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On the Study of Pantograph Differential Equations with Proportional Fractional Derivative

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Abstract
This manuscript is devoted to investigate the existence, uniqueness and stability of pantograph equations with Hilfer generalized proportional fractional derivative. The concerned results are obtained using standard theorems.

Keywords: *Pantograph differential equation; generalized proportional fractional derivative; existence; stability.*

AMS Subject Classification (2020): *34A08; 46E35; 58C30.*

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1. Introduction

In the present paper, we will derive some sufficient conditions on existence and stability results for pantograph equation involving fractional order of the form

$$\begin{cases} \mathscr{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t) = \mathfrak{g}(t, \mathfrak{h}(t), \mathfrak{h}(\lambda t)), \quad t \in J := [a, b], \\ \mathscr{I}^{1-\nu,\vartheta;\psi}\mathfrak{h}(t)| = \mathfrak{h}_0 \end{cases} \tag{1.1}$$

where, $\mathscr{D}^{\alpha,\beta,\vartheta;\psi}$ is ψ-Hilfer proportional fractional derivative of orders $\alpha \in (0, 1)$, $\beta \in [0, 1]$ and $\vartheta \in (0, 1]$, $\mathscr{I}^{1-\nu,\vartheta:\psi}$ is ψ-fractional integral of orders $1 - \nu(\nu = \alpha + \beta - \alpha\beta)$. Let $\mathfrak{g}$ be the continuous function from J into $R \times R$ and $\mathfrak{h}$ is the given function.

Fractional calculus is extension of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations (FDE) arise naturally in various fields such as science and engineering. Theory of FDE has been extensively studied by many authors, see [1–9].

It is renowned that, within the settled scenario, there's an awfully special delay equation called the pantograph equations. In the following years, the pantograph equation became a prime example for a delay differential equation. The pantograph equations have been well studied over the last several decades, refer to [10–12].

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Most recently a fractional derivative with kernel of function is introduced by Vanterler Da C. Sousa and the classical properties with transformation of existing fractional derivative is discussed in [8, 13]. Motivated by the above mentioned work, we introduce a new generalized fractional calculus based on a special case of the proportional derivatives discussed in [14]. There are three features for our new generalized proportional fractional (GPF) derivative that make it different and distinctive: the kernel of the fractional operator contains exponential function, the generated fractional integrals possess a semi-group property and the obtained operators provide undeviating generalization to the existing Riemann-Liouville and Caputo fractional derivatives and integrals when the order 0 tends to 1.

The paper is organized as follows. In section 2, we declare the weighted spaces, basic definitions and results for proportional derivatives and their corresponding integral equation. In Section 3, we analyze the existence, uniqueness and stability results for proposed problem.

2. Preliminaries

Let $J(0 \leq a \leq b)$ be a finite interval. The space of continuous function $\mathfrak{h}$, defined by C associated with the norm

$$\|\mathfrak{h}\|_{C_{\nu,\psi}} = \sup\left\{|\mathfrak{h}(t)| : t \in J\right\}.$$

We denote the weighted spaces of all continuous functions defined by

$$C_{\nu,\psi} = \left\{\mathfrak{g} : J \to R : (\psi(t)-\psi(a))^{\nu}\,\mathfrak{g}(t) \in C\right\}, 0 \leq \nu < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{\nu,\psi}} = \sup_{t\in J}\left|(\psi(t)-\psi(a))^{\nu}\,\mathfrak{g}(t)\right|.$$

The weighted space $C^{n}_{\nu,\psi}$ of functions $\mathfrak{g}$ on J is defined by

$$C^{n}_{\nu,\psi} = \left\{\mathfrak{g} : J \to R : \mathfrak{g}(t) \in C^{n-1}; \mathfrak{g}(t) \in C_{\nu,\psi}\right\}, 0 \leq \nu < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C^{n}_{\nu,\psi}} = \sum_{k=0}^{n-1}\left\|\mathfrak{g}^{k}\right\|_{C} + \|\mathfrak{g}^{n}\|_{C_{\nu,\psi}}.$$

For $n = 0$, we have, $C^{0}_{\nu} = C_{\nu}$.
Here, we present the following weighted space for our problem as follows

$$C^{\alpha,\beta}_{1-\nu;\psi} = \left\{\mathfrak{g} \in C_{1-\nu;\psi}, \mathscr{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{g} \in C_{\nu;\psi}\right\},$$

and

$$C^{\nu}_{1-\nu;\psi} = \left\{\mathfrak{g} \in C_{1-\nu;\psi}, \mathscr{D}^{\nu,\vartheta;\psi}\mathfrak{g} \in C_{1-\nu;\psi}\right\}.$$

It is obvious that

$$C^{\nu}_{1-\nu;\psi} \subset C^{\alpha,\beta}_{1-\nu;\psi}.$$

Definition 2.1. [14] If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$. then the fractional integral

$$\left(\mathscr{I}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \int_0^t \psi'(s)e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\frac{(\psi(t)-\psi(s))^{\alpha-1}}{\vartheta^{\alpha}\Gamma(\alpha)}\mathfrak{h}(s)ds. \tag{2.1}$$

Definition 2.2. [14] If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a,b]$, where $\psi'(s) > 0$, the GPF derivative of order α of the function $\mathfrak{h}$ with respect to another function isdefined by with $\psi'(t) \neq 0$ is describe as

$$\left(\mathscr{D}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\int_0^t \psi'(s)e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\frac{(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\mathfrak{h}(s)ds. \tag{2.2}$$

Definition 2.3. [14] If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a,b]$, where $\psi'(s) > 0$, the GPF derivative in Caputo sence of order α of the function $\mathfrak{h}$ with respect to another function isdefined by with $\psi'(t) \neq 0$ is describe as

$$\left(\mathscr{D}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \mathscr{I}^{n-\alpha,\vartheta;\psi}\left(\mathscr{D}^{n,\vartheta;\psi}\mathfrak{h}\right)(t). \tag{2.3}$$

Definition 2.4. The ψ-Hilfer GPF derivative of order α and type β over $\mathfrak{h}$ with respect to another function is defined by

$$\left(\mathscr{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}\right)(t)=\mathscr{I}^{\beta(1-\alpha),\vartheta;\psi}\left(\mathscr{D}^{1,\vartheta;\psi}\right)\mathscr{I}^{(1-\beta)(1-\alpha),\vartheta;\psi}\mathfrak{h}(t). \tag{2.4}$$

Next, we shall give the definitions and the criteria of generalized Ulam-Hyers-Rassias(UHR) stability. Let $\epsilon>0$ be a positive real number and $\varphi: J\to R^{+}$ be a continuous function. We consider the following inequalities:

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{v}(t)-\mathfrak{g}\left(t,\mathfrak{v}(t),\mathfrak{v}(t)\right)\right|\le\varphi(t). \tag{2.5}$$

Definition 2.5. Eq. (1.1) is generalized UHR stable with respect to $\varphi\in C_{1-\nu,\psi}$ if there exists a real number $C_{\mathfrak{g},\varphi}>0$ such that for each solution $\mathfrak{v}\in C_{1-\nu,\psi}$ of the inequality (2.5) there exists a solution $\mathfrak{h}\in C_{1-\nu,\psi}$ of Eq. (1.1) with

$$|\mathfrak{v}(t)-\mathfrak{h}(t)|\le C_{\mathfrak{g},\varphi}\varphi(t).$$

Lemma 2.1. *Let $\alpha,\beta>0$, then we have the following semigroup property*

$$(\mathscr{I}^{\alpha,\vartheta;\psi}\mathscr{I}^{\beta,\vartheta;\psi}\mathfrak{g})(t)=(\mathscr{I}^{\alpha+\beta,\vartheta;\psi}\mathfrak{g})(t),$$

and

$$(\mathscr{D}^{\alpha,\vartheta;\psi}\mathscr{I}^{\alpha,\vartheta;\psi}\mathfrak{g})(t)=\mathfrak{g}(t).$$

Lemma 2.2. *Let $n-1<\alpha<n$ where $n\in N$, $\vartheta\in(0,1]$, $0\le\beta\le1$, with $\nu=\alpha+\beta(n-\alpha)$, such that $n-1<\nu<n$. If $\mathfrak{g}\in C_{\nu}$ and $\mathfrak{I}^{n-\nu,\vartheta;\psi}\mathfrak{g}\in C_{\nu}^{n}$, then*

$$(\mathscr{I}^{\alpha,\vartheta;\psi}\mathscr{I}^{\alpha,\beta,\vartheta;\psi}\mathfrak{g})(t)=\mathfrak{g}(t)-\sum_{k=1}^{n}\frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\nu-k}}{\vartheta^{\nu-k}\Gamma\nu-k+1}\mathscr{I}^{k-\nu,\vartheta;\psi}(a),$$

Lemma 2.3. *(Grönwall's Lemma [13]) Let $\alpha>0$, $a(t)>0$ is locally integrable function on J and if $\mathfrak{g}(t)$ be a increasing and nonnegative continuous function on J, such that $|\mathfrak{g}(t)|\le K$ for some constant K. Moreover if $\mathfrak{h}(t)$ be a nonnegative locally integrable function on J with*

$$\mathfrak{h}(t)\le a(t)+g(t)\int_0^t\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{h}(s)ds,\quad(t)\in J,$$

with some $\alpha>0$. Then

$$\mathfrak{h}(t)\le a(t)+\int_0^t\left[\sum_{n=1}^{\infty}\frac{(\mathfrak{g}(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{n\alpha-1}\right]a(s)ds,\quad(t)\in J.$$

Theorem 2.1. *(Schauder fixed point theorem, [15]) Let B be closed, convex and nonempty subset of a Banach space C. Let $\mathscr{T}: B\to B$ be a continuous mapping such that $\mathscr{T}(B)$ is a relatively compact subset of C. Then $\mathscr{T}$ has at least one fixed point in B.*

Lemma 2.4. *A function $\mathfrak{h}$ is the solution of (1.1), if and only if $\mathfrak{h}$ satisfies the random integral equation*

$$\begin{aligned}\mathfrak{h}(t)=&\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1}\\&+\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds.\end{aligned} \tag{2.6}$$

3. Main results

Utilizing the concept of Theorem 2.1, we obtain the following results for the proposed problem (1.1). First, we declare the hypotheses used to obtain the result:

(H1) There exists a constant ℓ, such that

$$|\mathfrak{g}(\cdot,\mathfrak{h}_1(\cdot),\mathfrak{h}_2(\cdot))-\mathfrak{g}(\cdot,\mathfrak{y}_1(\cdot),\mathfrak{y}_2(\cdot))|\le\ell\left(|\mathfrak{h}_1-\mathfrak{y}_1|+|\mathfrak{h}_2-\mathfrak{y}_2|\right).$$

(H2) There exists an increasing function $\varphi \in C_{1-\nu,\psi}$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$\mathfrak{I}^{\alpha;\psi}\varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 3.1. *Assume that hypothesis $(H1)$ is satisfied. Then, Eq.(1.1) has at least one solution.*

Proof. Consider the operator $\mathscr{T} : C_{1-\nu,\psi} \to C_{1-\nu,\psi}$. Hence $\mathfrak{h}$ is a solution for the problem (1.1) if and only if $\mathfrak{h}(t) = (\mathscr{T}\mathfrak{h})(t)$, where the equivalent integral Eq. (2.6) which can be written in the operator form

$$\begin{aligned}(\mathscr{T}\mathfrak{h})(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} \\ &+ \frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds. \end{aligned} \tag{3.1}$$

Clearly, the fixed points of the operator $\mathscr{T}$ is solution of the problem (1.1). Set $\tilde{\mathfrak{g}} = \mathfrak{g}(s,0,0)$. For any $\mathfrak{h}$, we have

$$\begin{aligned}&\left|(\mathscr{T}\mathfrak{h})(t)(\psi(t)-\psi(a))^{1-\nu}\right| \\ &\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} + \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right| ds \\ &\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} \\ &\quad+ \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s)) - \mathfrak{g}(s,0,0) + \mathfrak{g}(s,0,0)\right| ds \\ &\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} + \frac{2\ell(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\|\mathfrak{h}\|_{C_{1-\nu,\psi}} \\ &\quad+ \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\|\tilde{\mathfrak{g}}\|_{C_{1-\nu,\psi}} \\ &\leq \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} + \frac{\ell}{\vartheta^\alpha\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^\alpha\|\mathfrak{h}\|_{C_{1-\nu,\psi}} \\ &\quad+ \frac{B(\nu,\alpha)}{\vartheta^\alpha\Gamma(\alpha)}(\psi(b)-\psi(a))^\alpha\|\tilde{\mathfrak{g}}\|_{C_{1-\nu,\psi}}\end{aligned}$$

This proves that $\mathscr{T}$ transforms the ball $B_r = \left\{\mathfrak{h} \in C_{1-\nu,\psi} : \|\mathfrak{h}\|_{C_{1-\nu,\psi}} \leq r\right\}$, into itself. We shall show that the operator $\mathscr{T} : B_r \to B_r$ satisfies all the conditions of Theorem 2.1. The proof will be given in the following steps.
Step 1: $\mathscr{T}$ is continuous.

Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n \to \mathfrak{h}$ in $C_{1-\nu,\psi}$. Then, for each $t \in J$,

$$\begin{aligned}&\left|((\mathscr{T}\mathfrak{h}_n)(t) - (\mathscr{T}\mathfrak{h})(t))(\psi(t)-\psi(a))^{1-\nu}\right| \\ &\leq \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}_n(s),\mathfrak{h}_n(\lambda s)) - \mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right| ds \\ &\leq \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^\alpha\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(s))^{\alpha+\nu-1}\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot),\mathfrak{h}_n(\cdot)) - \mathfrak{g}(\cdot,\mathfrak{h}(\cdot),\mathfrak{h}(\cdot))\|_{C_{1-\nu,\psi}} \\ &\leq \frac{1}{\vartheta^\alpha\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^\alpha\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot),\mathfrak{h}_n(\cdot)) - \mathfrak{g}(\cdot,\mathfrak{h}(\cdot),\mathfrak{h}(\cdot))\|_{C_{1-\nu,\psi}}.\end{aligned}$$

Due to continuity of $\mathfrak{g}$, we have

$$\|\mathscr{T}\mathfrak{h}_n - \mathscr{T}\mathfrak{h}\|_{C_{1-\nu,\psi}} \to 0 \text{ as } n \to \infty.$$

Step 2: $\mathscr{T}(B_r)$ is uniformly bounded.
This is clear since $\mathscr{T}(B_r) \subset B_r$ is bounded.

Step 3: We show that $\mathscr{T}(B_r)$ is equi-continuous.
Let $t_1 > t_2 \in J$ with B_r be a bounded set of $C_{1-\nu,\psi}$ as in Step 2, and $\mathfrak{h} \in B_r$. Then

$$\begin{aligned}
&\left|(\psi(t_1)-\psi(a))^{1-\nu}(\mathscr{T}\mathfrak{h})(t_1)-(\psi(t_2)-\psi(a))^{1-\nu}(\mathscr{T}\mathfrak{h})(t_2)\right| \\
&\leq \left|\frac{(\psi(t_1)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^{t_1} e^{\frac{\vartheta-1}{\vartheta}(\psi(t_1)-\psi(a))}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds\right. \\
&\quad \left. -\frac{(\psi(t_2)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^{t_2} e^{\frac{\vartheta-1}{\vartheta}(\psi(t_2)-\psi(a))}\psi^{'}(s)(\psi(t_2)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds\right| \\
&\leq \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^{\tau_1}\left[(\psi(\tau_1)-\psi(a))^{1-\nu}(\psi(\tau_1)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(a))^{1-\nu}(\psi(\tau_2)-\psi(s))^{\alpha-1}\right] \\
&\quad \times \psi^{'}(s)\left|\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right| ds \\
&\quad + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_{\tau_2}^{\tau_1}(\psi(\tau_2)-\psi(a))^{1-\nu}(\psi(\tau_2)-\psi(s))^{\alpha-1}\psi^{'}(s)\left|\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right| ds
\end{aligned}$$

right hand side of the inequality approaches to zero, as $t_1 \to t_2$. Therefore by Steps 1-3 together with the Arzela-Ascoli theorem, we say that $\mathscr{T}$ is continuous and compact. Hence by Theorem 2.1, the operator $\mathscr{T}$ has a fixed point which is a solution of the problem (1.1). □

Lemma 3.1. *Assume that the hypothesis $(H1)$ is satisfied. If*

$$\frac{2\ell}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^{\alpha} < 1.$$

Then, problem (1.1) has a unique fixed point.

Proof. Consider the operator $\mathscr{T}: C_{1-\nu,\psi} \to C_{1-\nu,\psi}$ defined by

$$\begin{aligned}
(\mathscr{T}\mathfrak{h})(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} \\
&+ \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds.
\end{aligned}$$

Clearly the operator $\mathscr{T}$ is well defined. Now for any $\mathfrak{h}_1, \mathfrak{h}_2 \in C_{1-\nu}$, we attain

$$\begin{aligned}
&\left|((\mathscr{T}\mathfrak{h}_1)(t)-(\mathscr{T}\mathfrak{h}_2)(t))(\psi(t)-\psi(a))^{1-\nu}\right| \\
&\leq \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}_1(s),\mathfrak{h}_1(\lambda s))-\mathfrak{g}(s,\mathfrak{h}_2(s),\mathfrak{h}_2(\lambda s))\right| ds \\
&\leq \frac{2\ell(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\left\|\mathfrak{h}_1-\mathfrak{h}_2\right\|_{C_{1-\nu,\psi}} \\
&\leq \frac{2\ell}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^{\alpha}\left\|\mathfrak{h}_1-\mathfrak{h}_2\right\|_{C_{1-\nu,\psi}}.
\end{aligned}$$

It follows that $\mathscr{T}$ has a contraction map, there exists a unique solution of problem (1.1). □

Theorem 3.2. *The hypotheses $(H1)$ and $(H2)$ are satisfied. Then Eq. (1.1) is g-UHR stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality (2.5) and by Theorem 3.1, $\mathfrak{h}$ is a unique solution of Eq. (1.1) is as follows

$$\begin{aligned}
\mathfrak{h}(t) =& \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} \\
&+ \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds.
\end{aligned}$$

By inequality (2.5), we obtain

$$\left|\mathfrak{v}(t) - \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} - \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{v}(s),\mathfrak{v}(\lambda s))ds\right| \leq \lambda_{\varphi}\varphi(t).$$

Hence for every $t \in J$, we have

$$\begin{aligned}
&|\mathfrak{v}(t)-\mathfrak{h}(t)| \\
&\leq \left|\mathfrak{v}(t) - \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} - \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))ds\right| \\
&\leq \left|\mathfrak{v}(t) - \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} - \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{v}(s),\mathfrak{v}(\lambda s))ds\right| \\
&\quad + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{v}(s),\mathfrak{v}(\lambda s))-\mathfrak{g}(s,\mathfrak{h}(s),\mathfrak{h}(\lambda s))\right|ds \\
&\leq \lambda_{\varphi}\varphi(t) + \frac{2\ell}{\Gamma(\alpha)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{v}(s)-\mathfrak{h}(s)\right|ds.
\end{aligned}$$

By Lemma 2.3, there exists a constant $c > 0$ such that

$$|\mathfrak{v}(t)-\mathfrak{h}(t)| \leq C_{\mathfrak{g},\varphi}\lambda_{\varphi}\varphi(t).$$

Hence, Eq. (1.1) is g-UHR stable. □

4. Conclusion

We have studied a nonlinear fractional differential equation with unknown function together with its lower-order fractional derivative. Several existence and uniqueness results have been derived by applying different tools of the fixed point theory. Our results are quite general and give rise to many new cases by assigning different values to the parameters involved in the problem.

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