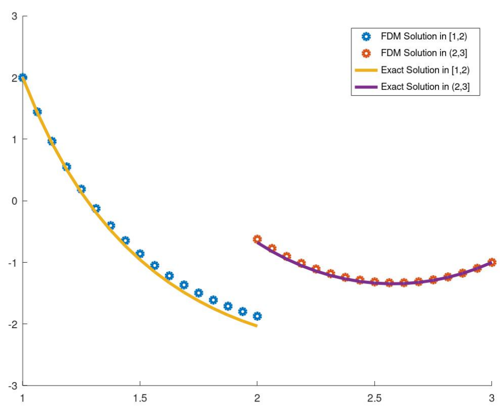

FIGURE 7. The FDM-solution and exact solution for the problem (5.1)-(5.3) where N=32

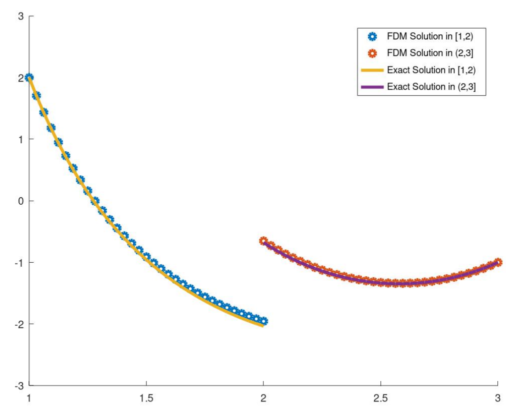

FIGURE 8. The FDM-solution and exact solution for the problem (5.1)-(5.3) where N=64

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

AUTHORS CONTRIBUTION STATEMENT

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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