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Ostrowski type inequalities via exponentially s-convexity on time scales

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Abstract

We introduce the concept of exponentially s-convexity in the second sense on a time scale interval. We prove among other things that if $f : [a, b] \to \mathbb{R}$ is an exponentially s-convex function, then

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\Delta t \\ &\leq \frac{f(a)}{e_\beta(a,x_0)(b-a)^{2s}}(h_2(a,b))^s + \frac{f(b)}{e_\beta(b,x_0)(b-a)^{2s}}(h_2(b,a))^s, \end{aligned}$$

where β is a positively regressive function. By considering special cases of our time scale, one can derive loads of interesting new inequalities. The results obtained herein are novel to best of our knowledge and they complement existing results in the literature.

Keywords: Ostrowski inequality Time scales Hölder's inequality Exponentially s-convexity.
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1. Time Scale Essentials

A nonempty subset of $\mathbb{R}$ is called a *time scale* if it is closed with respect to the standard topology inherited from the reals. The theory of time scale is a relatively new branch of mathematics that has gained

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a lot of interest from mathematicians working in various fields of the mathematical sciences. Initiated in 1988, the German mathematician Stefan Hilger [8] proposed a time scale as a unifier between the discrete and continuous calculus. A time scale is generically represented by $\mathbb{T}$. Since a time scale is not necessarily connected, we introduce the following operators on $\mathbb{T}$: forward jump operator σ and backward jump operator ρ.

The above operators are defined as follows:

$$\begin{aligned} \sigma: \quad & \mathbb{T} \to \mathbb{T} \\ & t \mapsto \sigma(t) := \inf\{x \in \mathbb{T} : x > t\} \end{aligned}$$

and

$$\begin{aligned} \rho: \quad & \mathbb{T} \to \mathbb{T} \\ & t \mapsto \rho(t) := \sup\{x \in \mathbb{T} : x < t\}. \end{aligned}$$

We put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. In view of the above definition, we collate the following list of vocabularies from [8]:

Definition 1. 1. If

$$\sigma(t) \begin{cases} = t \quad \text{and} \quad t < \sup \mathbb{T}, & \text{then } t \text{ is a right-dense point} \\ > t, & \text{then } t \text{ is a right-scattered point.} \end{cases}$$

2. If

$$\rho(t) \begin{cases} = t \quad \text{and} \quad t > \inf \mathbb{T}, & \text{then } t \text{ is a left-dense point} \\ < t, & \text{then } t \text{ is a left-scattered point.} \end{cases}$$

3. The graininess functions μ and ν are defined as follows: $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$, $t \in \mathbb{T}$.

We now give a quick introduction of the calculus on time scales.

Definition 2. Let $f : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood W of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t) [\sigma(t) - s] \right| \le \varepsilon \left| \sigma(t) - x \right| \quad \text{for all} \quad x \in W.$$

We call $f^\Delta(t)$ the delta derivative of f at t. We say that f is delta differentiable in $\mathbb{T}^\kappa$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is said to be the delta derivative of f in $\mathbb{T}^\kappa$.

The time scale version of the product rule is embedded in the succeeding theorem:

Theorem 1.1 ([3], Product rule)**.** If $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$, then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t and

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f(\sigma(t)) g^\Delta(t).$$

Definition 3. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous on $\mathbb{T}$ provided it is continuous at all points $t \in \mathbb{T}$ with $\sigma(t) = t$ and its left-sided limits exist at all points $t \in \mathbb{T}$ with $\rho(t) = t$. The set of functions that are rd-continuous from $\mathbb{T}$ into $\mathbb{R}$ is denoted by $\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$.

By [3], every rd-continuous function has an antiderivative.

Theorem 1.2 ([3])**.** If $f, g \in \mathrm{C_{rd}}$, $a, b, c \in \mathbb{T}$, and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b \Big[\alpha f(t) + \beta g(t)\Big] \Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t;$$

$$\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$$

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$$

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$$

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\, \Delta t.$$

Definition 4 ([3])**.** For all $x, t \in \mathbb{T}$, we define the functions $h_k, g_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, by

$$h_0(t,x) := g_0(t,x) := 1$$

and then recursively by

$$g_{k+1}(t,x) = \int_x^t g_k\left(\sigma(\tau), x\right) \Delta\tau, \quad h_{k+1}(t,x) = \int_x^t h_k(\tau, s)\Delta\tau.$$

Definition 5 ([3])**.** A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if the following relation holds for all $t \in \mathbb{T}^{\kappa}$:

$$1 + \mu(t)p(t) \neq 0.$$

Let $\mathcal{R}$ represents the set of all regressive and rd-continuous function and define the set of all positively regressive functions by

$$\mathcal{R}^+ := \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}\}.$$

Given a regressive function p, then for all $x, t \in \mathbb{T}$ one defines the exponential function by

$$e_p(t,x) = \exp\left[\int_x^t \Lambda_{\mu(r)}(p(r))\, \Delta r\right],$$

where $\Lambda_q(w)$ is the cylinder transformation defined by

$$\Lambda_q(w) = \begin{cases} \frac{1}{q}\mathrm{Log}(1+wq), & \text{if } q > 0 \\ w, & \text{if } q = 0, \end{cases}$$

where Log is the principal logarithmic function. It is known (see [3]) that the function $e_p(t, t_0)$ is the unique solution of the IVP:

$$y^{\Delta} = p(t)y, \qquad y(t_0) = 1.$$

If we let $\mathbb{T} = \mathbb{R}$, then $p(t) = c \in \mathbb{R}$ and

$$e_p(t,t_0) = \int_t^{t_0} p(r)\, dr, \quad e_p(t,t_0) = e^{c(t-t_0)}, \quad e_p(t,0) = e_c(t) = e^{ct}.$$

Also, for $\mathbb{T} = \mathbb{Z}$, we get

$$e_p(t,t_0) = \prod_{r=t_0}^{t-1}[1+p(r)], \quad e_c(t,t_0) = (1+c)^{t-t_0}, \quad e_c(t,0) = (1+c)^t$$

for $t, t_0 \in \mathbb{Z}$ with $t > t_0$, where $c \neq -1$ is a constant and $p : \mathbb{Z} \to \mathbb{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$.

Theorem 1.3 ([2], Hölder's inequality)**.** Let $a, b \in \mathbb{T}$ and $f, g : [a, b] \to \mathbb{R}$ be rd-continuous. Then

$$\int_a^b |f(t)g(t)| \,\Delta t \leq \left(\int_a^b |f(t)|^p \,\Delta t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \,\Delta t\right)^{\frac{1}{q}},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Extending classical integral inequalities to an arbitrary time scale has been an active area of research in recent years. Many of such extensions are bound in the literature. For example, the classical Ostrowski inequality, Grüss inequality, Ostrowski–Grüss inequality, Jensen's inequality, Gronwall's inequality, Bihari's inequality, Opial inequality, Wirtinger's inequality, Lyapunov's inequality, etc. For more on time scales inequalities, the interested reader is invited to see the following articles [2, 1, 6] and the references cited therein. In 2008, Bohner and Matthews [4] established the following time scale version of the Ostrowski inequality:

Theorem 1.4. Let $a, b, x, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Then

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(x)) \,\Delta x \right| \leq \frac{M}{b-a} \Big[h_2(t, a) + h_2(t, b) \Big],$$

where

$$M = \sup_{a<t<b} |f^{\Delta}(t)|.$$

It is our purpose in this present article to first introduce the notion of exponentially s-convexity in the second sense on a time scale interval. For this class of functions, we establish loads of variant inequalities of the Ostrowski type. To the best of our knowledge, this is the first work in this direction. In the section following, we start by presenting the definition of our newly introduced concept. Thereafter, the main results are framed and justified anchored on loads of newly established lemmas.

2. Main Results

Let $x_0, a, b \in \mathbb{T}$ with $a < b$. The time scale interval $[a, b]$ represents the intersection between $[a, b]$ and $\mathbb{T}$.

Definition 6. Let $s \in (0, 1]$. A function $f : [a, b] \to \mathbb{R}$ is called exponentially s-convex in the second sense if

$$f(t) \leq \left(\frac{b-t}{b-a}\right)^s \frac{f(a)}{e_\beta(a, x_0)} + \left(\frac{t-a}{b-a}\right)^s \frac{f(b)}{e_\beta(b, x_0)} \tag{1}$$

for any $t \in [a, b]$ and for some $\beta \in \mathcal{R}^+$. If (1) holds in the reverse sense, then we say that f is exponentially s-concave in the second sense.

Remark 2.1. By taking $\mathbb{T} = \mathbb{R}$ and $x_0 = 0$, Definition 6 reduces to [10, Definition 3.1]. See also [7, 9].

Throughout this paper, without loss of generality, we assume that $s \in (0, 1)$. We now start with the following technical lemma.

Lemma 2.2. The following estimates

$$\int_a^b (b-t)^s \Delta t \leq (b-a)^{1-s} (h_2(a, b))^s \quad \text{and} \quad \int_a^b (t-a)^s \Delta t \leq (b-a)^{1-s} (h_2(b, a))^s$$

hold.

Proof. Using Hölder's inequality on time scales, we get:

$$\begin{aligned}\int_a^b (b-t)^s \Delta t &\le \left(\int_a^b (b-t)\Delta t\right)^s \left(\int_a^b \Delta t\right)^{1-s}\\ &= (b-a)^{1-s}\left(\int_b^a (t-b)\Delta t\right)^s\\ &= (b-a)^{1-s}(h_2(a,b))^s\end{aligned}$$

and

$$\begin{aligned}\int_a^b (t-a)^s \Delta t &\le \left(\int_a^b (t-a)\Delta t\right)^s \left(\int_a^b \Delta t\right)^{1-s}\\ &= (b-a)^{1-s}(h_2(b,a))^s.\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be an exponentially s-convex function. Then*

$$\frac{1}{b-a}\int_a^b f(t)\Delta t \le \frac{f(a)}{e_\beta(a,x_0)(b-a)^{2s}}(h_2(a,b))^s + \frac{f(b)}{e_\beta(b,x_0)(b-a)^{2s}}(h_2(b,a))^s.$$

Proof. Applying the definition for exponentially s-convex function in the second sense and Lemma 2.2, we find

$$\begin{aligned}\int_a^b f(t)\Delta t &\le \frac{f(a)}{e_\beta(a,x_0)(b-a)^s}\int_a^b (b-t)^s\Delta t + \frac{f(b)}{e_\beta(b,x_0)(b-a)^s}\int_a^b (t-a)^s\Delta t\\ &\le \frac{f(a)(b-a)}{e_\beta(a,x_0)(b-a)^{2s}}(h_2(a,b))^s + \frac{f(b)(b-a)}{e_\beta(b,x_0)(b-a)^{2s}}(h_2(b,a))^s.\end{aligned}$$

Hence,

$$\frac{1}{b-a}\int_a^b f(t)\Delta t \le \frac{f(a)}{e_\beta(a,x_0)(b-a)^{2s}}(h_2(a,b))^s + \frac{f(b)}{e_\beta(b,x_0)(b-a)^{2s}}(h_2(b,a))^s.$$

This completes the proof. □

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$, $|f|\ge 1$ on $[a,b]$ and $|f|^q$ be an exponentially s-convex function for some $q\ge 1$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\Delta t \le \frac{|f(a)|^q}{e_\beta(a,x_0)(b-a)^{2s}}(h_2(a,b))^s + \frac{|f(b)|^q}{e_\beta(b,x_0)(b-a)^{2s}}(h_2(b,a))^s.$$

Proof. Using Lemma 2.2, we arrive at

$$\begin{aligned}\int_a^b f(t)\Delta t &\le \int_a^b |f(t)|\Delta t\\ &\le \int_a^b |f(t)|^q\Delta t\\ &\le \frac{|f(a)|^q}{e_\beta(a,x_0)(b-a)^s}\int_a^b (b-t)^s\Delta t + \frac{|f(b)|^q}{e_\beta(b,x_0)(b-a)^s}\int_a^b (t-a)^s\Delta t\\ &\le \frac{|f(a)|^q(b-a)}{e_\beta(a,x_0)(b-a)^{2s}}(h_2(a,b))^s + \frac{|f(b)|^q(b-a)}{e_\beta(b,x_0)(b-a)^{2s}}(h_2(b,a))^s.\end{aligned}$$

Hence,

$$\frac{1}{b-a}\int_a^b f(t)\Delta t \le \frac{|f(a)|^q}{e_\beta(a,x_0)(b-a)^{2s}}(h_2(a,b))^s + \frac{|f(b)|^q}{e_\beta(b,x_0)(b-a)^{2s}}(h_2(b,a))^s.$$

This completes the proof. □

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$, $|f|^q$ be exponentially s-convex function for some $q\ge 1$. Then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)\Delta t\\ &\le \frac{2^{\frac{1}{q}}}{(b-a)^{\frac{2s}{q}}}\left(\frac{|f(a)|}{(e_\beta(a,x_0))^{\frac{1}{q}}}(h_2(a,b))^{\frac{s}{q}} + \frac{|f(b)|}{(e_\beta(b,x_0))^{\frac{1}{q}}}(h_2(b,a))^{\frac{s}{q}}\right).\end{aligned}$$

Proof. By the absolute value property and Hölder's inequality (with congugate pair (p,q)), one gets:

$$\begin{aligned}\int_a^b f(t)\Delta t &\le \int_a^b |f(t)|\Delta t\\ &\le (b-a)^{\frac{1}{p}}\left(\int_a^b |f(t)|^q\Delta t\right)^{\frac{1}{q}}\\ &\le (b-a)^{\frac{1}{p}}\left(\frac{|f(a)|^q}{(b-a)^s e_\beta(a,x_0)}\int_a^b (b-t)^s\Delta t + \frac{|f(b)|^q}{(b-a)^s e_\beta(b,x_0)}\int_a^b (t-a)^s\Delta t\right)^{\frac{1}{q}}\\ &\le (b-a)^{\frac{1}{p}}\left(\frac{(b-a)|f(a)|^q}{(b-a)^{2s}e_\beta(a,x_0)}(h_2(a,b))^s + \frac{(b-a)|f(b)|^q}{(b-a)^{2s}e_\beta(b,x_0)}(h_2(b,a))^s\right)^{\frac{1}{q}}\\ &\le 2^{\frac{1}{q}}(b-a)^{\frac{1}{p}}\Bigg((b-a)^{\frac{1}{q}}\frac{|f(a)|}{(b-a)^{\frac{2s}{q}}(e_\beta(a,x_0))^{\frac{1}{q}}}(h_2(a,b))^{\frac{s}{q}}\\ &\qquad + \frac{(b-a)^{\frac{1}{q}}|f(b)|}{(b-a)^{\frac{2s}{q}}(e_\beta(b,x_0))^{\frac{1}{q}}}(h_2(b,a))^{\frac{s}{q}}\Bigg),\end{aligned}$$

whereupon we get the desired result. This completes the proof. □

Lemma 2.6. The succeeding inequalities hold:

$$\begin{aligned}\int_a^b \sigma(t)(b-t)^s\Delta t &\le (b-a)^s(g_2(b,a)+a(b-a)),\\ \int_a^b \sigma(t)(t-a)^s\Delta t &\le (b-a)^s(g_2(b,a)+a(b-a)).\end{aligned}$$

Proof. To establish the first inequality, we proceed as follows:

$$\begin{aligned}\int_a^b \sigma(t)(b-t)^s\Delta t &\le (b-a)^s\int_a^b \sigma(t)\Delta t\\ &= (b-a)^s\int_a^b (\sigma(t)-a)\Delta t + a(b-a)^{1+s}\\ &= (b-a)^s g_2(b,a) + a(b-a)^{1+s}\\ &= (b-a)^s(g_2(b,a)+a(b-a)).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_a^b \sigma(t)(t-a)^s\Delta t &\le (b-a)^s\int_a^b \sigma(t)\Delta t\\ &\le (b-a)^s(g_2(b,a)+a(b-a)).\end{aligned}$$

This completes the proof. □

Let

$$G(a,b)=(b-a)^s(g_2(b,a)+a(b-a)).$$

Then, by Lemma 2.6, we have

$$\int_a^b \sigma(t)(b-t)^s\Delta t\le G(a,b),\quad \int_a^b \sigma(t)(t-a)^s\Delta t\le G(a,b).$$

For the next theorem, the following lemma will be needed.

Lemma 2.7. Let $f\in\mathcal{C}^1_{\mathrm{rd}}([a,b])$. Then

$$\int_a^b \sigma(t)f^\Delta(t)\Delta t=bf(b)-af(a)-\int_a^b f(t)\Delta t.$$

Proof. Let $g(t)=t$, $t\in\mathbb{T}$. Then $g(\sigma(t))=\sigma(t)$ and $g^\Delta(t)=1$, $t\in\mathbb{T}^\kappa$. Using the product rule on time scales, we have:

$$\begin{aligned}(fg)^\Delta(t)&=f^\Delta(t)g(\sigma(t))+f(t)g^\Delta(t)\\ &=\sigma(t)f^\Delta(t)+f(t).\end{aligned}$$

Hence, for $t\in\mathbb{T}^\kappa$,

$$\sigma(t)f^\Delta(t)=(fg)^\Delta(t)-f(t)$$

and

$$\begin{aligned}\int_a^b \sigma(t)f^\Delta(t)\Delta t&=\int_a^b\left[(fg)^\Delta(t)-f(t)\right]\,\Delta t\\ &=\int_a^b (fg)^\Delta(t)-\int_a^b f(t)\Delta t\\ &=(fg)(b)-(fg)(a)-\int_a^b f(t)\Delta t\\ &=f(b)g(b)-f(a)g(a)-\int_a^b f(t)\Delta t\\ &=bf(b)-af(a)-\int_a^b f(t)\Delta t.\end{aligned}$$

That completes the proof. □

Theorem 2.8. *Let $f\in\mathcal{C}^1_{\mathrm{rd}}([a,b])$ and $|f^\Delta|^q$ be an exponentially s-convex function for some $q\ge1$. Then*

$$\begin{aligned}&\left|\frac{bf(b)-af(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\Delta t\right|\\ &\le\frac{2^{\frac1q}\sigma(b)}{(b-a)^{\frac{2s}{q}}}\left(\frac{|f^\Delta(a)|}{(e_\beta(a,x_0))^{\frac1q}}(h_2(a,b))^{\frac sq}+\frac{|f^\Delta(b)|}{(e_\beta(b,x_0))^{\frac1q}}(h_2(b,a))^{\frac sq}\right).\end{aligned}$$

Proof. Using Lemmas 2.2 and 2.7, and the exponentially s-convexity of $|f^{\Delta}|^q$, we obtain:

$$\begin{aligned}
&\left|\frac{bf(b)-af(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\Delta t\right| \\
\leq\ & \frac{1}{b-a}\int_a^b \sigma(t)|f^{\Delta}(t)|\Delta t \\
\leq\ & \frac{\sigma(b)}{b-a}\int_a^b |f^{\Delta}(t)|\Delta t \\
\leq\ & \frac{\sigma(b)}{b-a}(b-a)^{1-\frac{1}{q}}\left(\int_a^b |f^{\Delta}(t)|^q\Delta t\right)^{\frac{1}{q}} \\
\leq\ & \frac{\sigma(b)}{(b-a)^{\frac{1}{q}}}\left(\frac{|f^{\Delta}(a)|^q}{(b-a)^s e_{\beta}(a,x_0)}\int_a^b (b-t)^s\Delta t\right. \\
&\left.+\frac{|f^{\Delta}(b)|^q}{(b-a)^s e_{\beta}(b,x_0)}\int_a^b (t-a)^s\Delta t\right)^{\frac{1}{q}} \\
\leq\ & \frac{\sigma(b)}{(b-a)^{\frac{1}{q}}}\left(\frac{|f^{\Delta}(a)|^q}{e_{\beta}(a,x_0)}(b-a)^{1-2s}(h_2(a,b))^s\right. \\
&\left.+\frac{|f^{\Delta}(b)|^q}{e_{\beta}(b,x_0)}(b-a)^{1-2s}(h_2(b,a))^s\right)^{\frac{1}{q}} \\
=\ & \frac{\sigma(b)}{(b-a)^{\frac{2s}{q}}}\left(\frac{|f^{\Delta}(a)|^q}{e_{\beta}(a,x_0)}(h_2(a,b))^s+\frac{|f^{\Delta}(b)|^q}{e_{\beta}(b,x_0)}(h_2(b,a))^s\right)^{\frac{1}{q}} \\
\leq\ & \frac{2^{\frac{1}{q}}\sigma(b)}{(b-a)^{\frac{2s}{q}}}\left(\frac{|f^{\Delta}(a)|}{(e_{\beta}(a,x_0))^{\frac{1}{q}}}(h_2(a,b))^{\frac{s}{q}}+\frac{|f^{\Delta}(b)|}{(e_{\beta}(b,x_0))^{\frac{1}{q}}}(h_2(b,a))^{\frac{s}{q}}\right).
\end{aligned}$$

This completes the proof. □

For the next results, we recall the following time scale version of Taylor's formula:

Lemma 2.9 ([5], Taylor's Formula)**.** Let $n\in\mathbb{N}$. Suppose that g is $n+1$ times differentiable on $\mathbb{T}^{\kappa^{n+1}}$. Let $a\in\mathbb{T}^{\kappa^n}$ and $t\in\mathbb{T}$. Then,

$$g(t)=\sum_{k=0}^{n}h_k(t,a)g^{\Delta^k}(a)+\int_a^{\rho^n(t)}h_n(t,\sigma(\tau))g^{\Delta^{n+1}}(\tau)\Delta\tau.$$

Let f be n-times differentiable on $\mathbb{T}^{\kappa^n}$. Define $g(t)=\int_a^t f(x)\,\Delta x$. Then, applying Lemma 2.9, we get

$$\int_a^t f(x)\,\Delta x=\sum_{k=1}^{n}h_k(t,a)f^{\Delta^{k-1}}(a)+\int_a^{\rho^n(t)}h_n(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau.$$

In particular, if we let $t=b$, then the above equation becomes:

$$\int_a^b f(x)\,\Delta x=\sum_{k=1}^{n}h_k(b,a)f^{\Delta^{k-1}}(a)+\int_a^{\rho^n(b)}h_n(b,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau. \tag{2}$$

Setting

$$I(a,b,n,f)=\int_a^b f(x)\Delta x-\sum_{k=1}^{n}h_k(b,a)f^{\Delta^{k-1}}(a),$$

we get that (2) is reduced to:

$$I(a,b,n,f)=\int_a^{\rho^n(b)} h_n(b,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau. \tag{3}$$

Theorem 2.10. *Let $a>0$, $f\in\mathcal{C}^n_{\mathrm{rd}}([a,b])$, $|f^{\Delta^n}|^q$ be an exponentially s-convex function, where $q\geq 1$, and $|f^{\Delta^n}|\geq 1$ on $[a,b]$. Then*

$$\frac{1}{b-a}\left|I(a,b,n,f)\right|\leq\frac{(b-a)^{n-2s}}{n!}\left(\frac{|f^{\Delta^n}(a)|^q}{e_\beta(a,x_0)}(h_2(a,b))^s+\frac{|f^{\Delta^n}(b)|^q}{e_\beta(b,x_0)}(h_2(b,a))^s\right).$$

Proof. By (3), we have

$$\begin{aligned}
\frac{1}{b-a}\left|I(a,b,n,f)\right| &= \frac{1}{b-a}\left|\int_a^{\rho^n(b)} h_n(b,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau\right|\\
&\leq \frac{1}{b-a}\int_a^{\rho^n(b)} h_n(b,\sigma(\tau))|f^{\Delta^n}(\tau)|\Delta\tau\\
&\leq \frac{1}{b-a}\int_a^{b}\frac{(b-\sigma(\tau))^n}{n!}|f^{\Delta^n}(\tau)|\Delta\tau\\
&\leq \frac{(b-a)^{n-1}}{n!}\int_a^b|f^{\Delta^n}(\tau)|^q\Delta\tau\\
&\leq \frac{(b-a)^{n-1}}{n!}\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^s e_\beta(a,x_0)}\int_a^b(b-t)^s\Delta t\right.\\
&\quad\left.+\frac{|f^{\Delta^n}(b)|^q}{(b-a)^s e_\beta(b,x_0)}\int_a^b(t-a)^s\Delta t\right)\\
&\leq \frac{(b-a)^{n-2s}}{n!}\left(\frac{|f^{\Delta^n}(a)|^q}{e_\beta(a,x_0)}(h_2(a,b))^s+\frac{|f^{\Delta^n}(b)|^q}{e_\beta(b,x_0)}(h_2(b,a))^s\right).
\end{aligned}$$

This completes the proof. □

Theorem 2.11. *Let $a>0$, $f\in\mathcal{C}^n_{\mathrm{rd}}([a,b])$, $|f^{\Delta^n}|^q$ is an exponentially s-convex function, where $q\geq 1$. Then*

$$\begin{aligned}
&\frac{1}{b-a}|I(a,b,n,f)|\\
&\leq\frac{2^{\frac{1}{q}}(b-a)^{n-\frac{2s}{q}}}{n!}\left(\frac{|f^{\Delta^n}(a)|}{(e_\beta(a,x_0))^{\frac{1}{q}}}(h_2(a,b))^{\frac{s}{q}}+\frac{|f^{\Delta^n}(b)|}{(e_\beta(b,x_0))^{\frac{1}{q}}}(h_2(b,a))^{\frac{s}{q}}\right).
\end{aligned}$$

Proof. Let $p \geq 1$ be chosen so that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned}
\frac{1}{b-a}|I(a,b,n,f)| &\leq \frac{(b-a)^{n-1}}{n!}\int_a^b |f^{\Delta^n}(t)|\Delta t \\
&\leq \frac{(b-a)^{n-1}}{(n-1)!}\left(\int_a^b \Delta t\right)^{\frac{1}{p}}\left(\int_a^b |f^{\Delta^n}(t)|^q \Delta t\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^{n-1+\frac{1}{p}}}{n!}\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^s e_\beta(a,x_0)}\int_a^b (b-t)^s \Delta t\right. \\
&\quad \left. + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^s e_\beta(b,x_0)}\int_a^b (t-a)^s \Delta t\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^{n-\frac{1}{q}}}{n!}\left(\frac{|f^{\Delta^n}(a)|^q}{(b-a)^{2s-1} e_\beta(a,x_0)}(h_2(a,b))^s\right. \\
&\quad \left. + \frac{|f^{\Delta^n}(b)|^q}{(b-a)^{2s-1} e_\beta(b,x_0)}(h_2(b,a))^s\right)^{\frac{1}{q}} \\
&\leq \frac{2^{\frac{1}{q}}(b-a)^{n-\frac{1}{q}}}{n!}\left(\frac{|f^{\Delta^n}(a)|}{(b-a)^{\frac{2s-1}{q}}(e_\beta(a,x_0))^{\frac{1}{q}}}(h_2(a,b))^{\frac{s}{q}}\right. \\
&\quad \left. + \frac{|f^{\Delta^n}(b)|}{(b-a)^{\frac{2s-1}{q}}(e_\beta(b,x_0))^{\frac{1}{q}}}(h_2(b,a))^{\frac{s}{q}}\right) \\
&= \frac{2^{\frac{1}{q}}(b-a)^{n-\frac{2s}{q}}}{n!}\left(\frac{|f^{\Delta^n}(a)|}{(e_\beta(a,x_0))^{\frac{1}{q}}}(h_2(a,b))^{\frac{s}{q}}\right. \\
&\quad \left. + \frac{|f^{\Delta^n}(b)|}{(e_\beta(b,x_0))^{\frac{1}{q}}}(h_2(b,a))^{\frac{s}{q}}\right).
\end{aligned}$$

This completes the proof. □

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