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Research Article

Compact Embedding Theorems for The Space of Functions with Wavelet Transform in Amalgam Spaces

Öznur Kulak[1*]

[1*] Amasya University, Faculty of Arts and Sciences, Department of Mathematics, Amasya, Turkey, (ORCID: 0000-0003-1433-3159)
oznur.kulak@amasya.edu.tr

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Abstract

This paper is motivated to define the space $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ using the wavelet transform, and is also motivated to consider the inclusion and compact embedding theorems in this space.

Keywords: Wavelet transform, Dilation operator, Translation operator, Compact embedding, Inclusion.

Dalgacık Dönüşümleri Amalgam Uzaylarında Olan Fonksiyon Uzayları için Kompakt Gömülme Teoremleri

Öz

Bu çalışma dalgacık dönüşümü kullanarak $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ uzayını tanımlamak ve ayrıca bu uzayda kapsama, kompakt gömülme teoremlerini incelemek için motive edilmiştir.

Anahtar Kelimeler: Dalgacık dönüşümü, Açılma operatörü, Öteleme operatörü, Kompakt gömülme, Kapsama.

[*] Corresponding Author: oznur.kulak@amasya.edu.tr

1. Introduction

In this paper, the space $C_c^\infty(\mathbb{R})$ denotes the space of infinitely differentiable complex-valued functions with compact supported on $\mathbb{R}$ (Gasquet & Witomski, 1999). The parameters in wavelet theory are "*time*" x and "*scale*" s. The translation operator T_x is defined by

$$T_x f(t) = f(t-x)$$

for all $x, t \in \mathbb{R}$. The dilation operator D_s is given by

$$D_s f(t) = |s|^{-\frac{1}{2}} f\left(\frac{t}{s}\right)$$

for all $t \in \mathbb{R}$, $0 \neq s \in \mathbb{R}$. The dilation operator preserves the shape of signals and also changes the scale. This operator acts as a microscope. The continuous wavelet transform of a function f with respect to wavelet g is defined by for $x \in \mathbb{R}$ and $0 \neq s \in \mathbb{R}$

$$W_g f(x,s) = |s|^{-\frac{1}{2}} \int_{\mathbb{R}} f(t) \overline{g\left(\frac{t-x}{s}\right)} dt$$
$$= f * D_s g^*(x),$$

where $g^*(t) = \overline{g(-t)}$ (Gröchenig, 2001). This transform gives local information of signals of any size and any time. The continuous wavelet transform of the translation of a signal is

$$W_g(T_z f) = T_{(z,0)} W_g f$$

for any $z \in \mathbb{R}$ (Kulak & Gürkanlı, 2011).

Let $g_1, g_2 \in L^2(\mathbb{R})$. Then the conditions

$$\int_0^\infty \frac{|\widehat{g_1(s\omega)}\widehat{g_2(s\omega)}|}{s} ds < \infty$$

and

$$\int_0^\infty \frac{\overline{\widehat{g_1(s\omega)}}\widehat{g_2(s\omega)}}{s} ds = K \text{ (independent of } \omega)$$

is said the wavelet admissibility condition (Daubechies, 1992; Gröchenig, 2001; Mallat, 1998). Assume that $g_1, g_2 \in L^2(\mathbb{R})$ satisfy the admissibility condition. Then we have

$$\int_0^\infty \int_{\mathbb{R}} W_{g_1} f_1(x,s) \overline{W_{g_2} f_2(x,s)} \frac{dxds}{s^2} = K\langle f_1, f_2\rangle$$

for all $f_1, f_2 \in L^2(\mathbb{R})$ (Daubechies, 1992; Gröchenig, 2001; Mallat, 1998). Also the function $f \in L^2(\mathbb{R})$ is reconstructed from it's the wavelet transform by

$$f = \frac{1}{K} \int_{\mathbb{R}} \int_0^\infty W_{g_1} f(x,s) T_x D_s\, g_2 \frac{dxds}{s^2},$$

where $g_1, g_2 \in L^2(\mathbb{R})$ satisfy the admissibility condition (Daubechies, 1992; Gröchenig, 2001; Mallat, 1998). If the function ω is positive real valued, measurable and locally bounded on $\mathbb{R}$ and satisfies the following inequalities

$$\omega(x) \geq 1, \omega(x+y) \leq \omega(x)\omega(y)$$

for all $x, y \in \mathbb{R}$ (Reiter, 1968), then the function ω is called weight function. Also the weight function

$$\omega(x) = (1+|x|)^a$$

is called weight of polynomial type such that $x \in \mathbb{R}$ and $a \geq 0$. If the weight functions ω_1 and ω_2 satisfy the condition $\omega_1(x) \leq C\omega_2(x)$, $(C > 0)$ for all $x \in \mathbb{R}$, then this condition is denoted by the symbol $\omega_1 \prec \omega_2$. Moreover if the weight functions ω_1 and ω_2 are equivalent, we write that $\omega_1 \approx \omega_2$ if and only if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$. The space $L^p(\mathbb{R})$, $(1 \leq p \leq \infty)$ denotes the usual Lebesgue space (Reiter, 1968). For $1 \leq p \leq \infty$, the weighted Lebesgue space is defined by $L_\omega^p(\mathbb{R}) = \{f : f\omega \in L^p(\mathbb{R})\}$ (Reiter, 1968). The space $\left(L^p(\mathbb{R})\right)_{loc}$ consists of classes of measurable functions f on $\mathbb{R}$ such that $f\chi_K \in L^p(\mathbb{R})$ for any compact subset $K \subset \mathbb{R}$. Fix a compact $Q \subset \mathbb{R}$ and $Q^o \neq \emptyset$. Assume that ϑ is weight function. The weighted weighted amalgam space $W(L^q, L_\vartheta^r)$ is space of all $f \in \left(L^q(\mathbb{R})\right)_{loc}$, where $F_f(z) = \left\|f\chi_{z+Q}\right\|_q$ is in $L_\vartheta^r(\mathbb{R})$. The norm of this space is given by

$$\|f\|_{W(L^q, L_\vartheta^r)} = \left\|F_f\right\|_{r,\vartheta} = \left\|\left\|f\chi_{z+Q}\right\|_q\right\|_{r,\vartheta}$$

for all $f \in W(L^q, L_\vartheta^r)$ (Heil, 2003).

2. On The Space $A_s(W)_{\omega,\vartheta}^{p,q,r}(\mathbb{R})$

Let $1 \leq p, q, r < \infty$ and ω, ϑ be weight functions on $\mathbb{R}$. Suppose that $0 \neq g \in S(\mathbb{R})$ and $s \in \mathbb{R}^+$. We set

$$A_s(W)_{\omega,\vartheta}^{p,q,r}(\mathbb{R}) = \left\{f \in L_\omega^p(\mathbb{R}) : W_g f \in W(L^q, L_\vartheta^r)\right\}$$

and equip this vector space with the following norm

$$\|f\|_{A_s(W)_{\omega,\vartheta}^{p,q,r}} = \|f\|_{p,\omega} + \left\|W_g f\right\|_{W(L^q, L_\vartheta^r)}$$

The space $A_s(W)_{\omega,\vartheta}^{p,q,r}(\mathbb{R})$ is normed Banach space with this norm.

Theorem 2.1

Let ω, ϑ be weight functions of polynomial type. The space $C_c^\infty(\mathbb{R})$ is dense in $A_s(W)_{\omega,\vartheta}^{p,q,r}(\mathbb{R})$.

Proof. Take any $h \in C_c^\infty(\mathbb{R})$. Then we have $h \in L_\omega^p(\mathbb{R})$. On the other hand by (Feichtinger, 1980), we get

$$\left\|W_g h\right\|_{W(L^q, L_\vartheta^r)} = \|h * D_s g^*\|_{W(L^q, L_\vartheta^r)}$$
$$\leq \|h\|_{W(L^q, L_\vartheta^r)} \|D_s g^*\|_{1,\vartheta} < \infty$$

for some $C > 0$. So we write $W_g f \in W(L^q, L_\vartheta^r)$. Then we find $h \in A_s(W)_{\omega,\vartheta}^{p,q,r}(\mathbb{R})$ and

$$C_c^\infty(\mathbb{R}) \subset A_s(W)_{\omega,\vartheta}^{p,q,r}(\mathbb{R}).$$

Assume that $h \in A_s(W)_{\omega,\vartheta}^{p,q,r}(\mathbb{R})$. That means $h \in L_\omega^p(\mathbb{R})$ and $W_g h \in W(L^q, L_\vartheta^r)$. It is known that $C_c^\infty(\mathbb{R})$ is dense in the spaces $L_\omega^p(\mathbb{R})$ and $W(L^q, L_\vartheta^r)$ (Kulak & Gürkanlı, 2013; Kulak & Gürkanlı, 2014). Then there exist $(h_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that

$$\|h_n - h\|_{p,\omega} \to 0, \left\|f_n - W_g h\right\|_{W(L^q, L_\vartheta^r)} \to 0.$$

From the subsequence property, we achieve a subsequence $\left(f_{n_k}\right)_{n_k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that $f_{n_k} = W_g h_{n_k}$ and

$$\|f_{n_k} - W_g h\|_{W(L^q,L^r_\vartheta)} \to 0,$$

where $(h_{n_k})_{n_k\in\mathbb{N}} \subset (h_n)_{n\in\mathbb{N}}$. Hence we get

$$\|h_{n_k} - f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}} \to 0$$

and $(h_{n_k})_{n_k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$. Finally we obtain

$$\overline{C_c^\infty(\mathbb{R})} = A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R}).$$

Theorem 2.2

$A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ is Banach function space.

Proof. Take any $f \in A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$. Then there exists $C > 0$ such that

$$\int_K |f(x)|dx \le C\|f\|_p \le C\left\{\|f\|_{p,\omega_1} + \|W_g f\|_{W(L^q,L^r_\vartheta)}\right\}$$
$$= C\|f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}$$

On the other hand since $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ is Banach space and by the above inequality, we obtain that the space $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ is Banach function space.

Theorem 2.3

The space $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ is invariant under translations. Moreover For every $0 \ne f \in A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ and $s \in \mathbb{R}_+$, there exists $C(f) > 0$ such that

$$C(f)\omega(z) \le \|T_z f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}$$
$$\le (\omega(z) + \vartheta(z,s))\|f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}.$$

Proof. Let $f \in A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ be given. Then we have $f \in L^p_\omega(\mathbb{R})$ and $W_g f \in W(L^q, L^r_\vartheta)$. It is known that

$$\|T_z f\|_{p,\omega} \le \omega(z)\|f\|_{p,\omega} \tag{2.1}$$

and $T_z f \in L^p_\omega(\mathbb{R})$ for all $z \in \mathbb{R}^d$ (Fischer, Gürkanlı & Liu, 1996). From the equality $W_g(T_z f) = T_{(z,0)}W_g f$, we achieve

$$\|W_g(T_z f)\|_{W(L^q,L^r_\vartheta)} \le \vartheta(z)\|W_g f\|_{W(L^q,L^r_\vartheta)} \tag{2.2}$$

for all $z \in \mathbb{R}$ (Heil, 2003). So by (2.1) and (2.2), we get

$$\|T_z f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}} = \|T_z f\|_{p,\omega} + \|W_g T_z f\|_{W(L^q,L^r_\vartheta)}$$
$$\le \omega(z)\|f\|_{p,\omega} + \vartheta(z)\|W_g f\|_{W(L^q,L^r_\vartheta)}. \tag{2.3}$$

Hence we find $T_z f \in A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$.

Now we take $0 \ne f \in A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$. We know by (Fischer, Gürkanlı & Liu, 1996) that there exists $C(f) > 0$ such that

$$C(f)\omega(z) \le \|T_z f\|_{p,\omega} \le \omega(z)\|f\|_{p,\omega}. \tag{2.4}$$

Using by the inequalities (2.3) and (2.4), we have

$$C(f)\omega_1(z) \le \omega(z)\|f\|_{p,\omega} + \vartheta(z)\|W_g f\|_{W(L^q,L^r_\vartheta)}$$
$$\le \omega(z)\|f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}} + \vartheta(z)\|f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}$$
$$= (\omega(z) + \vartheta(z))\|f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}.$$

3. Compact Embedding and Inclusion Theorems

Lemma 3.1

Assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$. If $(f_n)_{n\in\mathbb{N}}$ converges to zero in $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$, then

$$\int_{\mathbb{R}} f_n(x)g(x)dx \to 0$$

for $n \to \infty$ and for all $g \in C_c(\mathbb{R})$.

Proof. Let $g \in C_c(\mathbb{R})$ and $\frac{1}{p} + \frac{1}{s} = 1$. Then we write

$$\left|\int_{\mathbb{R}} f_n(x)g(x)dx\right| \le \|g\|_s\|f_n\|_p$$
$$\le \|g\|_s\|f_n\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}} \tag{3.1}$$

Since $(f_n)_{n\in\mathbb{N}}$ is a sequence in $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ and by (3.1), we obtain

$$\int_{\mathbb{R}} f_n(x)g(x)dx \to 0$$

for $n \to \infty$ and for all $g \in C_c(\mathbb{R})$.

Theorem 3.2 If $\omega_2 \prec \omega_1$ and $\vartheta_2 \prec \vartheta_1$, then

$$A_s(W)^{p,q,r}_{\omega_1,\vartheta_1}(\mathbb{R}) \subset A_s(W)^{p,q,r}_{\omega_2,\vartheta_2}(\mathbb{R})$$

holds.

Proof. Since $\omega_2 \prec \omega_1$ and $\vartheta_2 \prec \vartheta_1$, there exist $C_1, C_2 > 0$ such that

$$\omega_2(x) \le C_1\omega_1(x)$$

and

$$\vartheta_2(x) \le C_2\vartheta_1(x)$$

for all $x \in \mathbb{R}$. Take any $f \in A_s(W)^{p,q,r}_{\omega_1,\vartheta_1}(\mathbb{R})$. Then we have

$$f \in L^p_{\omega_1}(\mathbb{R}) \text{ and } W_g f \in W(L^q, L^r_{\vartheta_1}).$$

Thus we get

$$\|f\|_{p,\omega_2} \le C_1\|f\|_{p,\omega_1}$$

and

$$\|W_g f\|_{W(L^q,L^r_{\vartheta_2})} \le C_2\|W_g f\|_{W(L^q,L^r_{\vartheta_1})}.$$

So we find $f \in A_s(W)^{p,q,r}_{\omega_1,\vartheta_1}(\mathbb{R})$. Hence we achieve

$$A_s(W)^{p,q,r}_{\omega_1,\vartheta_1}(\mathbb{R}) \subset A_s(W)^{p,q,r}_{\omega_2,\vartheta_2}(\mathbb{R}).$$

The compact embedding theorems have been studied in a number of papers (Gürkanlı, 2008; Kulak & Gürkanlı, 2011; Ünal & Aydın, 2019). Now let's give theorems that show in which cases compact embedding will not occur.

Theorem 3.3

Let ω, ϑ be weight functions of polynomial type and let φ be weight function on $\mathbb{R}$. If $\varphi \prec \omega$ and $\frac{\varphi(x)}{\omega(x)+\vartheta(x,s)} \nrightarrow 0$ for $x \to \infty$, then the embedding of the space $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ into $L^p_\varphi(\mathbb{R})$ is never compact.

Proof. If we use the assumption $\prec \omega$, we say that there exists $C_1>0$ such that $\varphi(x) \le C_1\omega(x)$ for all $x \in \mathbb{R}$. So we have

$$A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R}) \subset L^p_\varphi(\mathbb{R}).$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence with $u_n \to \infty$ as $n \to \infty$ in $\mathbb{R}$. From the assumption $\frac{\varphi(x)}{\omega(x)+\vartheta(x,s)}$ does not tend to zero as $x \to \infty$, there exists $\delta > 0$ such that $\frac{\varphi(x)}{\omega(x)+\vartheta(x,s)} \ge \delta > 0$ for $x \to \infty$. Fixed $f \in A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$ and $u_0 \in \mathbb{R}_+$. Now we take a sequence $(f_n)_{n\in\mathbb{N}}$ such that

$$f_n = \left(\omega(u_n) + \vartheta(u_n, u_0)\right)^{-1} T_{u_n} f.$$

By Theorem 2.3, we achieve

$$\|f_n\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}$$
$$= \left\| \left(\omega(u_n) + \vartheta(u_n, u_0)\right)^{-1} T_{u_n} f \right\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}$$
$$= \left(\omega(u_n) + \vartheta(u_n, u_0)\right)^{-1} \left\|T_{u_n} f\right\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}$$
$$\le \left(\omega(u_n) + \vartheta(u_n, u_0)\right)^{-1} (\omega(u_n) +$$
$$+\left(\vartheta(u_n, u_0)\right) \|f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}$$
$$= \|f\|_{A_s(W)^{p,q,r}_{\omega,\vartheta}}.$$

So we find that this sequence is bounded in $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$. Now we will show that there wouldn't exist norm convergence subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L^p_\varphi(\mathbb{R})$. Therefore we find

$$\left|\int_{\mathbb{R}} f_n(x) g(x) dx\right|$$
$$\le \frac{1}{\omega(u_n)+\vartheta(u_n,u_0)} \int_{\mathbb{R}} \left|T_{u_n} f(x)\right| |g(x)| dx$$
$$\le \frac{1}{\omega(u_n)+\vartheta(u_n,u_0)} \|g\|_{p'} \left\|T_{u_n} f\right\|_p$$
$$= \frac{1}{\omega(u_n)+\vartheta(u_n,u_0)} \|g\|_{p'} \left\|T_{u_n} f\right\|_p, \qquad (3.2)$$

where $\frac{1}{p'} + \frac{1}{p} = 1$ for all $g \in C_c(\mathbb{R})$. So by inequality (3.2) tends zero for $n \to \infty$, then we obtain

$$\int_{\mathbb{R}} f_n(x) g(x) dx \to 0.$$

From Lemma 3.1, the only possible limit of $(f_n)_{n\in\mathbb{N}}$ in $L^p_\varphi(\mathbb{R})$ is zero. Since the following equivalence

$$\left\|T_{u_n} f\right\|_p \approx \varphi(u),$$

there exist $C_1, C_2 > 0$ such that

$$C_1\varphi(u_n) \le \left\|T_{u_n} f\right\|_{p,\varphi} \le C_2\varphi(u_n). \qquad (3.3)$$

If we use the inequality (3.3) and Lemma 3.1, then we achieve

$$\|f_n\|_{p,\varphi} = \left\| \left(\omega(u_n) + \vartheta(u_n, u_0)\right)^{-1} T_{u_n} f \right\|_{p,\varphi}$$
$$= \left(\omega(u_n) + \vartheta(u_n, u_0)\right)^{-1} \left\|T_{u_n} f\right\|_{p,\varphi}$$
$$\ge C_1\varphi(u_n)\left(\omega(u_n) + \vartheta(u_n, u_0)\right)^{-1}. \qquad (3.4)$$

On the other hand since the following inequality

$$\frac{\varphi(x)}{\omega(x)+\vartheta(x,s)} \ge \delta > 0$$

for all u_n, and by the inequality (3.4), we find

$$\|f_n\|_{p,\varphi} \ge C_1\varphi(u_n)\left(\omega(u_n) + \vartheta(u_n, u_0)\right)^{-1} > \delta C_1 > 0.$$

That means there would not be possible to find norm convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $L^p_\varphi(\mathbb{R})$.

Theorem 3.4

Suppose that ω_1, ϑ_1 are weight functions of polynomial type and ω_2, ϑ_2 are any weight functions. If $\omega_2 \prec \omega_1$, $\vartheta_2 \prec \vartheta_1$ and

$$\frac{\omega_2(x)}{\omega_1(x)+\vartheta_1(x,s)} \nrightarrow 0$$

for $x \to \infty$, then the embedding of the space $A_s(W)^{p,q,r}_{\omega_1,\vartheta_1}(\mathbb{R})$ into $A_s(W)^{p,q,r}_{\omega_2,\vartheta_2}(\mathbb{R})$ is never compact.

Proof. Using by assumptions $\omega_2 \prec \omega_1$ and $\vartheta_2 \prec \vartheta_1$ and by Theorem 3.2, we write $A_s(W)^{p,q,r}_{\omega_1,\vartheta_1}(\mathbb{R}) \subset A_s(W)^{p,q,r}_{\omega_2,\vartheta_2}(\mathbb{R})$. The unit map is a continuous from $A_s(W)^{p,q,r}_{\omega_1,\vartheta_1}(\mathbb{R})$ into $A_s(W)^{p,q,r}_{\omega_2,\vartheta_2}(\mathbb{R})$. Assume that the unit map is compact. Let $(f_n)_{n\in\mathbb{N}}$ in $A_s(W)^{p,q,r}_{\omega_1,\vartheta_1}(\mathbb{R})$ be arbitrary bounded sequence. If there exists convergent subsequence of $(f_n)_{n\in\mathbb{N}}$ in $A_s(W)^{p,q,r}_{\omega_2,\vartheta_2}(\mathbb{R})$, this sequence also converges in $L^p_{\omega_2}(\mathbb{R})$. But this is not possible by Theorem 3.3. This completes the proof..

5. Conclusion

In this paper, we assume that the scale s of the wavelet transform which is important tool for signal analysis and time-frequency analysis is fixed. Then we considered the space of functions $f \in L^p_\omega(\mathbb{R})$ such that their wavelet transforms $W_g f$ in $W(L^q, L^r_\vartheta)$. Also we denoted this space with the symbol $A_s(W)^{p,q,r}_{\omega,\vartheta}(\mathbb{R})$. Finally, we proved the inclusion and compact embedding theorems.

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