



Spinor Q-Equations in Lorentzian 3-space \mathbb{E}_1^3

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Abstract

In this paper, hyperbolic spinor representations of space curves are studied according to the q-frame in \mathbb{E}_1^3 . The spinor formulations of curves are calculated for the q-frame according to the spacelike and timelike tangent vector cases of the curves in \mathbb{E}_1^3 . Moreover, the relationships of spinor equations between q-frame and Frenet frame in Lorentz space are expressed. The results are supported with some theorems.

1. Introduction

Spinors, as two-component complex column vectors, were discovered by E. Cartan during his researches on linear representations of groups in 1910 [1]. A geometrical interpretation of spinors is based on 3-dimensional Cartesian space with three complex-valued components. The usage of geometrical notions in physics seems clearly in particular physics so that bosons and fermions are described by spinors while bosons are only characterized by tensors [2]. $Spin(2)$, and $Spin(3) = SU(2)$ as special cases of special unitary groups are composed by spinors algebra and the Pauli matrices admit to present a clearer characterization of three-dimensional real space rotation rather than the classic definition [3].

Geometrical notions such as curves, and surfaces represented by spinors are available in literature. In particular, spinor modelling of curves was firstly put forward by Castillo et al. by means of Frenet frame vectors in Euclidean 3-space [4]. Then from the point of view of Bishop frame called also as alternative frame, spinor representation was examined in the study [5]. Spinor analysis of Darboux frame on oriented a surface and its relationship with some types of tetrads such as Frenet and Darboux frames were studied in [6].

Lorentz space is a great important setting in which the relativity theory is established in physics. Geometrical studies of the notion spinors were taken into consideration as in the following

works. By means of hyperbolic spinor representation stemmed from the different structure of Lorentz 3-space, non-null regular curves were characterized in the research [7]. The spinor analysis of Darboux frame for non-null curves lying in non-degenerate surface was given in the same space by the work [8].

The quasi-normal vector of a space curve was defined for construction of three-dimensional offset curves by Coquillart [9]. The definition of the quasi-normal vector is as follows: for each point of the curve, the vector lies in the plane perpendicular to the tangent of the curve at this point [10]. The q-frame along a space curve was proposed in Euclidean 3-space by means of the quasi-normal vector [11]. The quasi frame fields along space curves were also studied in Lorentz 3-space in the work [12]. Some advantages of using the q-frame are related to curves to be lines, that is, their first curvatures vanish, in this condition, the q-frame can be constructed, and also the q-frame does not change even though curves to be unit or non-unit speed ones.

The motivation of our study is to characterize hyperbolic spinor representations of space curves in 3-dimensional Lorentz space according to q-frame and to research the differences between q-frame and Frenet frame from the spinor point of view. Accordingly, our investigation consists of two parts occurred because of the causal characters of space curves in Lorentz space. The derivative formulas of q-frame

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in terms of hyperbolic spinors are provided to represent the oriented triad of q-frame. The relationship between Frenet and q-frame formulas have been obtained with respect to the hyperbolic rotation angles relative to each frame by using hyperbolic spinors representations of Frenet and q-frames. Thus, it is aimed to use the advantages of q-frame in future studies.

2. Preliminaries

The Lorentzian space is equipped with the standard metric that is

$$g(u, v) = u_1v_1 + u_2v_2 - u_3v_3, \tag{1}$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two arbitrary vectors in \mathbb{E}_1^3 . If $g(u, u) > 0$ or $u = 0$, $g(u, u) < 0$ and $g(u, u) = 0$ ($u \neq 0$), the vector $u \in \mathbb{E}_1^3$ is called spacelike, timelike and lightlike (null) vectors, respectively. From the Eq. (1) the norm of the vector $u \in \mathbb{E}_1^3$ is obtained like as below,

$$\|u\|_L = \sqrt{|g(u, u)|}. \tag{2}$$

For the same vectors u, v , Lorentzian cross product is defined by

$$u \wedge_L v = \begin{bmatrix} e_1 & e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}, \tag{3}$$

where $e_1 \wedge e_2 = e_3, e_2 \wedge e_3 = -e_1, e_3 \wedge e_1 = -e_2$, [13]. The tangent vector $\alpha'(s)$ of a space curve $\alpha(s)$ can be spacelike, timelike or null (lightlike) so the curve is called with these names [14]. If $\langle \alpha'(s), \alpha'(s) \rangle = \mp 1$, a non-null curve α is parameterized by arc-length parameter s [15].

The Frenet frame $\{t, n, b\}$ of a non-null curve in Lorentz space is given by

$$t = \alpha', \quad n = \frac{\alpha''}{\|\alpha''\|}, \quad b = \varepsilon_b(t \wedge n). \tag{4}$$

Additionally, the Frenet derivative formulas are expressed as

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \varepsilon_b \kappa & 0 & \tau \\ 0 & \varepsilon_t \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{5}$$

where $\varepsilon_t = g(t, t)$, $\varepsilon_b = g(b, b)$ and κ, τ are the Lorentzian curvature and torsion functions, respectively [16].

The q-frame is an alternative way to defining a moving frame of a space curve with three orthonormal basis which are the unit tangent vector t , the quasi-normal n_q and the quasi-binormal vector b_q . The q-frame $\{t, n_q, b_q, k\}$ is expressed by,

$$t = \frac{\alpha'}{\|\alpha'\|}, \quad n_q = \frac{t \wedge k}{\|t \wedge k\|}, \quad b_q = t \wedge n_q, \tag{6}$$

where k is the projection vector which is chosen as $(0,1,0)$ spacelike or $(0,0,1)$ timelike. The q-frame has many dominances from other frames (Frenet, Bishop, Sabban) for characterizing a space curve. Such that the q-frame can be defined even when Lorentzian curvature vanished (along a line) and the space curve does not need to has unit speed for characterizing [12].

The q-frame derivative formulation in Lorentz space must be handled for two different cases which are the tangent vector of space curve is spacelike or timelike, seperately. In the case of tangent vector is spacelike (so quasi-normal or quasi-binormal is timelike), the derivative formulas of q-frame are expressed by

$$\begin{bmatrix} t' \\ n_q' \\ b_q' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_{n_q} k_1 & \varepsilon_{b_q} k_2 \\ -k_1 & 0 & \varepsilon_{b_q} k_3 \\ -k_2 & \varepsilon_{b_q} k_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}, \tag{7}$$

where $\varepsilon_{n_q} = \langle n_q, n_q \rangle$, $\varepsilon_{b_q} = \langle b_q, b_q \rangle$ and Lorentzian q-curvatures are

$$\begin{aligned} k_1 &= \kappa \cosh \theta, \quad k_2 = -\kappa \sinh \theta, \\ k_3 &= \varepsilon_{b_q}(d\theta + \tau). \end{aligned} \tag{8}$$

Also, the rotation matrix between Frenet frame and q-frame of a spacelike curve in Lorentz space can be given by the hyperbolic angle θ which is between the principal normal n and the quasi-normal n_q as follows:

$$\begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{9}$$

and

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & -\sinh \theta \\ 0 & -\sinh \theta & \cosh \theta \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}, \quad (10)$$

where the projection vector $k = (0,1,0)$ is spacelike [17]. With this transformation, timelike vectors turn into timelike vectors, spacelike vectors turn into spacelike vectors [18]. Moreover, the Euler formula is expressed by

$$e^{j\theta} = \cosh \theta + j \sinh \theta, \quad (11)$$

for the hyperbolic rotation [19].

In the case that the curve has timelike tangent vector, the derivation formulation does not depend on projection vector being timelike or spacelike. Then the derivative formulas of q-frame are obtained by

$$\begin{bmatrix} t' \\ n_q' \\ b_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}, \quad (12)$$

where Lorentzian q-curvatures are defined by

$$k_1 = \kappa \cos \theta, k_2 = -\kappa \sin \theta, k_3 = d\theta + \tau, \quad (13)$$

where θ is the angle between the vectors which are the principal normal n and the quasi-normal n_q . Also, the rotation matrix between Frenet frame and q-frame of a timelike curve in Lorentz space is

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}, \quad (14)$$

[12].

The group $U(n, \mathbb{H})$ is said to be hyperbolic unitary group which is established by Hermitian $n \times n$ matrices set. The subgroup $SO(1, 3)$ is a special Lorentzian group which is composed by Lorentzian transformation whose determinant is +1 [20]. The relation between the groups $SO(1, 3)$ and $SU(2, \mathbb{H})$ is a special one called as homomorphism. By means of this homomorphism, Hermitian matrices of the group $SU(2, \mathbb{H})$ serve as hyperbolic spinors while the elements of the subgroup $SO(1, 3)$ express vectors in Lorentz space [21].

A hyperbolic spinor can be defined as

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad (15)$$

by means of three vectors $a, b, c \in \mathbb{E}_1^3$ such that

$$a + jb = \Psi^t \sigma \Psi, \quad c = -\widehat{\Psi}^t \sigma \Psi, \quad (16)$$

where $j^2 = 1$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a vector whose cartesian components are the hyperbolic symmetric 2×2 matrices

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \\ \sigma_3 &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \quad (17)$$

which are the products of the matrix

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (18)$$

by the Pauli matrices employed in physics [22].

If hyperbolic spinor $\widehat{\Psi}$ be the mate and $\bar{\Psi}$ be the conjugation of hyperbolic spinor Ψ . Then,

$$\begin{aligned} \widehat{\Psi} &= -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{\Psi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\Psi}_1 \\ \bar{\Psi}_2 \end{pmatrix} \\ &= \begin{pmatrix} -\bar{\Psi}_2 \\ \bar{\Psi}_1 \end{pmatrix}. \end{aligned} \quad (19)$$

If it is chosen as $a + jb = (x_1, x_2, x_3)$, it is obtained from the Eq. (17) and (19)

$$\begin{aligned} x_1 &= \Psi^t \sigma_1 \Psi = \Psi_1^2 - \Psi_2^2, \\ x_2 &= \Psi^t \sigma_2 \Psi = j(\Psi_1^2 + \Psi_2^2), \\ x_3 &= \Psi^t \sigma_3 \Psi = -2\Psi_1 \Psi_2, \end{aligned}$$

where superscript t means transposition of hyperbolic spinor Ψ . Then,

$$\begin{aligned} a + jb &= \Psi^t \sigma \Psi \\ &= (\Psi_1^2 - \Psi_2^2, j(\Psi_1^2 + \Psi_2^2), -2\Psi_1 \Psi_2), \end{aligned} \quad (20)$$

is obtained. Likewise, it can be seen

$$\begin{aligned} c &= (c_1, c_2, c_3) \\ &= (\Psi_1 \bar{\Psi}_2 + \bar{\Psi}_1 \Psi_2, i(\Psi_1 \bar{\Psi}_2 - \bar{\Psi}_1 \Psi_2), |\Psi_1|^2 - |\Psi_2|^2). \end{aligned}$$

The norms $\|a\|_L = \|b\|_L = \|c\|_L = \bar{\Psi}^t \Psi$ are obtained by using the vector $a + jb$ which is an isotropic one, that is, $g(a + jb, a + jb) = 0$.

The equality $\bar{\Psi}^t \Psi' = \bar{\Psi}^t \Psi$ is satisfied for the equation $\Psi' = U\Psi$ such that a matrix $U \in$

$SU(2, \mathbb{H})$. It means that the norms of the vectors a', b', c' are equal to the ones of the vectors a, b, c , so that these two sets of the vectors correspond to Ψ' , and Ψ , respectively. Thus, all transformations from the orthogonal basis of \mathbb{E}_1^3 to another orthogonal basis of the same space are the elements of $SU(2, \mathbb{H})$.

The elements U , and $-U$ of $SU(2, \mathbb{H})$ match the same ordered set of \mathbb{E}_1^3 . This occurs because the transformation from $SU(2, \mathbb{H})$ to $SO(1, 3)$ is a two-to-one homomorphism. On the other hand, the sets $\{a, b, c\}$ performs to the spinor Ψ . So it can be expressed that the different ordered sets of \mathbb{E}_1^3 represent to the different hyperbolic spinors. But the same set can be shown by the spinors Ψ , and $-\Psi$. Then, the equalities below are satisfied for hyperbolic spinors ϕ and Ψ

$$\begin{aligned} \overline{\phi^t \sigma \Psi} &= -\hat{\phi}^t \sigma \hat{\Psi}, \\ a\overline{\phi} + b\overline{\Psi} &= \bar{a}\hat{\phi} + \bar{b}\hat{\Psi}, \\ \hat{\Psi} &= -\Psi, \end{aligned} \tag{21}$$

where a and b are hyperbolic numbers [23]. Moreover, the ordered sets $\{a, b, c\}, \{b, c, a\}, \{c, a, b\}$ represent different hyperbolic spinors. The following equation is satisfied for any pair of hyperbolic spinors ϕ and Ψ

$$\phi^t \sigma \Psi = \Psi^t \sigma \phi, \tag{22}$$

where the matrices σ (given by the Eq. (17)) are symmetric.

3. Spinor Q-Equations in \mathbb{E}_1^3

In this section of the study, it is obtained that the hyperbolic spinor representations of spacelike and timelike curves according to q-frame and the relationships of these representations between q-frame and the Frenet frame of the spacelike and timelike curves in \mathbb{E}_1^3 , separately.

3.1. Hyperbolic Spinor Q-Equations of Spacelike Curves

Let $\alpha: I \rightarrow \mathbb{E}_1^3$ be a spacelike curve and $\{n, b, t\}$ be Frenet vector fields in \mathbb{E}_1^3 . It is known that

$$n + jb = \Psi^t \sigma \Psi, \quad t = -\hat{\Psi}^t \sigma \Psi, \tag{23}$$

equations of Frenet frame are obtained as

$$\frac{d\Psi}{ds} = \frac{1}{2} (-j\tau\Psi + \varepsilon_b \kappa \hat{\Psi}), \tag{24}$$

where Ψ is a hyperbolic spinor which is represent Frenet frame, $\bar{\Psi}^t \Psi = 1$ and κ, τ is Lorentzian curvature and torsion of the spacelike curve α , respectively [24].

On the other hand, it is chosen that the q-frame $\{n_q, b_q, t\}$ of the spacelike curve in \mathbb{E}_1^3 corresponds to the hyperbolic spinor ϕ . Then, it can be written as

$$n_q + jb_q = \phi^t \sigma \phi, \quad t = -\hat{\phi}^t \sigma \phi, \tag{25}$$

where $\bar{\phi}^t \phi = 1$. If $\frac{d\phi}{ds}$ means the change of the q-frame along the spacelike curve, differentiating the first equation in the Eq. (25) and considering the Eq. (7), it is obtained

$$\begin{aligned} -k_1 t + \varepsilon_{b_q} k_3 b_q + j(-k_2 t + \varepsilon_{b_q} k_3 n_q) \\ = \left(\frac{d\phi}{ds}\right)^t \sigma \phi + \phi \sigma \left(\frac{d\phi}{ds}\right). \end{aligned} \tag{26}$$

Since the set $\{\phi, \hat{\phi}\}$ composes a basis for the hyperbolic spinors, the following equation

$$\frac{d\phi}{ds} = f\phi + g\hat{\phi}, \tag{27}$$

is achieved where f and g are two arbitrary hyperbolic functions. Then, considering the Eqs. (25), (26) and (27), it is obtained

$$f = -\frac{j}{2} \varepsilon_{b_q} k_3, \quad g = \frac{1}{2} (k_1 + j k_2). \tag{28}$$

Thus, the following theorem can be given with the aid of the Eq. (27) and (28).

Theorem 1. Let the hyperbolic spinor ϕ represents the q-frame $\{n_q, b_q, t\}$ of the spacelike curve α in Lorentz space \mathbb{E}_1^3 . The q-frame derivative equations are given via a hyperbolic spinor as

$$\frac{d\phi}{ds} = -\frac{j}{2} \varepsilon_{b_q} k_3 \phi + \frac{1}{2} (k_1 + j k_2) \hat{\phi}, \tag{29}$$

where k_1, k_2 and k_3 are q-curvatures of the spacelike curve.

Other than this, if it is approached the relationship between the spinors Ψ and ϕ , considering the Eq. (9), it is obtained

$$\begin{aligned} t &= t \\ n_q &= n \cosh \theta + b \sinh \theta, \\ b_q &= n \sinh \theta + b \cosh \theta \end{aligned}$$

and

$$n_q + j b_q = (n + j b)(\cosh \theta + j \sinh \theta). \quad (30)$$

From the Eqs. (23), (25) and (30), it can be seen that

$$\begin{aligned} \phi^t \sigma \phi &= e^{j\theta} (\Psi^t \sigma \Psi), \\ t &= t. \end{aligned} \quad (31)$$

Then, it is given by the following theorem:

Theorem 2. The relationships of the hyperbolic spinor formulas between q-frame and Frenet frame of the spacelike curve are

$$\begin{aligned} \phi^t \sigma \phi &= e^{j\theta} (\Psi^t \sigma \Psi), \\ t &= t. \end{aligned}$$

where $e^{j\theta} = \cosh \theta + j \sinh \theta$ and the spinors Ψ and ϕ represent the Frenet frame $\{n, b, t\}$ and q-frame $\{n_q, b_q, t\}$, respectively.

Lemma 1. Let $\alpha: I \rightarrow \mathbb{E}_1^3$ be a spacelike curve in Lorentz space \mathbb{E}_1^3 and the angle θ be hyperbolic rotation angle between the triads $\{n, b, t\}$ and $\{n_q, b_q, t\}$. Then, the hyperbolic rotation angle is also same between the hyperbolic spinors $\Psi^t \sigma \Psi$ and $\phi^t \sigma \phi$. Furthermore, the hyperbolic rotation angle is equal to $\theta/2$ between the hyperbolic spinors Ψ and ϕ .

Proof. For the isotropic vector $n + j b = (1, j, 0)$, the following equation can be written from the Eq. (20)

$$\begin{aligned} n + j b &= (a_1, a_2, a_3) = \Psi^t \sigma \Psi \\ &= (\Psi_1^2 - \Psi_2^2, j(\Psi_1^2 + \Psi_2^2), -2\Psi_1 \Psi_2), \end{aligned} \quad (32)$$

and

$$\Psi_1 = \mp \sqrt{\frac{a_1 + j a_2}{2}}, \quad \Psi_2 = \mp \sqrt{\frac{-a_1 + j a_2}{2}}, \quad (33)$$

are obtained. Then, it can be calculated the hyperbolic spinor Ψ corresponds to the triad $\{n, b, t\}$ as $\Psi = (\Psi_1, \Psi_2) = (\pm 1, 0)$. So, the triad

$\{n, b, t\}$ is rotated with the hyperbolic angle θ . From the Eq. (9), it can be written

$$\begin{aligned} n_q &= n \cosh \theta + b \sinh \theta, \\ b_q &= n \sinh \theta + b \cosh \theta. \end{aligned}$$

with the aid of the Eq. (30), since the hyperbolic spinor Ψ rotates to the hyperbolic spinor ϕ when the Frenet frame $\{n, b, t\}$ rotates to the q-frame $\{n_q, b_q, t\}$, it is obtained

$$\begin{aligned} n_q + j b_q &= (n + j b) e^{j\theta} \\ &= (a_1, a_2, a_3) e^{j\theta} = \phi^t \sigma \phi \\ &= (\phi_1^2 - \phi_2^2, j(\phi_1^2 + \phi_2^2), -2\phi_1 \phi_2) \end{aligned}$$

and

$$\begin{aligned} \phi_1 &= \mp e^{j\theta/2} \sqrt{\frac{a_1 + j a_2}{2}}, \\ \phi_2 &= \mp e^{j\theta/2} \sqrt{\frac{-a_1 + j a_2}{2}}. \end{aligned} \quad (34)$$

Then, it can be written

$$\phi_1 = \mp e^{j\theta/2}, \quad \phi_2 = 0.$$

and

$$\begin{aligned} \phi &= (\phi_1, \phi_2) = (\mp e^{j\theta/2}, 0) \\ &= e^{j\theta/2} (\mp 1, 0) = e^{j\theta/2} \Psi. \end{aligned} \quad (35)$$

So, the hyperbolic rotation angle is equal to $\theta/2$ between the hyperbolic spinors Ψ and ϕ .

3.2. Hyperbolic Spinor Q-Equations of Timelike Curves

Let $\alpha: I \rightarrow \mathbb{E}_1^3$ be a timelike curve and $\{n_q, b_q, t\}$ be the q-frame of timelike curve in Lorentz space. Then, for the hyperbolic spinor λ which represents q-frame $\{n_q, b_q, t\}$, it is written

$$n_q + j b_q = \lambda^t \sigma \lambda, \quad t = -\hat{\lambda}^t \sigma \lambda$$

where $\bar{\lambda}^t \lambda = 1$. Differentiating the first part of last equation and from the equation of $\frac{d\lambda}{ds} = m\lambda + n\hat{\lambda}$, it can be obtained

$$(k_1 + jk_2 + 2n)t - (jk_3 + 2m)n_q + (k_3 - j2m)b_q = 0 \quad (36)$$

and

$$m = -\frac{jk_3}{2}, \quad m = \frac{jk_3}{2}, \quad n = -\frac{k_1 + jk_2}{2}.$$

where m, n are two hyperbolic functions. In this case, it can be seen as $m = 0$ and so $k_3 = 0$. From here, it can be given with the following theorem:

Theorem 3. Let the hyperbolic spinor λ represents q-frame triad $\{n_q, b_q, t\}$ of the timelike curve in Lorentz space \mathbb{E}_1^3 . The q-frame derivative equations are given via a hyperbolic spinor as

$$\frac{d\lambda}{ds} = -\frac{1}{2}(k_1 + jk_2)\hat{\lambda} \quad (37)$$

where $k_3 = 0$. Then, q-frame $\{n_q, b_q, t\}$ turns into the Bishop frame [24].

Remark 1. If a timelike space curve represents by hyperbolic spinor via q-frame in Lorentz space \mathbb{E}_1^3 has a vanishing of third q-curvature k_3 , then the q-frame turns into the Bishop frame.

4. Conclusion

The spinors play an important role in Mathematics and Physics. These two-component complex column vectors render possible to describe some notions like as fermions. Spinors are characterized by a projective representation of the rotation group in Geometry. It is seemed to increase also the

studies on spinors from many different fields recently. The representations of this notion can also be used to give the characterizations of space curves according to Frenet and Bishop frames in Euclidean and Lorentz 3-spaces.

In this study, we handled the spinor representation of a space curve according to q-frame in \mathbb{E}_1^3 , so this is a hyperbolic spinor representation. This frame can be defined even when Lorentzian curvature vanished (along a line) and the space curve does not need to have unit speed for characterization. We firstly defined the q-frame $\{n_q, b_q, t\}$ of the spacelike curve in \mathbb{E}_1^3 which corresponds to the hyperbolic spinor ϕ . Then, we obtained the q-frame derivative equations which correspond to a hyperbolic spinor equation. Also, the relations of spinor formulations between q-frame and Frenet frame of the spacelike curve are given with the aid of Euler formula. The hyperbolic rotation angle between Ψ and ϕ is achieved as $\theta/2$ where hyperbolic spinors Ψ and ϕ represent the Frenet frame $\{n, b, t\}$ and the q-frame $\{n_q, b_q, t\}$ of the spacelike curve, respectively. On the other hand, in the case that the space curve has timelike tangent vector, we investigated the derivative formulas of q-frame, but the third q-curvature k_3 vanished and the frame turned into Bishop frame. As a result, our findings can be used for future works on this subject. It is clearly seen that there is a wide study area for spinor representations along space curves according to q-frame.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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