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Fixed Point Theorems in $\mathscr{G}$ - Fuzzy Convex Metric Spaces

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Abstract

This work introduces a new three-step iteration process and shows that the same leads to a unique fixed point with the help of theorems under different conditions of contractive mappings over-generalized $\mathscr{G}$ - fuzzy metric spaces in the convex structure. Also, we investigate the data dependence result of this iterative process in the generalized $\mathscr{G}$ - fuzzy convex metric spaces.

1. Introduction

The fuzzy set was released in 1965 by the pioneer scientist Zadeh [1] as a class of objects with a continuum of grades of membership. After Zadeh's paper [1], many scientists employed the notion of fuzzy sets in many subjects of sciences such as fuzzy metric space, fuzzy topology, fuzzy decisions, fuzzy set theory, etc. Kramosil and Michalek [2] paved a way for further work by introducing the concept of fuzzy metric spaces which then modified by George and Veeramani [3]. After that, several fixed point theorems were proved in fuzzy metric spaces.
Mustafa and Sims [4] brought out the concept of generalized metric space, shortly known as $\mathscr{G}$-metric space, and came out with interesting properties with its topology. Sun and Yang [5] also generalized the definition of fuzzy metric space in their way. In 2016, Jeyaraman et al. [6] proved a result that lead to a unique common fixed point theorem with six weakly compatible mappings in $\mathscr{G}$-fuzzy metric spaces. We introduce a new three-step iteration process and show the convergence of the iteration process to a unique fixed point using theorems under different conditions of contractive mappings on the $\mathscr{G}$-fuzzy metric spaces in the convex structure. Also, we investigate the data dependence result of this iterative process in the generalized $\mathscr{G}$ - fuzzy convex metric spaces.

2. Preliminaries

Definition 2.1. Let $(X,\mathscr{G},*)$ be a $\mathscr{G}$-fuzzy metric space and $I=[0,1]$. A continuous mapping $\Delta : X\times X\times I\to X$ is said to be a convex structure on X if for each $(x,y,k)\in X\times X\times I$ and $u\in X$,

$$\mathscr{G}(u,\Delta(x,y,k),\Delta(x,y,k),t)\geq k\mathscr{G}(u,x,x,t)+(1-k)M(u,y,y,t)$$

A space X together with a convex structure Δ is called a $\mathscr{G}$-fuzzy convex metric space($\mathscr{G}$-FCMS).

Definition 2.2. Let X be a $\mathscr{G}$-FCMS. A nonempty subset C of X is said to be generalized convex if $\Delta(x,y,z;a_1,a_2,a_3)\in C$ whenever $(x,y,z;a_1,a_2,a_3)\in C\times C\times C\times[0,1]\times[0,1]\times[0,1]$.

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Definition 2.3. Let $(X,\mathscr{G},*)$ be a $\mathscr{G}$-FCMS. A mapping $\Delta : X\times X\times X\times[0,1]\times[0,1]\to X$ is said to be $\mathscr{G}$-fuzzy convex structure on X if for each $(x,y,z,a_1,a_2)\in X\times X\times X\times[0,1]\times[0,1], a_1\geq a_2$ and $u,v\in X$,

$$\mathscr{G}(u,v,\Delta(x,y,z;a_1,a_2),t)\geq(a_1-a_2)\mathscr{G}(u,v,x,t)+(1-a_1)\mathscr{G}(u,v,y,t)+a_2\mathscr{G}(u,v,z,t).$$

Lemma 2.4. *Let a_n be a nonnegative sequence in $\mathscr{G}$-FCMS and let ρ is a real number satisfying $0\leq\rho<1$ and $(\in_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $\lim_{n\to\infty}\in_n=1$, then for any sequence of positive numbers $(\in_n)_{n\in\mathbb{N}}$ satisfying $a_{n+1}\geq\rho a_n+\in_n, n=1,2,\ldots,$ one has $\lim_{n\to\infty}a_n=1$.*

3. Main result

Theorem 3.1. *Let C be a nonempty closed convex subset of a $(X,\mathscr{G},*)$ complete $\mathscr{G}$-FCMS with Δ convex structure and $\Gamma:X\to X$ be a mapping satisfying the following conditions:*

$$\mathscr{G}\big(\Gamma x,\Gamma y,\Gamma z,t\big)\geq\big\{a_1\mathscr{G}(x,y,z,t)+a_2\mathscr{G}(x,\Gamma x,\Gamma x,t)+a_3\mathscr{G}(y,\Gamma y,\Gamma y,t)+a_4\mathscr{G}(z,\Gamma z,\Gamma z,t)\big\} \tag{3.1}$$

for all $x,y,z\in X$ where $0\leq a_1,a_2,a_3,a_4<1$ and $\{x_n\}_{n\geq0}$ ie the iterative scheme given by

(i) $x_0\in X$, *for all* $n\in N$,
(ii) $x_{n+1}=\Delta(\Gamma y_n,\Gamma y_n,\Gamma y_n:\gamma_n,\gamma_n)$,
(iii) $y_n=\Delta(z_n,\Gamma z_n,\Gamma x_n:\alpha_n,\beta_n)$,
(iv) $z_n=\Delta(\Gamma x_n,x_n,\Gamma x_n:\theta_n,\theta_n)$ *such that* $\lim_{n\to\infty}\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)=1$ *with* $\{\gamma_n\},\{\alpha_n\},\{\beta_n\}$ *and* $\{\theta_n\}\subset[0,1]$

Then $\{x_n\}_{n\geq0}$ $\mathscr{G}$-converges to unique fixed point $\dot{p}$ of Γ.

Proof: Suppose that Γ satisfies condition (i)-(iv), we have

$$\begin{aligned}\mathscr{G}(x_{n+1},\dot{p},\dot{p},t) &= \mathscr{G}(\Delta(\Gamma y_n,\Gamma y_n,\Gamma y_n;\gamma_n,\gamma_n),\dot{p},\dot{p},t)\\ &\geq\big\{(\gamma_n-\gamma_n)\mathscr{G}(\Gamma y_n,\dot{p},\dot{p},t)+(1-\gamma_n)\mathscr{G}(\Gamma y_n,\dot{p},\dot{p},t)+\gamma_n\mathscr{G}(\Gamma y_n,\dot{p},\dot{p},t)\big\}\\ &\geq\big\{a_1\mathscr{G}(y_n,\dot{p},\dot{p},t)+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+a_3\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)+a_4\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)\big\}\\ &=\big\{a_1\mathscr{G}(y_n,\dot{p},\dot{p},t)+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+(a_3+a_4)\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)\big\}\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\mathscr{G}\big(y_n,\dot{p},\dot{p},t\big) &= \mathscr{G}\Big(\Delta(z_n,\Gamma z_n,\Gamma x_n;\alpha_n,\beta_n),\dot{p},\dot{p},t\Big)\\ &\geq\Big\{\big(\alpha_n-\beta_n\big)\mathscr{G}(z_n,\dot{p},\dot{p},t)+\big(1-\alpha_n\big)\mathscr{G}(\Gamma z_n,\dot{p},\dot{p},t)+\beta_n\mathscr{G}(\Gamma x_n,\dot{p},\dot{p},t)\Big\}\\ &\geq\Bigg\{\big(\alpha_n-\beta_n+a_1(1-\alpha_n)\big)\mathscr{G}(z_n,\dot{p},\dot{p},t)+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &\quad+\big(1-\alpha_n\big)a_2\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)+\beta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\ &\quad+\big(1-(\alpha_n-\beta_n)\big)\big(a_3+a_4\big)\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)\Bigg\}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\mathscr{G}\big(z_n,\dot{p},\dot{p},t\big) &= \mathscr{G}\Big(\Delta(\Gamma x_n,x_n,\Gamma x_n;\theta_n,\theta_n),\dot{p},\dot{p},t\Big)\\ &\geq\Big\{\big(1-\theta_n(1-a_1)\big)\mathscr{G}(x_n,\dot{p},\dot{p},t)+\theta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\ &\quad+\theta_n\big(a_3+a_4\big)\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)\Big\}\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) in (3.2), we obtain

$$\begin{aligned}\mathscr{G}\left(x_{n+1},\dot{p},\dot{p},t\right) \geq\ & a_1\Bigg\{\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\Big(\left(1-\theta_n(1-a_1)\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &+\theta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)+\theta_n(a_3+a_4)\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)\Big)+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &+(1-\alpha_n)a_2\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)+\beta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\ &+\left(1-(\alpha_n-\beta_n)\right)(a_3+a_4)\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)\Bigg\}+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)\\ &+(a_3+a_4)\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)\\ &=a_1\Big(\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\Big)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+a_1\left((1-\alpha_n)a_2\right)\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)\\ &+a_1\left((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2\right)\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\ &+\Bigg\{a_1\Big[\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\theta_n(a_3+a_4)+\left(1-(\alpha_n-\beta_n)\right)(a_3+a_4)\Big]\\ &+(a_3+a_4)\Bigg\}\mathscr{G}(\dot{p},\dot{p},\Gamma\dot{p},t)\end{aligned}$$

Since $\mathscr{G}(\dot{p},\Gamma\dot{p},\Gamma\dot{p},t)=1$, we obtain,

$$\begin{aligned}\mathscr{G}(x_{n+1},\dot{p},\dot{p},t) \geq\ & \Bigg\{a_1\Big[\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\Big]\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &+a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+a_1\Big[(1-\alpha_n)a_2\Big]\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)\\ &+a_1\Big[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2\Big]\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\Bigg\}\end{aligned}$$

In order to satisfy the conditions of Lemma 2.4, we take δ,ε_n and κ_n as follows:

$$\begin{aligned}0\leq\delta &= a_1\left[(\alpha_n-\beta_n+(1-\alpha_n)a_1)(1-\theta_n(1-a_1))+\beta_n a_1\right]<1\\ \varepsilon_n &= \Big\{a_2\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)+a_1\left[\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\theta_n a_2+\beta_n a_2\right]\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\ &\quad +a_1\left((1-\alpha_n)a_2\right)\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)\Big\}\\ \kappa_n &= \mathscr{G}(x_n,\dot{p},\dot{p},t).\end{aligned}$$

Since

$$\lim_{n\to\infty}\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)=\lim_{n\to\infty}\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t)=\lim_{n\to\infty}\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t)=1$$

By Lemma 2.4, we have $\lim_{n\to\infty}\mathscr{G}(x_n,\dot{p},\dot{p},t)=1$. □

Example 3.2. *Let $X=[-1,1]$ and the $\mathscr{G}$ fuzzy metric is defined by $\mathscr{G}(x,y,z,t)=\dfrac{t}{t+\mathscr{G}(x,y,z)}$, where $\mathscr{G}(x,y,z)=|x-y|+|y-z|+|z-x|$. The $\mathscr{G}$-fuzzy convex structure Δ is defined by $\Delta(x,y,z,a_1,a_2)=(a_1-a_2)x+(1-a_1)y+a_2z$ and the self map $\Gamma(x)=\dfrac{x}{4}$. Clearly, $(X,\mathscr{G},*)$ is a complete $\mathscr{G}$-FCMS. The sequences are defined by $\alpha_n=\dfrac{n}{n+1}$, $\beta_n=\dfrac{n}{n+2}$, $\gamma_n=\dfrac{n}{n+3}$ and $theta_n=\dfrac{n}{n+4}$. Thus, the sequence $\{x_n\}_{n\geq 0}$ is satisfied all the conditions of the Theorem 3.1 and the sequence $\mathscr{G}$-converges to unique fixed point 0 of Γ.*

Theorem 3.3. *Let C be a non empty closed convex subset of a $(X,\mathscr{G},*)$ complete $\mathscr{G}$-FCMS with Δ convex structure and $\Gamma:X\to X$ be a mapping satisfying the following conditions:*

$$\mathscr{G}\left(\Gamma x,\Gamma y,\Gamma z,t\right)\geq\Big\{a_1\mathscr{G}(x,y,z,t)+a_2\mathscr{G}(x,\Gamma x,\Gamma x,t)+a_3\mathscr{G}(y,\Gamma y,\Gamma y,t)+a_4\mathscr{G}(\Gamma x,\Gamma z,\Gamma z,t)\Big\} \tag{3.5}$$

for all $x,y,z\in X$ where $0\leq a_1,a_2\leq\frac{1}{4},a_3,a_4\in[0,1]$ and $\{x_n\}_{n\geq 0}$ is given by

(i) $x_0\in X$,

(ii) $x_{n+1} = \Delta(\Gamma y_n, \Gamma y_n, \Gamma y_n : \gamma_n, \gamma_n)$,
(iii) $y_n = \Delta(z_n, \Gamma z_n, \Gamma x_n : \alpha_n, \beta_n)$,
(iv) $z_n = \Delta(\Gamma x_n, x_n, \Gamma x_n : \theta_n, \theta_n)$ *with*
(v) $\{\theta_n\}_{n\geq 0} \subset [0, \frac{1}{4})$,
(vi) $\beta_n \leq (1-\alpha_n)a_1 \leq \alpha_n$

Then $\{x_n\}_{n\geq 0}$ *converges to unique fixed point* $\dot{p}$ *of* Γ.

Proof: Suppose that Γ satisfies condition (i) - (iv), we have

$$\begin{aligned}
\mathscr{G}\left(x_{n+1}, \dot{p}, \dot{p}, t\right) &= \mathscr{G}\left(\Delta(\Gamma y_n, \Gamma y_n, \Gamma y_n; \gamma_n, \gamma_n), \dot{p}, \dot{p}, t\right) \\
&\geq \Big\{(\gamma_n - \gamma_n)\mathscr{G}(\Gamma y_n, \dot{p}, \dot{p}, t) + (1-\gamma_n)\mathscr{G}(\Gamma y_n, \dot{p}, \dot{p}, t) + \gamma_n \mathscr{G}(\Gamma y_n, \dot{p}, \dot{p}, t)\Big\} \\
&\geq \Big\{a_1\mathscr{G}(y_n, \dot{p}, \dot{p}, t) + a_2\mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) + a_3\mathscr{G}(\dot{p}, \dot{p}, \Gamma\dot{p}, t) + a_4\mathscr{G}(\dot{p}, \dot{p}, \Gamma\dot{p}, t)\Big\} \\
&= \Big\{a_1\mathscr{G}(y_n, \dot{p}, \dot{p}, t) + a_2\mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) + (a_3 + a_4)\mathscr{G}(\dot{p}, \dot{p}, \Gamma\dot{p}, t)\Big\} \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
\mathscr{G}\left(y_n, \dot{p}, \dot{p}, t\right) &= \mathscr{G}\left(\Delta(z_n, \Gamma z_n, \Gamma x_n; \alpha_n, \beta_n), \dot{p}, \dot{p}, t\right) \\
&\geq \Big\{(\alpha_n - \beta_n)\mathscr{G}(z_n, \dot{p}, \dot{p}, t) + (1-\alpha_n)\mathscr{G}(\Gamma z_n, \dot{p}, \dot{p}, t) + \beta_n \mathscr{G}(\Gamma x_n, \dot{p}, \dot{p}, t)\Big\} \\
&\geq \Big\{\big(\alpha_n - \beta_n + a_1(1-\alpha_n)\big)\mathscr{G}(z_n, \dot{p}, \dot{p}, t) + \beta_n a_1 \mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\
&\quad + (1-\alpha_n)a_2\mathscr{G}(z_n, \Gamma z_n, \Gamma z_n, t) + \beta_n a_2 \mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) \\
&\quad + (1-(\alpha_n - \beta_n))(a_3 + a_4)\mathscr{G}(\dot{p}, \Gamma\dot{p}, \Gamma\dot{p}, t)\Big\} \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
\mathscr{G}\left(z_n, \dot{p}, \dot{p}, t\right) &= \mathscr{G}\left(\Delta(\Gamma x_n, x_n, \Gamma x_n; \theta_n, \theta_n), \dot{p}, \dot{p}, t\right) \\
&\geq \Big\{(\theta_n - \theta_n)\mathscr{G}(\Gamma x_n, \dot{p}, \dot{p}, t) + (1-\theta_n)\mathscr{G}(x_n, \dot{p}, \dot{p}, t) + \theta_n \mathscr{G}(\Gamma x_n, \dot{p}, \dot{p}, t)\Big\} \\
&\geq \Big\{\big(1 - \theta_n(1-a_1)\big)\mathscr{G}(x_n, \dot{p}, \dot{p}, t) + \theta_n a_2 \mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) \\
&\quad + \theta_n(a_3 + a_4)\mathscr{G}(\dot{p}, \Gamma\dot{p}, \Gamma\dot{p}, t)\Big\} \qquad (3.8)
\end{aligned}$$

Substituting (3.7) and (3.8) in (3.6), we have

$$\begin{aligned}
\mathscr{G}\left(x_{n+1}, \dot{p}, \dot{p}, t\right) &\geq a_1\Bigg\{\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\Big(\big(1 - \theta_n(1-a_1)\big)\mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\
&\quad + \theta_n a_2 \mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) + \theta_n(a_3 + a_4)\mathscr{G}(\dot{p}, \Gamma\dot{p}, \Gamma\dot{p}, t)\Big) + \beta_n a_1 \mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\
&\quad + (1-\alpha_n)a_2\mathscr{G}(z_n, \Gamma z_n, \Gamma z_n, t) + \beta_n a_2 \mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) \\
&\quad + \big(1-(\alpha_n - \beta_n)\big)(a_3 + a_4)\mathscr{G}(\dot{p}, \Gamma\dot{p}, \Gamma\dot{p}, t)\Bigg\} + a_2\mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) \\
&\quad + (a_3 + a_4)\mathscr{G}(\dot{p}, \dot{p}, \Gamma\dot{p}, t) \\
&= a_1\Big(\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\big(1 - \theta_n(1-a_1)\big) + \beta_n a_1\Big)\mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\
&\quad + a_2\mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) + a_1\big((1-\alpha_n)a_2\big)\mathscr{G}(z_n, \Gamma z_n, \Gamma z_n, t) \\
&\quad + a_1\big((\alpha_n - \beta_n + (1-\alpha_n)a_1)\theta_n a_2 + \beta_n a_2\big)\mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t) \\
&\quad + \Big\{a_1\big[\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\theta_n(a_3 + a_4) + \big(1-(\alpha_n - \beta_n)\big)(a_3 + a_4)\big] \\
&\quad + (a_3 + a_4)\Big\}\mathscr{G}(\dot{p}, \dot{p}, \Gamma\dot{p}, t)
\end{aligned}$$

Since $\mathscr{G}(\dot{p}, \Gamma\dot{p}, \Gamma\dot{p}, t) = 1$, we obtain,

$$\begin{aligned}
\mathscr{G}\left(x_{n+1}, \dot{p}, \dot{p}, t\right) &\geq \Bigg\{a_1\big[\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\big(1 - \theta_n(1-a_1)\big) + \beta_n a_1\big]\mathscr{G}(x_n, \dot{p}, \dot{p}, t) \\
&\quad + a_2\mathscr{G}(y_n, \Gamma y_n, \Gamma y_n, t) + a_1\big[(1-\alpha_n)a_2\big]\mathscr{G}(z_n, \Gamma z_n, \Gamma z_n, t) \\
&\quad + a_1\big[\big(\alpha_n - \beta_n + (1-\alpha_n)a_1\big)\theta_n b + \beta_n a_2\big]\mathscr{G}(x_n, \Gamma x_n, \Gamma x_n, t)\Bigg\} \qquad (3.9)
\end{aligned}$$

Continuing the process,

$$\mathscr{G}\left(x_n,\Gamma x_n,\Gamma x_n,t\right) \geq \left(\frac{1+2a_1}{1-2a_2}\right)\mathscr{G}(x_n,\dot{p},\dot{p},t) \tag{3.10}$$

$$\begin{aligned}\mathscr{G}(z_n,\Gamma z_n,\Gamma z_n,t) &\geq \left(\frac{1+2a_1}{1-2a_2}\right)\mathscr{G}(z_n,\dot{p},\dot{p},t)\\ &\geq \left(\frac{1+2a_1}{1-2a_2}\right)(1-\theta_n(1-a_1))\mathscr{G}(x_n,\dot{p},\dot{p},t)+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\mathscr{G}(x_n,\Gamma x_n,\Gamma x_n,t)\\ &\geq \left(\frac{1+2a_1}{1-2a_2}\right)(1-\theta_n(1-a_1))\mathscr{G}(x_n,\dot{p},\dot{p},t)+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\mathscr{G}(x_n,\dot{p},\dot{p},t)\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathscr{G}(y_n,\Gamma y_n,\Gamma y_n,t) &\geq \left(\frac{1+2a_1}{1-2a_2}\right)\mathscr{G}(y_n,\dot{p},\dot{p},t)\\ &\geq \left(\frac{1+2a_1}{1-2a_2}\right)\Bigg\{\Bigg[\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\mathscr{G}(x_n,\dot{p},\dot{p},t)\Bigg]+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\beta_n a_2\left(\frac{1+2a_1}{1-2a_2}\right)\\ &\mathscr{G}(x_n,\dot{p},\dot{p},t)+(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Bigg[(1-\theta_n(1-a_1))\mathscr{G}(x_n,\dot{p},\dot{p},t)\\ &+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\mathscr{G}(x_n,\dot{p},\dot{p},t)\Bigg]\Bigg\}\end{aligned} \tag{3.12}$$

Substituting (3.10), (3.11) and (3.12) in (3.9), we obtain,

$$\begin{aligned}\mathscr{G}\left(x_{n+1},\dot{p},\dot{p},t\right) &\geq \Bigg\{\Big(a_1\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\mathscr{G}(x_n,\dot{p},\dot{p},t)\right)\Big)\\ &+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Bigg(\Big[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\Big]\Big[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\\ &+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\beta_n a_2\left(\frac{1+2a_1}{1-2a_2}\right)+(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Big[(1-\theta_n(1-a_1))\\ &+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\Bigg)+a_1\Big((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2\Big)\left(\frac{1+2a_1}{1-2a_2}\right)\\ &+a_1\Bigg((1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Big[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\Bigg)\Bigg\}\mathscr{G}(x_n,\dot{p},\dot{p},t)\end{aligned}$$

$$\begin{aligned}\mathscr{G}\left(x_{n+1},\dot{p},\dot{p},t\right) &\geq \Bigg\{a_1\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\Big)+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\\ &\Bigg(\Big[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\Big]\Big[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\\ &+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\beta_n a_2\left(\frac{1+2a_1}{1-2a_2}\right)+(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\\ &\Big[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\Bigg)+a_1\big((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2\\ &+\beta_n a_2)\left(\frac{1+2a_1}{1-2a_2}\right)+a_1(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Big[(1-\theta_n(1-a_1))\end{aligned}$$

$$
\begin{aligned}
\mathscr{G}\left(x_{n+1},\dot{p},\dot{p},t\right) \geq\; & a_1\left(\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\left[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\right]\left(1-\theta_n(1-a_1)\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+\left(\frac{1+2a_1}{1-2a_2}\right)^2\left[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\right]\theta_n a_2^2\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\left(\frac{1+2a_1}{1-2a_2}\right)^2\beta_n a_2^2\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+(1-\alpha_n)a_2^2\left(\frac{1+2a_1}{1-2a_2}\right)^2\left(1-\theta_n(1-a_1)\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+\left(\frac{1+2a_1}{1-2a_2}\right)^3(1-\alpha_n)\theta_n a_2^3\mathscr{G}(x_n,\dot{p},\dot{p},t)+a_1\left((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2\right)\\
&\left(\frac{1+2a_1}{1-2a_2}\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)+a_1\left((1-\alpha_n)a_2\right)\left(\frac{1+2a_1}{1-2a_2}\right)\left(1-\theta_n(1-a_1)\right)\mathscr{G}(x_n,\dot{p},\dot{p},t)\\
&+a_1\left((1-\alpha_n)a_2\right)\left(\frac{1+2a_1}{1-2a_2}\right)^2\theta_n a_2\mathscr{G}(x_n,\dot{p},\dot{p},t).
\end{aligned}
$$

Since

$$
\begin{aligned}
0 \leq\; & \Bigg\{a_1\Big(\left(\alpha_n-\beta_n+(1-\alpha_n)a_1\right)\left(1-\theta_n(1-a_1)\right)+\beta_n a_1\Big)\\
&+a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Bigg(\left[(\alpha_n-\beta_n+(1-\alpha_n)a_1)\right]\left[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\right]\\
&+\beta_n a_1\mathscr{G}(x_n,\dot{p},\dot{p},t)+\beta_n a_2\left(\frac{1+2a_1}{1-2a_2}\right)+(1-\alpha_n)a_2\left(\frac{1+2a_1}{1-2a_2}\right)\Big[(1-\theta_n(1-a_1))\\
&+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\Big]\Bigg)+a_1((\alpha_n-\beta_n+(1-\alpha_n)a_1)\theta_n a_2+\beta_n a_2)\left(\frac{1+2a_1}{1-2a_2}\right)\\
&+a_1\left((1-\alpha_n)a_2\right)\left(\frac{1+2a_1}{1-2a_2}\right)\left[(1-\theta_n(1-a_1))+\left(\frac{1+2a_1}{1-2a_2}\right)\theta_n a_2\right]\Bigg\}<1
\end{aligned}
$$

By Lemma 2.4, we have $\lim_{n\to\infty}\mathscr{G}(x_n,\dot{p},\dot{p},t)=1$. □

4. Conclusion

In this paper, we obtain the sequence using three step iteration process and convergence of iteration process to unique fixed point under conditions of contractive mappings on the G -fuzzy metric spaces in convex structure.

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References

[1] L. A. Zadeh, *Fuzzy sets*, Inf. Comput., **8** (1965) 338-353.
[2] I. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces*, Kubernetika, **11** (1975), 336-344.
[3] A. George, P. Veeramani, *On some result in Fuzzy metric spaces*, Fuzzy Sets Syst., **64** (1994), 395-399.
[4] Z. Mustafa, B. Sims, *A new approach to generalized metric space*, J. Nonlinear Convex Anal., **7**(2) (2006), 289 - 297.
[5] G. Sun, K. Yang, *Generalized fuzzy metric spaces with properties*, Res. J. App. Sci. Engg. Tech., **2** (2010), 673 - 678.
[6] M. Jeyaraman, R. Muthuraj, M. Jeyabharathi, M. Sornavalli, *Common fixed point theorems in $\mathscr{G}$-fuzzy metric spaces*, J. New Theory, **10** (2016), 12 - 18.