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Unrestricted Pell and Pell – Lucas 2^N-ons

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Abstract

In this study, we define unrestricted Pell and Pell – Lucas hyper-complex numbers. We choose arbitrary Pell and Pell – Lucas numbers for the coefficients of the ordered basis $\{e_0, e_1, \cdots, e_{N-1}\}$ of hyper-complex 2^N-ons where $N \in \{0,1,2,3,4\}$ and call these hyper-complex numbers unrestricted Pell and Pell-Lucas 2^N-ons. We give generating functions and Binet formulas for these type of hyper-complex numbers. We also obtain some generalization of well – known identities such as Catalan's, Cassini's and d'Ocagne's identities.

1. Introduction

Pell numbers and Pell – Lucas numbers are defined by the following recursive relations

$P_0 = 0,\ P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$,

and

$Q_0 = 1,\ Q_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$

respectively. Pell numbers take their name from English mathematician John Pell after his studies on the equation $x^2 - dy^2 = (-1)^n$ where d is not a perfect square integer. Generating functions for the sequences $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ are

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2} \quad \text{and} \quad \sum_{n=0}^{\infty} Q_n x^n = \frac{2-x}{1-2x-x^2}$$

respectively. Binet formulas for the Pell and Pell – Lucas numbers are

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ and } Q_n = \frac{\gamma^n + \delta^n}{2}$$

respectively, where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. The positive root γ is known as "silver ratio" and plays a similar role to the golden ratio of Fibonacci and Lucas numbers.

There are some interesting applications of Fibonacci and Pell numbers. For example, all repdigits are expressed as the products of a Fibonacci or a Pell number [1]. Pell sequence is used for solving some Diophantine equations [2].

Hyper-complex numbers are usually constructed by using Cayley-Dickson Process. Complex numbers, quaternions, octonions and sedenions with Pell and Pell-Lucas numbers' coefficients are investigated in this study. There are many studies about Pell and Pell-Lucas hyper-complex numbers. We can refer to [3, 4, 5, 6, 7, 8, 9, 10, 11] for Pell and Pell-Lucas quaternions, to [5, 8, 10] for Pell and Pell-Lucas octonions and to [5] for Pell and Pell-Lucas sedenions. In all of these studies, authors choose the consecutive Pell and Pell-Lucas numbers as coefficients. The difference between this study and previous studies is that we choose random Pell and Pell-Lucas numbers as coefficients of hyper-complex numbers. A similar idea can be seen in [4]. In that study, the authors investigated the

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unrestricted Pell and Pell-Lucas quaternions, which is a special case of our study.

For $N = 0,1,2,3,4$ and $\vec{c} = (c_0 = 0, c_1, \dots, c_{2^N-1})$ where $c_1, c_2, \dots, c_{2^N-1}$ are integers, unrestricted Pell and Pell-Lucas 2^N-ons are defined by

$$P_{N,r}^{\vec{c}} = \sum_{i=0}^{2^N-1} P_{r+ci}\, e_i \text{ and } Q_{N,r}^{\vec{c}} = \sum_{i=0}^{2^N-1} Q_{r+ci}\, e_i \qquad (1)$$

respectively. Here multiplication rules of the standard basis $\{e_0 = 1, e_1, e_2, \dots, e_{2^N-1}\}$ of the hyper-complex numbers for $N = 0,1,2,3,4$ are in the following table [12]. We set $i \equiv e_i$ for $i = 0,1, \dots, 15$.

Table 1. Multiplication rules of the 2^N-ons for $N = 0,1,2,3,4$.

.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	-0	3	-2	5	-4	-7	6	9	-8	-11	10	-13	12	15	-14
2	2	-3	-0	1	6	7	-4	-5	10	11	-8	-9	-14	-15	12	13
3	3	2	-1	-0	7	-6	5	-4	11	-10	9	-8	-15	14	-13	12
4	4	-5	-6	-7	-0	1	2	3	12	13	14	15	-8	-9	-10	-11
5	5	4	-7	6	-1	-0	-3	2	13	-12	15	-14	9	-8	11	-10
6	6	7	4	-5	-2	3	-0	-1	14	-15	-12	13	10	-11	-8	9
7	7	-6	5	4	-3	-2	1	-0	15	14	-13	-12	11	10	-9	-8
8	8	-9	-10	-11	-12	-13	-14	-15	-0	1	2	3	4	5	6	7
9	9	8	-11	10	-13	12	15	-14	-1	-0	-3	2	-5	4	7	-6
10	10	11	8	-9	-14	-15	12	13	-2	3	-0	-1	-6	-7	4	5
11	11	-10	9	8	-15	14	-13	12	-3	-2	1	-0	-7	6	-5	4
12	12	13	14	15	8	-9	-10	-11	-4	5	6	7	-0	-1	-2	-3
13	13	-12	15	-14	9	8	11	-10	-5	-4	7	-6	1	-0	3	-2
14	14	-15	-12	13	10	-11	8	9	-6	-7	-4	5	2	-3	-0	1
15	15	14	-13	-12	11	10	-9	8	-7	6	-5	-4	3	2	-1	-0

According to the context mentioned above, we regard 1-ons: real numbers, 2-ons: complex numbers, 3-ons: quaternions, 4-ons: octonions and 5-ons: sedenions. From the definition (1) and the definitions of Pell and Pell-Lucas numbers following recursive relations can be found easily:

$$P_{N,r}^{\vec{c}} = 2P_{N,r-1}^{\vec{c}} + P_{N,r-2}^{\vec{c}} \text{ and } Q_{N,r}^{\vec{c}} = 2Q_{N,r-1}^{\vec{c}} + Q_{N,r-2}^{\vec{c}}. \quad (2)$$

The special cases for unrestricted Pell and Pell-Lucas 2^N-ons are in the following table.

Table 2. Special cases

N	$\vec{c}$	Sequences
0	(0)	Classical Pell and Pell-Lucas numbers
1	(0,1)	Gaussian Pell and Pell-Lucas numbers
2	(0,1,2,3)	Pell and Pell-Lucas quaternions
3	$(0,1,\dots,7)$	Pell and Pell-Lucas octonions
4	$(0,1,\dots,15)$	Pell and Pell-Lucas sedenions

Example 1. The octonion $P_8 + P_{12}e_1 + P_{-16}e_3 + P_{21}e_7$ can be represent by $P_{3,8}^{(0,4,-8,-24,-8,-8,-8,13)}$

The well-known identities $P_{-n} = (-1)^{n+1}P_n$ and $Q_{-n} = (-1)^n Q_n$ give

$$P_{N,-r}^{\vec{c}} = (-1)^{r+1}\left[\sum_{i=0}^{2^N-1} (-1)^{c_i} P_{r-c_i} e_i\right]$$

and

$$Q_{N,-r}^{\vec{c}} = (-1)^{r}\left[\sum_{i=0}^{2^N-1} (-1)^{c_i} Q_{r-c_i} e_i\right].$$

2. Binet Formulas and Generating Functions

The next theorem gives the Binet formulas for the unrestricted Pell and Pell – Lucas 2^N-ons.

Theorem 2.1. For $N = 0,1,2,3,4$ and any integers $c_1, c_2, \dots, c_{2^N-1}$, the r-th unrestricted Pell and Pell-Lucas 2^N-on are

$$P_{N,r}^{\vec{c}} = \frac{\breve{\gamma}\gamma^r - \breve{\delta}\delta^r}{\gamma - \delta} \text{ and } Q_{N,r}^{\vec{c}} = \frac{\breve{\gamma}\gamma^r + \breve{\delta}\delta^r}{2}$$

where

$$\breve{\gamma} = \sum_{i=0}^{2^N-1} \gamma^{c_i} e_i \text{ and } \breve{\delta} = \sum_{i=0}^{2^N-1} \delta^{c_i} e_i.$$

respectively.

Proof. From the definitions of unrestricted Pell 2^N-ons and the Binet formula for the Pell numbers, we have

$$P_{N,r}^{\vec{c}} = P_r + P_{n+c_1}e_1 + \dots + P_{n+c_{2^N-1}}e_{2^N-1}$$

$$= \frac{1}{\gamma-\delta}(\gamma^r - \delta^r + (\gamma^{r+c_1} - \delta^{r+c_1})e_1 + (\gamma^{r+c_2} - \delta^{r+c_2})e_2 + \cdots + (\gamma^{r+c_{2^N-1}} - \delta^{r+c_{2^N-1}})e_{2^N-1}$$

$$= \frac{1}{\gamma-\delta}\left[\gamma^r\left(\gamma^{c_1} + \gamma^{c_2} + \cdots + \gamma^{c_{(2^N-1)}}\right) + \delta^r\left(\delta^{c_1} + \delta^{c_2} + \cdots + \delta^{c_{(2^N-1)}}\right)\right]$$

The last equation gives the Binet formula for the unrestricted Pell 2^N-ons. Binet formula for the unrestricted Pell-Lucas 2^N-ons can be obtained similarly. ■

Generating functions for the unrestricted Pell and Pell – Lucas 2^N-ons sequences are given in the next theorem.

Theorem 2.2. The generating functions for the sequences $\{P_{N,r}^{\vec{c}}\}_{r=0}^{\infty}$ and $\{Q_{N,r}^{\vec{c}}\}_{r=0}^{\infty}$ are

$$\sum_{i=0}^{\infty} P_{N,i}^{\vec{c}}\, x^i = \frac{P_{N,0}^{\vec{c}} + x\left(P_{N,1}^{\vec{c}} - 2P_{N,0}^{\vec{c}}\right)}{1-2x-x^2}$$

and

$$\sum_{i=0}^{\infty} Q_{N,i}^{\vec{c}}\, x^i = \frac{Q_{N,0}^{\vec{c}} + x(Q_{N,1}^{\vec{c}} - 2Q_{N,0}^{\vec{c}})}{1-2x-x^2}.$$

respectively.

Since the proofs are very straightforward, we don't give the proofs. Now we need to define the following set for later use. For $i \in \{1,2,\ldots,2^{N-1}-1\}$, we define the set

$$S_i = \{(j,k) : e_i e_j = e_k, 1 \le j,k \le 2^{N-1}-1, i \ne j, i \ne k \ ve\ j \ne k\}. \quad (3)$$

By using this set, we give the following lemma.

Lemma 2.3. For $N \in \{0,1,2,3,4\}$, we have

$$\breve{\gamma}\breve{\delta} = Y_N^{\vec{c}} + 2\sqrt{2}Z_N^{\vec{c}} \text{ and } \breve{\delta}\breve{\gamma} = Y_N^{\vec{c}} - 2\sqrt{2}Z_N^{\vec{c}} \quad (4)$$

where

$$Y_N^{\vec{c}} = 2Q_{N,0}^{\vec{c}} - \sum_{i=0}^{2^N-1}(-1)^{c_i}$$

and

$$Z_N^{\vec{c}} = \sum_{i=1}^{2^N-1} e_i \sum_{(j,k)\in S_i} (-1)^{c_k} P_{c_j-c_k}.$$

Proof. We prove the case N = 4. The others can be proved similarly. We have

$$Y_4^{\vec{c}} = Q_{4,0}^{\vec{c}} + (-1)^{c_1+1} + (-1)^{c_2+1} + \cdots + (-1)^{c_{15}+1} - 1. \quad (5)$$

Each versor e_i (i = 1,…,15) in $Z_4^{\vec{c}}$ contains seven terms. We have to calculate the sets S_i for each versor e_i. From Table 1, we obtain

$S_1 = \{(2,3),(4,5),(7,6),(8,9),(11,10),(13,12),(14,15)\}$,
$S_2 = \{(3,1),(4,6),(5,7),(8,10),(9,11),(14,12),(15,13)\}$,
$S_3 = \{(1,2),(6,5),(4,7),(10,9),(8,11),(15,12),(13,14)\}$,
$S_4 = \{(5,1),(6,2),(7,3),(8,12),(9,13),(10,14),(11,15)\}$,
$S_5 = \{(7,2),(1,4),(3,6),(12,9),(14,11),(8,13),(10,15)\}$,
$S_6 = \{(5,3),(2,4),(1,7),(15,9),(12,10),(11,13),(8,14)\}$
$S_7 = \{(6,1),(3,4),(2,5),(13,10),(12,11),(9,14),(8,15)\}$,
$S_8 = \{(9,1),(10,2),(11,3),(12,4),(13,5),(14,6),(15,7)\}$,
$S_9 = \{(11,2),(13,4),(14,7),(1,8),(3,10),(5,12),(6,15)\}$,
$S_{10} = \{(9,3),(14,4),(15,5),(2,8),(1,11),(6,12),(7,13)\}$,
$S_{11} = \{(10,1),(15,4),(13,6),(3,8),(2,9),(7,12),(5,14)\}$,
$S_{12} = \{(9,5),(10,6),(11,7),(4,8),(1,13),(2,14),(3,15)\}$,
$S_{13} = \{(12,1),(14,3),(10,7),(5,8),(4,9),(6,11),(2,15)\}$,
$S_{14} = \{(15,1),(12,2),(11,5),(6,8),(7,9),(4,10),(3,13)\}$,
and
$S_{15} = \{(13,2),(12,3),(9,6),(7,8),(5,10),(4,11),(1,14)\}$.

So we have

$$\begin{aligned}
Z_4^{\vec{c}} = &\big[(-1)^{c_3}P_{c_2-c_3} + (-1)^{c_5}P_{c_4-c_5} + (-1)^{c_6}P_{c_7-c_6} + (-1)^{c_9}P_{c_8-c_9} + (-1)^{c_{10}}P_{c_{11}-c_{10}} + (-1)^{c_{12}}P_{c_{13}-c_{12}} + (-1)^{c_{15}}P_{c_{14}-c_{15}}\big]e_1 \\
&+\big[(-1)^{c_1}P_{c_3-c_1} + (-1)^{c_6}P_{c_4-c_6} + (-1)^{c_7}P_{c_5-c_7} + (-1)^{c_{10}}P_{c_8-c_{10}} + (-1)^{c_{11}}P_{c_9-c_{11}} + (-1)^{c_{12}}P_{c_{14}-c_{12}} + (-1)^{c_{13}}P_{c_{15}-c_{13}}\big]e_2 \\
&+\big[(-1)^{c_2}P_{c_1-c_2} + (-1)^{c_5}P_{c_6-c_5} + (-1)^{c_7}P_{c_4-c_7} + (-1)^{c_9}P_{c_{10}-c_9} + (-1)^{c_{11}}P_{c_8-c_{11}} + (-1)^{c_{12}}P_{c_{15}-c_{12}} + (-1)^{c_{14}}P_{c_{13}-c_{14}}\big]e_3 \\
&+\big[(-1)^{c_1}P_{c_5-c_1} + (-1)^{c_2}P_{c_6-c_2} + (-1)^{c_3}P_{c_7-c_3} + (-1)^{c_{12}}P_{c_8-c_{12}} + (-1)^{c_{13}}P_{c_9-c_{13}} + (-1)^{c_{14}}P_{c_{10}-c_{14}} + (-1)^{c_{15}}P_{c_{11}-c_{15}}\big]e_4 \\
&+\big[(-1)^{c_2}P_{c_7-c_2} + (-1)^{c_4}P_{c_1-c_4} + (-1)^{c_6}P_{c_3-c_6} + (-1)^{c_9}P_{c_{12}-c_9} + (-1)^{c_{13}}P_{c_{14}-c_{11}} + (-1)^{c_{13}}P_{c_8-c_{13}} + (-1)^{c_{15}}P_{c_{10}-c_{15}}\big]e_5 \\
&+\big[(-1)^{c_3}P_{c_5-c_3} + (-1)^{c_4}P_{c_2-c_4} + (-1)^{c_7}P_{c_1-c_7} + (-1)^{c_9}P_{c_{15}-c_9} + (-1)^{c_{10}}P_{c_{12}-c_{10}} + (-1)^{c_{13}}P_{c_{11}-c_{13}} + (-1)^{c_{14}}P_{c_8-c_{14}}\big]e_6 \\
&+\big[(-1)^{c_1}P_{c_6-c_1} + (-1)^{c_4}P_{c_3-c_4} + (-1)^{c_5}P_{c_2-c_5} + (-1)^{c_{10}}P_{c_{13}-c_{10}} + (-1)^{c_{11}}P_{c_{12}-c_{11}} + (-1)^{c_{14}}P_{c_9-c_{14}} + (-1)^{c_{15}}P_{c_8-c_{15}}\big]e_7.
\end{aligned}$$

The last equation and Eq.(5) give the first equation in Eq.(4) for N = 4. ■

3. Some Identities

In this section, we give generalizations for some well-known identities about Pell and Pell-Lucas hyper-complex numbers. We use $P_r^{\vec{c}}$ and $Q_r^{\vec{c}}$ instead of $P_{N,r}^{\vec{c}}$ and $Q_{N,r}^{\vec{c}}$ respectively for abbreviation.

Theorem 3.1. (Vajda's identity). For any integers $n, r, s, c_0, c_1, c_2, \dots, c_{2^N-1}$, we have

$$P_{n+r}^{\vec{c}} P_{n+s}^{\vec{c}} - P_n^{\vec{c}} P_{n+r+s}^{\vec{c}} = (-1)^n P_r (P_s Y_N^{\vec{c}} - 2Q_s Z_N^{\vec{c}}) \quad (6)$$

and

$$Q_{n+r}^{\vec{c}} Q_{n+s}^{\vec{c}} - Q_n^{\vec{c}} Q_{n+r+s}^{\vec{c}} = 2(-1)^{n+1} P_r \big(P_s Y_N^{\vec{c}} - 2Q_s Z_N^{\vec{c}}\big). \quad (7)$$

Proof. From the Binet formula for the unrestricted Pell 2^N-ons, we get

$$\begin{aligned}
&P_{n+r}^{\vec{c}} P_{n+s}^{\vec{c}} - P_n^{\vec{c}} P_{n+r+s}^{\vec{c}} \\
&= \frac{1}{(\gamma-\delta)^2}\big[\big(\breve{\gamma}\gamma^{n+r} - \breve{\delta}\delta^{n+r}\big)\big(\breve{\gamma}\gamma^{n+s} - \breve{\delta}\delta^{n+s}\big) \\
&\qquad - \big(\breve{\gamma}\gamma^{n} - \breve{\delta}\delta^{n}\big)\big(\breve{\gamma}\gamma^{n+r+s} - \breve{\delta}\delta^{n+r+s}\big)\big] \\
&= \frac{1}{(\gamma-\delta)^2}\big[(\breve{\gamma})^2\gamma^{2n+r+s} + (\breve{\delta})^2\delta^{2n+r+s} \\
&\qquad - \breve{(\gamma\delta)}\gamma^{n+r}\delta^{n+s} - \breve{(\delta\gamma)}\gamma^{n+s}\delta^{n+r} \\
&\qquad - (\breve{\gamma})^2\gamma^{2n+r+s} - \breve{(\gamma)}^2\delta^{2n+r+s} \\
&\qquad + \breve{(\gamma\delta)}\gamma^{n}\delta^{n+r+s} + \breve{(\delta\gamma)}\gamma^{n+r+s}\delta^{n}\big] \\
&= \frac{(\gamma\delta)^n}{(\gamma-\delta)^2}\big[-\breve{(\gamma\delta)}\gamma^{r}\delta^{s} - \breve{(\delta\gamma)}\gamma^{s}\delta^{r} + \breve{(\gamma\delta)}\delta^{r+s} \\
&\qquad + \breve{(\delta\gamma)}\gamma^{r+s}\big] \\
&= \frac{(-1)^n}{(\gamma-\delta)^2}\big[(\gamma^r - \delta^r)(-\breve{(\gamma\delta)}\delta^s + \breve{(\delta\gamma)}\gamma^s)\big] \\
&= \frac{(-1)^n P_r}{(\gamma-\delta)}\big[\breve{(\delta\gamma)}\gamma^s - \breve{(\gamma\delta)}\delta^s\big] \\
&= \frac{(-1)^n P_r}{2\sqrt{2}}\big[(Y_N^{\vec{c}} - 2\sqrt{2}Z_N^{\vec{c}})\gamma^s - (Y_N^{\vec{c}} - 2\sqrt{2}Z_N^{\vec{c}})\delta^s\big].
\end{aligned}$$

The last equation gives Eq.(6). Eq.(7) can proved similarly. ■

If we take $s = -r$ and use the identities $P_r P_{-r} = -(-1)^r P_r^2$ and $2P_r Q_{-r} = (-1)^r P_{2r}$, we obtain Catalan's identities for the unrestricted Pell and Pell – Lucas 2^N-ons given in the next theorem.

Theorem 3.2. (Catalan's identities) For $N \in \{0,1,2,3,4\}$ and any integers $n, r, c_1, c_2, \cdots, c_{2^N-1}$, we have

$$P_{n+r}^{\vec{c}} P_{n-r}^{\vec{c}} - \big[P_n^{\vec{c}}\big]^2 = (-1)^{n+r+1}(Y_N^{\vec{c}} P_r^2 + Z_N^{\vec{c}} P_{2r}) \quad (8)$$

and

$$Q_{n+r}^{\vec{c}} Q_{n-r}^{\vec{c}} - \big[Q_n^{\vec{c}}\big]^2 = 2(-1)^{n+r}\big(Y_N^{\vec{c}} P_r^2 + Z_N^{\vec{c}} P_{2r}\big). \quad (9)$$

If we take $r = 1$ in Theorem 3.2, we obtain Cassini's identities the unrestricted Pell and Pell – Lucas 2^N-ons.

Theorem 3.3. (Cassini's identities) For $N \in \{0,1,2,3,4\}$ and any integers $r, c_1, c_2, \cdots, c_{2^N-1}$, we have

$$P_{n+1}^{c} P_{n-1}^{c} - [P_n^{c}]^2 = (-1)^n (Y_N^{\vec{c}} + 2Z_N^{\vec{c}}) \quad (10)$$

and

$$Q_{n+1}^{c} Q_{n-1}^{c} - [Q_n^{c}]^2 = -2(-1)^n (Y_N^{\vec{c}} + 2Z_N^{\vec{c}}) \quad (11)$$

The following theorem gives the d'Ocagne's identities for the unrestricted Pell and Pell – Lucas 2^N-ons.

Theorem 3.4. (d'Ocagne's identities) For $N \in \{0,1,2,3,4\}$ and any integers $m, n, c_1, c_2, \cdots, c_{2^N-1}$, we have

$$P_m^{c} P_{n+1}^{c} - P_{m+1}^{c} P_n^{c} = (-1)^n (Y_N^{\vec{c}} p_{m-n} + 2Z_N^{\vec{c}} q_{m-n}) \quad (12)$$

and

$$Q_m^{c} Q_{n+1}^{c} - Q_{m+1}^{c} Q_n^{c} = -2(-1)^n \big(Y_N^{\vec{c}} p_{m-n} + 2Z_N^{\vec{c}} q_{m-n}\big). \quad (13)$$

Proof. By using the Binet formula for the unrestricted Pell 2^N-ons, we obtain

$$\begin{aligned}
&P_m^{c} P_{n+1}^{c} - P_{m+1}^{c} P_n^{c} \\
&\qquad = \frac{1}{8}\big[\big(\breve{\gamma}\gamma^{m} - \breve{\delta}\delta^{m}\big)\big(\breve{\gamma}\gamma^{n+1} \\
&\qquad - \breve{\delta}\delta^{n+1}\big) \\
&\qquad - \big(\breve{\gamma}\gamma^{m+1} - \breve{\delta}\delta^{m+1}\big)\big(\breve{\gamma}\gamma^{n} - \breve{\delta}\delta^{n}\big)\big] \\
&= \frac{1}{8}\big(-\breve{\gamma}\breve{\delta}\gamma^{m}\delta^{n+1} - \breve{\delta}\breve{\gamma}\gamma^{n+1}\delta^{m} + \breve{\gamma}\breve{\delta}\gamma^{m+1}\delta^{n} \\
&\qquad + \breve{\delta}\breve{\gamma}\gamma^{n}\delta^{m+1}\big) \\
&= \frac{(-1)^n}{8}\big[\breve{\gamma}\breve{\delta}(\gamma-\delta)\gamma^{m-n} - \breve{\delta}\breve{\gamma}(\gamma-\delta)\delta^{m-n}\big] \\
&= \frac{(-1)^n}{2\sqrt{2}}\big[\breve{\gamma}\breve{\delta}\gamma^{m-n} - \breve{\delta}\breve{\gamma}\delta^{m-n}\big] \\
&= \frac{(-1)^n}{2\sqrt{2}}\big[\big(Y_N^{\vec{c}} + 2\sqrt{2}Z_N^{\vec{c}}\big)\gamma^{m-n} - \big(Y_N^{\vec{c}} - 2\sqrt{2}Z_N^{\vec{c}}\big)\delta^{m-n}\big]
\end{aligned}$$

The last identity gives Eq.(12). Eq.(13) can be obtained similarly. ■

We give many identities in the next theorem, which can be proved by using the Binet formulas or definitions of the unrestricted Pell and Pell – Lucas 2^N-ons and well-known identities for the classical Pell and Pell-Lucas numbers.

Theorem 3.5. For $N \in \{0,1,2,3,4\}$ and any integers $m, n, c_1, c_2, \cdots, c_{2^N-1}$, we have

$$\begin{gathered}
P_m^{\vec{c}} + P_{m-1}^{\vec{c}} = Q_m^{\vec{c}}, \\
Q_m^{\vec{c}} + Q_{m-1}^{\vec{c}} = 2P_m^{\vec{c}}, \\
P_m^{\vec{c}} + Q_m^{\vec{c}} = P_{m+1}^{\vec{c}}, \\
P_{m+1}^{\vec{c}} + P_{m-1}^{\vec{c}} = 2Q_m^{\vec{c}} \\
Q_{m+1}^{\vec{c}} + Q_{m-1}^{\vec{c}} = 4P_m^{\vec{c}}, \\
Q_{m+n}^{\vec{c}} + (-1)^n Q_{m-n} = 2q_n Q_m^{\vec{c}}, \\
P_{m+n}^{\vec{c}} + (-1)^n P_{m-n}^{\vec{c}} = 2q_n P_m^{\vec{c}}, \\
P_{m-n}^{\vec{c}} = (-1)^n \big(P_{n-1} P_m^{\vec{c}} - P_n P_{m-1}^{\vec{c}}\big), \\
\big[Q_m^{\vec{c}}\big]^2 - 2\big[P_m^{\vec{c}}\big]^2 = (-1)^m Y_N^{\vec{c}}
\end{gathered}$$

$$P_n^{\vec{c}} + P_{n-1}^{\vec{c}} = Q_n^{\vec{c}},$$
$$Q_n^{\vec{c}} + Q_{n-1}^{\vec{c}} = 2P_n^{\vec{c}},$$
$$P_n^{\vec{c}} + Q_n^{\vec{c}} = P_{n+1}^{\vec{c}},$$
$$2P_n^{\vec{c}} + Q_n^{\vec{c}} = Q_{n+1}^{\vec{c}},$$
$$2Q_n^{\vec{c}} + 3P_n^{\vec{c}} = P_{n+2}^{\vec{c}}$$
$$3Q_n^{\vec{c}} + 4P_n^{\vec{c}} = Q_{n+2}^{\vec{c}}$$
$$Q_{n+1}^{\vec{c}} - Q_n^{\vec{c}} = 2P_n^{\vec{c}},$$
$$P_{n+1}^{\vec{c}} + P_{n-1}^{\vec{c}} = 2Q_n^{\vec{c}},$$
$$Q_{n+1}^{\vec{c}} + Q_{n-1}^{\vec{c}} = 4P_n^{\vec{c}}$$
$$P_n^{\vec{c}} + P_{n+1}^{\vec{c}} + P_{n+3}^{\vec{c}} = 2P_{n+2}^{\vec{c}}$$
$$Q_n^{\vec{c}} + Q_{n+1}^{\vec{c}} + Q_{n+3}^{\vec{c}} = 3Q_{n+2}^{\vec{c}},$$
$$P_{n+1}^{\vec{c}} - P_{n-1}^{\vec{c}} = 2P_n^{\vec{c}}$$
$$Q_{n+1}^{\vec{c}} - Q_{n-1}^{\vec{c}} = 2Q_n^{\vec{c}},$$
$$P_{n+2}^{\vec{c}} + P_{n-2}^{\vec{c}} = 6P_n^{\vec{c}},$$
$$Q_{n+2}^{\vec{c}} + Q_{n-2}^{\vec{c}} = 6Q_n^{\vec{c}}$$
$$P_{n+2}^{\vec{c}} - P_{n-2}^{\vec{c}} = 4Q_n^{\vec{c}},$$
$$Q_{n+2}^{\vec{c}} - Q_{n-2}^{\vec{c}} = 8P_n^{\vec{c}},$$
$$2P_n^{\vec{c}} + Q_n^{\vec{c}} = Q_{n+1}^{\vec{c}},$$
$$2P_n^{\vec{c}} + Q_{n+2}^{\vec{c}} = 3Q_{n+1}^{\vec{c}},$$
$$P_{n+1}^{\vec{c}} + Q_{n-1}^{\vec{c}} = 3P_n^{\vec{c}}$$
$$Q_n^{\vec{c}}Q_{n+1}^{\vec{c}} - 2P_n^{\vec{c}}P_{n+1}^{\vec{c}} = (-1)^n\left[Y_N^{\vec{c}} - 4Z_N^{\vec{c}}\right],$$
$$P_n^{\vec{c}}P_{n+3}^{\vec{c}} - P_{n+1}^{\vec{c}}P_{n+2}^{\vec{c}} = (-1)^{n+1}\left[2Y_N^{\vec{c}} - 6Z_N^{\vec{c}}\right],$$
$$Q_n^{\vec{c}}Q_{n+3}^{\vec{c}} - Q_{n+1}^{\vec{c}}Q_{n+2}^{\vec{c}} = (-1)^n\left[4Y_N^{\vec{c}} - 12Z_N^{\vec{c}}\right],$$
$$P_n^{\vec{c}}Q_{n-1}^{\vec{c}} - Q_n^{\vec{c}}P_{n-1}^{\vec{c}} = (-1)^{n-1}\left[Y_N^{\vec{c}} + 2Z_N^{\vec{c}}\right].$$

4. Conclusions

There are many studies on hyper-complex numbers, such as quaternions, octonions and sedenions, whose coefficients are Pell and Pell-Lucas numbers. Current study differs from all of them by the choice of coefficients. Placing successive Pell and Pell-Lucas numbers in order for the coefficients of versors is common. Our definition provides to select arbitrary Pell or Pell-Lucas numbers for the coefficients of versors. We call this kind of hyper-complex numbers unrestricted Pell and unrestricted Pell-Lucas hyper-complex numbers. After introducing these numbers, we present Binet-like formulas for them and using Binet formulas, we obtain a numbers of identities for Pell and Pell-Lucas quaternions, octonions and sedenion. Although we limit N to 1 to 4, one can easily realize that there is no need such a restriction actually.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Çağman, A., 2021. Repdigits as product of Fibonacci and Pell numbers. Turkish Journal of Science, **6**(1), pp. 31-35.

[2] Çağman, A., 2021. An approach to Pillai's problem with the Pell sequence and the powers of 3. Miskolc Mathematical Notes, **22**(2), pp. 599-610.

[3] Aydın F. T., Köklü K., Yüce S., 2017. Generalized dual Pell quaternions. Notes on Number Theory and Discrete Mathematics, **23**(4), pp. 66-84.

[4] Bilgici G., Catarino P., 2018. Unrestricted Pell and Pell-Lucas quaternions. International Journal of Mathematics and Systems Science, **1**(3), pp. 1-10.

[5] Catarino P., 2016. The modified Pell and the modified k-Pell quaternions and octonions. Advances in Applied Clifford Algebras, **26**(2), pp. 577-590.

[6] Catarino P., 2018. Bicomplex k-Pell quaternions. Computational Methods and Function Theory, pp. 1-12.

[7] Catarino P., 2018. k-Pell, k-Pell–Lucas and modified k-Pell sedenions. Asian-European Journal of Mathematics, **12**(2), 1950018, pp. 1-10.

[8] Catarino P., Vasco P., 2017. On dual - Pell quaternions and octonions. Mediterranean Journal of Mathematics, **14**(2), 75.

[9] Çimen C.B., İpek A., 2016. On Pell quaternions and Pell-Lucas quaternions. Advances in Applied Clifford Algebras, **26**(1), pp. 39-51.

[10] Szynal-Liana A., Włoch I., 2016. The Pell quaternions and the Pell octonions. Advances in Applied Clifford Algebras, **26**(1), pp. 435-440.

[11] Tokeşer Ü., Ünal Z., Bilgici G., 2017. Split Pell and Pell–Lucas quaternions. Advances in Applied Clifford Algebras, **27**(2), pp. 1881-1893.

[12] Cawagas R.E., 2004. On the structure and zero divisors of the Cayley-Dickson sedenion algebra. Discussiones Mathematicae-General Algebra and Applications, **24**(2), pp. 251-265.