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The Physical Concepts on Special Tube Surfaces Generated by Normal Curves in Galilean 3-Space

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Abstract
In this study, we examine the tube surfaces formed by normal curves in Galilean 3-space, and we give Clairaut's theorem on the tube surfaces using geodesic normal curves. Also, we attempted to explain why the specific kinetic energy and angular momentum of particles may be on tube surfaces.

1. Introduction

The development of partial differential equations plays an important role in mathematics. A geodesic is a locally length-minimizing curve. The geodesic equation is expressed by establishing a one of the necessary theoretical foundations of relativity.

A Lagrangian formulation of the geodesic deviation equations is constructed for particle interpretations of quantum. The constants of motion that are of course constant along a geodesic are expressed by the geodesic equations. This constant is the result of one-parameter group of symmetries on the surface, and the surface is invariant under any one-parameter group of symmetries. In mathematical language, this quantity is a constant obtained by Clairaut for geodesic movement on a surface defined in a coordinate system adapted to this one-parameter group of symmetries [19].

Many studies of tube surfaces, including rectifying curves, the Darboux frame, the geodesic curve, the Mean curvature, the Gaussian curvature, have received much attention from our researchers. Among them, we can cite our work [2], in which we described the rotational surfaces using curves and matrices in Galilean 4-space. We examined the tube surfaces generated by special curves in G_3 and gave certain conditions describing the geodesics on the surfaces [3,5]. We studied Weingarten, HK-quadric, harmonic tubular surfaces, the conditions of geodesic on this surface using the help of Clairaut's theorem in G_3 [4]. We expressed the specific kinetic energy, the specific angular momentum, and the conditions of being geodesic on a rotational surface generated by a magnetic curve with the Killing magnetic field [6].

The tubular surface and the characterizations of the parameter curves of this surface have been investigated in Euclidean space, see [1,10-11]. In [8], the author defined the tubular surfaces in Galilean space and the differential features of tubular surfaces. In [12], they analyzed the problem of constructing a family of surfaces from a given space-like (or time-like) geodesic curve using the Frenet frame of the curve in Minkowski space and they expressed the family of surfaces as a linear combination of the components of this frame and the necessary and sufficient conditions for the coefficients to satisfy both the geodesic and the isoparametric requirements were given by the authors. In [6], the authors investigated some curves on a plane and in space and they stated the position vectors and gave some theorems about such curves in the Galilean plane G_2. Furthermore, the slant helices were given in G_3. In [20], the theory of the curves in Galilean space was studied. Also, some results were studied on surfaces in G_3 [9,13,18,22,25].

According to references [23, 24], the specific energy of the particle is constant because of its motion in space, which is very important in terms of its specific energy and angular momentum.

In this paper, the speed being constant along a geodesic is shown on the tube surface using Clairaut's

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theorem. Furthermore, the geodesic formulae are given using some parameters. Also, the energy and angular momentum on these surfaces that generated normal curves in G_3 are expressed.

2. Preliminaries

The classical context of Euclidean space is the origin of results that could be transferred to some other geometries. One way of defining new geometries is through Cayley-Klein spaces. They are expressed as projective spaces P_nR with an absolute figure, a subset of P_nR originating of a sequence of quadrics and planes 1. By means of the absolute figure, metric connections are also well defined, and they are invariant under the group of movements.
The scalar product and cross product of the vectors $U=(u_1,u_2,u_3)$, $V=(v_1,v_2,v_3)$ in G_3 is defined as

$$\langle U,V\rangle=\begin{cases}u_1v_1, & if\ u_1\neq 0 \text{ or } v_1\neq 0\\ u_2v_2+u_3v_3, & if\ u_1=0, v_1=0\end{cases} \tag{1}$$

and

$$U\times V=\begin{cases}\begin{pmatrix}0,\\ v_1u_3-v_3u_1,\\ v_2u_1-v_1u_2\end{pmatrix}, & \text{if } u_1\neq 0 \text{ or } v_1\neq 0\\ \begin{pmatrix}v_3u_2-v_2u_3,\\ 0,\\ 0\end{pmatrix}, & \text{if } u_1=0, v_1=0\end{cases} \tag{2}$$

[14].

Let $\varrho: I\subset\mathbb{R}\to G_3$ be a curve given by

$\varrho(s)=(s,y(s),z(s))$.

The vectors of the Frenet-Serret frame are defined by $t(s)=\left(1,y'(s),z'(s)\right)$; $n(s)=\frac{t'(s)}{\kappa(s)}$; $b(s)=\frac{n'(s)}{\tau(s)}$,

where the real valued functions $\kappa(s)=\| t'(s)\|$ is defined as the first curvature the curve and $\tau(s)=\| n'(s)\|$ is said to be as the second curvature function. Frenet-Serret equations are given by

$$t'=\kappa n, n'=\tau b, b'=-\tau n. \tag{3}$$

Let the equation of a surface $\Gamma=\Gamma(s,v)$ in G_3 be given by

$$\Gamma(s,v)=\left(x(s,v),y(s,v),z(s,v)\right), \tag{4}$$

and the unit normal vector field η on $\Gamma(s,v)$ is given by

$$\eta=\frac{\Gamma_{,1}\times\Gamma_{,2}}{\|\Gamma_{,1}\times\Gamma_{,2}\|}, \tag{5}$$

and the partial differentiations according to s and v are expressed by

$$\Gamma_{,1}=\frac{\partial\Gamma(s,v)}{\partial s}; \Gamma_{,2}=\frac{\partial\Gamma(s,v)}{\partial v}. \tag{6}$$

Also, δ is written as

$$\delta=\frac{x_{,2}\Gamma_{,1}-x_{,1}\Gamma_{,2}}{w}, \tag{7}$$

where $x_{,1}=\frac{\partial x(s,v)}{\partial s}$, $x_{,2}=\frac{\partial x(s,v)}{\partial v}$ and $w=\|\Gamma_{,1}\times\Gamma_{,2}\|$.

Let us define

$$g_1=x_{,1}, g_2=x_{,2}, g_{ij}=g_ig_j; g^1=\frac{x_{,2}}{w},$$

$$g^2=\frac{x_{,1}}{w}, g^{ij}=g^ig^j; i,j=1,2; \tag{8}$$

$$h_{11}=\langle\Gamma_{,1}^*,\Gamma_{,1}^*\rangle; h_{12}=\langle\Gamma_{,1}^*,\Gamma_{,2}^*\rangle; \quad h_{22}=\langle\Gamma_{,2}^*,\Gamma_{,2}^*\rangle, \tag{9}$$

where $\Gamma_{,1}^*$ and $\Gamma_{,2}^*$ are the projections of the vectors $\Gamma_{,1}$ and $\Gamma_{,2}$ on the yz-plane, respectively, and the corresponding matrix of the first fundamental form ds^2 of the surface $\Gamma(s,v)$ is given by

$$ds^2=ds_1^2+ds_2^2=(g_1ds+g_2dv)^2+\quad\varepsilon(h_{11}ds^2+2h_{12}dsdv+h_{22}dv^2) \tag{10}$$

where

$$\varepsilon=\begin{cases}0, & dw:dv_1 \text{non}-\text{isotropic}\\ 1, & dw:dv_1 \text{isotropic}\end{cases}, \quad [19]. \tag{11}$$

In this case, the coefficients of ds^2 are defined as g_{ij}^*. That is, it can be given in terms of g_i and h_{ij} by

$$w^2=g_1^2h_{22}-2g_1g_2h_{12}+g_2^2h_{11}.$$

The Gaussian and mean curvatures are expressed by means of the coefficients of L_{ij}, they are the normal components of $\Gamma_{,i,j}(i,j=1,2)$. That is,

$$\Gamma_{,i,j}=\sum^2\Gamma_{ij}^k\Gamma_{,k}+L_{ij}\eta, \tag{12}$$

where L_{ij} are written by

$$L_{ij}=\frac{1}{g_1}\langle g_1\Gamma_{,i,j}^*-g_{i,j}\Gamma_{,1}^*,\eta\rangle$$
$$=\frac{1}{g_2}\langle g_2\Gamma_{,i,j}^*-g_{i,j}\Gamma_{,2}^*,\eta\rangle, \tag{13}$$

and the curvatures K and H of the surface are wtitten as follows

$$K=\frac{L_{11}L_{22}-L_{12}^2}{w^2}, H=\frac{g_2^2L_{11}-2g_1g_2L_{12}+g_1^2L_{22}^2}{w^2}, \tag{14}$$

[21].

momentum on these surfaces generated normal curves in G_3 are expressed.

Definition 1. A surface generated by the rotation of a regular parametrized plane curve $\varrho(s) = (h(s), 0, h(s))$ around an z-axis in its plane. Then, the position vector of the rotational surface is

$$\sigma(s, v_i) = \big(\rho(s)\cos v_i, \rho(s)\sin v_i, h(s)\big); \tag{15}$$

$$s \in I, 0 \leq v_i \leq 2\pi, x = \rho(s) > 0, z = h(s),$$

where ρ is the distance between a point on the surface and the z-axis of rotation and v_i is the angle of rotation, [14, 19].

Definition 2. Let $\varrho: I \subset \mathbb{R} \to M$ be a curve given as

$$\varrho(s) = \big(x(w(s), v_i(s)), y(w(s), v_i(s)), z(w(s), v_i(s))\big),$$

which is an arc-length parametrized geodesic on an rotational surface for differential equations given by $(w(s), v_i(s))$. By using the Lagrangian, the line element of the rotational surface is:

$$L = \dot{w}^2 + \rho^2 \dot{v_i}^2,$$

and

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w}; \quad \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v_i}{\partial s}}\right) = \frac{\partial L}{\partial v_i};$$

$$\ddot{w} = \rho\rho' \dot{v}_i^2; \frac{d}{ds}(\rho \dot{v}_i^2) = 0 \tag{16}$$

so that is a constant of the motion and the previous equations are said to be as Euler-Lagrange equations, [14,19].

Definition 3. A vector $x = (x_1, x_2, x_3)$ is called a non-isotropic if $x_1 \neq 0$. All unit isotropic vectors are of the form $x = (1, x_2, x_3)$. For isotropic vectors $x_1 = 0$ hold, [14].

Theorem 1. (Clairaut's Theorem)Let M be a surface of rotation and let ϱ be a geodesic in M. Also, let ρ be the distance from the curve to axis of rotation and let θ express the angle between ϱ' and the longitudinal curve through ϱ. Then, if ϱ is a geodesic $\rho sin\theta$ is constant along the curve. On the contrary, if $\rho sin\theta$ is a constant, then the curve ϱ is a geodesic, [19].

Definition 4. A one-parameter group of diffeomorphisms of a manifold M is a smooth map $\psi: M \times \mathbb{R} \to M$, such that $\psi_t(x) = \psi(x, t)$, where

1) $\psi_t: M \to M$ *is a diffeomorphism,*
2) $\psi_o = id.$
3) $\psi_{s+t} = \psi_s o \psi_t.$

This group is associated with a vector field W given by $\frac{d}{dt}\psi_t(x) = W(x)$.

If a one-parameter group of isometries is formed by a vector field W and this vector is said to be as Killing vector field, [15].

3. Some Characteristics of Normal Curves in G_3

In this section, normal curves in G_3 are described using the components of their position vectors.

Theorem 2. Let $\varrho_i: I \subset \mathbb{R} \to G_3$ be a regular isotropic normal curve with curvatures $\kappa(w) \geq 0, \tau$ in $G_3, i = 1,2$. Then, the position vectors of ϱ_i hold following equalities*:*

1) If $\tau(w)$ =constant, the normal curve is given by

$$\varrho_1(w) = (2\eta_1 cos\tau w)\overleftarrow{n} - (2\eta_1 sin\tau w)\overleftarrow{b}; \ \eta_i \in \mathbb{R}, i \in \{1,2\}.$$

2) If $\tau(w)$ ≠constant, the normal curve is given by

$$\begin{aligned}\varrho_2(w) &= \gamma_3 e^{\frac{-\dot{\tau}}{2\tau}w}\left(cosh\left(\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w}\right)\right)\overleftarrow{n} \\ &+ \frac{\gamma_3 e^{\frac{-\dot{\tau}}{2\tau}w}}{\tau}\begin{pmatrix} \left(\frac{-\dot{\tau}}{2\tau}w\right)_w cosh\left(\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w}\right) \\ +\left(\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w}\right)_w sinh\left(\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w}\right)\end{pmatrix}\overleftarrow{b},\end{aligned}$$

where $\gamma_3, \eta_1, \eta_2 \in \mathbb{R}_0$.

Proof. Assume that $\varrho(w)$ is an normal curve with the curvature functions $\kappa(w)$, $\tau(w)$ in G_3 as follows

$$\varrho(w) = \Sigma_0 \overleftarrow{n} + \Sigma_1 \overleftarrow{b}, \tag{17}$$

for some differentiable functions $\Sigma_0(w)$, $\Sigma_1(w)$. Thus, differentiating (17) with respect to w and using (3), we get

$$\overleftarrow{t} = (\dot{\Sigma}_0 - \tau\Sigma_1)\overleftarrow{n} + (\tau\Sigma_0 + \dot{\Sigma}_1)\overleftarrow{b}, \tag{18}$$

by multipling both sides of (3.2) by t, n, b, we have

$$\dot{\Sigma}_0 - \tau\Sigma_1 = 0; \tau\Sigma_0 + \dot{\Sigma}_1 = 0, \tag{19}$$

respectively and using (19) and making necessary calculations, we can write

1) if $\tau(w)$ ≠constant, we get

$$\Sigma_0 = e^{\frac{-\dot{\tau}}{2\tau}w}\begin{pmatrix} \gamma_1 e^{\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2-\tau^2}w} \\ +\gamma_2 e^{-\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2-\tau^2}w} \end{pmatrix}, \tag{20}$$

and taking $\gamma_1 = \gamma_2 = \frac{\gamma_3}{2}$, we get

$$\Sigma_0 = \gamma_3 e^{\frac{-\dot{\tau}}{2\tau}w}(\cosh(\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w})) \tag{21}$$

and using the equation $\dot{\Sigma_0} = \tau\Sigma_1$, we obtain

$$\Sigma_1 = \frac{\gamma_3}{\tau}e^{\frac{-\dot{\tau}}{2\tau}w}\begin{pmatrix} (\frac{-\dot{\tau}}{2\tau}w)_w\cosh(\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w}) \\ +(\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w})_w\sinh(\sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w}) \end{pmatrix}. \tag{22}$$

Hence, from (21) and (22), the position vector is obtained.

2) if $\tau(w)$ =constant, we get

$$\Sigma_0 = (\eta_1 + \eta_2)\cos\tau w + i(\eta_1 - \eta_2)\sin\tau w; \tag{23}$$

$$\Sigma_1 = i\big((\eta_1 - \eta_2)\cos\tau w + i(\eta_1 + \eta_2)\sin\tau w\big) \tag{24}$$

and taking $\eta_1 = \eta_2$, we have

$$\varrho_1(w) = (2\eta_1\cos\tau w)\overleftarrow{n} - (2\eta_1\sin\tau w)\overleftarrow{b},$$

where $\eta_i, \gamma_i \in \mathbb{R}, i \in \{1,2,3\}$.

4. The Special Tube Surfaces Formed by Normal Curves in Galilean 3-space

In this part, special tube surfaces formed by a normal curve have been examined mathematically.

A canal surface is defined as a one-parameter set of spheres whose centres are described by a radius function ρ and the orbit $\varrho_i(w)$(spine curve), in addition to parametrizing the spine curve via the Frenet frame. If the radius function ρ is constant, the canal surface is said to be the tube or pipe surface [9].

Let us denote by ρ the vector connecting the point from the parametrized curve $\varrho_i(w)$ with the point from the surface. Afterwards, we have the position vector R of a point on the surface as

$$R = \varrho_i(w) + \rho = \varrho_i(w) + A\left(\cos v\vec{n} + \sin v\vec{b}\right), \tag{25}$$

where A is a constant radius of Euclidean circle of Frenet frame in G_3, v is the angle between n and ρ that ρ lies in the Euclidean normal plane of the curve $\varrho_i(w)$.

4.1 The Clairaut's Theorem on Special Tube Surface Formed by Normal Curve in G_3

In this subsection, using the Clairaut's theorem, the specific tube surfaces with normal curve are characterized. Also, the general equation of geodesics on the tube surfaces is given in G_3.

Theorem 3. Let $\varrho_i: I \subset \mathbb{R} \to G_3$ be a regular isotropic curve for $\kappa(w) \geq 0, \tau$ in $G_3, i = 1,2$ and let $\Gamma^i(w,v)$ be the tube surface formed by the normal curve. Then, the following statements hold:

a) If $\tau(w)$ =constant, there is no the tube surface generated by the normal curve.

b) If $\tau(w)$ ≠constant, the tube surface generated by the normal curve is given by

$$\Gamma^2(w,v) = (\gamma_3 e^f\cosh g + A\cos v)\overleftarrow{n}$$
$$+\left\{\frac{\gamma_3 e^f}{\tau}(f_w\cosh g + g_w\sinh g) + A\sin v\right\}\overleftarrow{b}$$

and $A = \frac{-\gamma_3\rho(w)\cos v}{\tau(w)}$, where $\gamma_3 \in \mathbb{R}$ and for isotropic vectors, the first fundamental form is given by

$$I = \begin{pmatrix}\tau^2(w)A^2 + \gamma_3^2\rho^2(w) \\ +2\gamma_3\tau(w)\rho(w)A\cos v\end{pmatrix}\dot{w}^2 + A^2\dot{v}^2.$$

b_1) For the equation $2\int E(w,v)\partial w = c_2 s + c_3$, $\varrho_2(w)$ is a geodesic on $\Gamma^2(w,v)$ necessary and sufficient condition the following equations satisfied

$$A^2\ddot{v} + A2\tau(w)\gamma_3\sin v\dot{w}^2 = 0;$$

$$v = \arccos\left(\frac{\tau\tau_w A^2 + \rho\rho_w}{A\gamma_3(\tau_w\rho + \tau\rho_w)}\right); w = \int \cos\theta ds,$$

where

$$\xi^1 = e^f\cosh g, \xi^2 = f_w\cosh g + g_w\sinh g,$$

$$\xi^3 = e^f\xi^2 = \xi^1_w; h(w) = \frac{d}{dw}\left(\frac{\xi^3}{\tau}\right);$$

$$\rho(w) = \tau(w)\xi^1(w) + h(w);$$

$$f = \frac{-\dot{\tau}}{2\tau}w, g = \cosh(\sqrt{(\frac{\dot{\tau}}{2\tau})^2 - \tau^2 w}).$$

Proof. The specific tube surface generated by normal curve $\varrho_2(w)$ is parametrized by

$$\Gamma^2(w,v) = \varrho_2(w) + A(\cos v\overleftarrow{n} + \sin v\overleftarrow{b}). \tag{26}$$

i) For $\tau(w) \neq$constant, by using the following equation

$$\varrho_2(w) = (\gamma_3 e^{f(w)}\cosh g(w))\overleftarrow{n} + \\ + \frac{\gamma_3 e^{f(w)}}{\tau}(f_w\cosh g(w) + g_w\sinh g(w))\overleftarrow{b}, \tag{27}$$

where

$$f(w) = -\frac{\dot{\tau}}{2\tau}w; g(w) = \sqrt{\left(\frac{\dot{\tau}}{2\tau}\right)^2 - \tau^2 w},$$

we can write the tube surface as

$$\Gamma^2(w,v) = \begin{matrix} \{\gamma_3 e^{f(w)}\cosh g(w) + A\cos v\}\overleftarrow{n} \\ +\left\{\begin{matrix}\frac{\gamma_3 e^{f(w)}}{\tau}\begin{pmatrix} f_w\cosh g(w) \\ +g_w\sinh g(w)\end{pmatrix} \\ +A\sin v\end{matrix}\right\}\overleftarrow{b},\end{matrix} \tag{28}$$

where v is the angle between $\overleftarrow{n}$ and $\overleftarrow{A}$, we can write the equation

$$A = \frac{-\gamma_3\rho(w)\cos v}{\tau(w)}, \tag{29}$$

where

$$\xi^1 = e^f\cosh g, \xi^2 = f_w\cosh g + g_w\sinh g,$$

$$\xi^3 = e^f\xi^2 = \xi^1_w;$$

$$h(w) = \frac{d\left(\frac{e^f}{\tau}(f_w\cosh g + g_w\sinh g)\right)}{dw} = \frac{d\left(\frac{\xi^3}{\tau}\right)}{dw};$$

$$\rho(w) = \tau(w)\xi^1(w) + h(w).$$

Recall, since $\tau \neq 0$, we can write

$$\Gamma^2_w = (-\tau A\sin v)\overleftarrow{n} + \{\gamma_3\rho(w) + \tau A\cos v\}\overleftarrow{b}) = N_w;$$

$$\Gamma^2_v = A(-\sin v\overleftarrow{n} + \cos v\overleftarrow{b}) = AN_v,$$

it follows that the vector cross product of these vectors is found out by

$$\Gamma^2_w \times \Gamma^2_v = A\gamma_3\rho(w)\sin v\overleftarrow{t} \tag{30}$$

and from (30) , the unit normal vector η of $\Gamma^2(w,v)$ is found as follows

$$\eta = \overleftarrow{t}. \tag{31}$$

Furthermore, from (7), we get

$$\delta = \frac{-\Gamma^2_v}{A} = \sin v\overleftarrow{n} - \cos v\overleftarrow{b}.$$

For the isotropic vectors $\overleftarrow{n}$ and $\overleftarrow{b}$ and by using the Frenet frame in Galilean space, we can find

$$x(w,v) = 0; x_w = g_1 = 0, x_{v_2} = g_2 = 0;$$

$$g_{11} = g_{12} = g_{22} = 0; g^1 = 0, g^2 = 0; \tag{32}$$

$$h_{11} = E(w,v) = \tau^2(w)A^2 + \gamma_3^2\rho^2(w) + \\ 2\gamma_3\tau(w)\rho(w)A\cos v; \tag{33}$$

$$h_{12} = 0, h_{22} = A^2. \tag{34}$$

Then, we substitute (33) and (34), (32) into the equation (10). Hence, the first fundamental form of the tube surface by generated normal curve in Galilean space can be written as

$$I = E(w,v)dw^2 + A^2dv^2. \tag{35}$$

Furthermore, since $\tau \neq 0$, we obtain the first fundamental form with two variable parameter. Hence, we write orthogonal coordinates of parametrization. So, by considering the first fundamental form, the Lagrangian can be written as

$$E(w,v)\dot{w}^2 + A^2\dot{v}^2 = L \tag{36}$$

and we know that a geodesic on the surface $\Gamma^2(w,v)$ can be found by using the Euler-Lagrangian equations

$$\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w}; \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v}. \tag{37}$$

a) If $\tau(w) \neq$constant, for the equation given by

$$A^2\ddot{v} + 2\tau(w)A\gamma_3\sin v\dot{w}^2 = 0, \tag{38}$$

the second Lagrangian equation is given by $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial v}{\partial s}}\right) = \frac{\partial L}{\partial v} \neq 0$ and for the equation

$$v = \arccos\left(\frac{\tau\tau_w A^2 + \rho\rho_w}{A\gamma_3(\tau_w\rho + \tau\rho_w)}\right),$$

the equation $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w} = 0$ holds. Thus, we can write $\frac{\partial L}{\frac{\partial w}{\partial s}} = 2E(w,v)\dot{w} =$ constant and which means

$$2\int E(w,v)\partial w = c_2 s + c_3 \tag{39}$$

and let $\varrho_2(w)$ be a geodesic on the surface of $\Gamma^2(w,v)$, so it is given by $(w(s),v(s))$. Also, let θ be the angle between $\dot{\varrho}_2$ and a meridian and N_w is the vector pointing along meridians of Γ^2; N_v is the vector pointing along parallels of Γ^2. Hence, we can say that orthonormal basis $\{N_w, N_v\}$ and $\dot{\varrho}_2$ is found out as

$$\dot{\varrho}_2 = N_w\cos\theta + N_v\sin\theta = \dot{w}\Gamma_w^2 + \dot{v}\Gamma_v^2 = \dot{w}N_w + \dot{v}AN_v.$$

We see that $\dot{w} = \cos\theta$, so we can write as $2E(w,v)\dot{w} = 2E(w,v)\cos\theta$=constant along $\varrho_2(w)$. Conversely, $\varrho_2(w)$ is the normal curve with $2E(w,v)\cos\theta =$ constant. Hence, the second Euler Lagrange equation is held. If the differential of the L value is taken and added to the first equation we have

$$w = \int \cos\theta ds. \tag{40}$$

$b)$ For τ =constant, the tube surface generated by the curve ϱ_1 is parametrized by

$$\Gamma^1(w,v) = \varrho_1(w) + A_1(\cos v\overleftarrow{n} + \sin v\overleftarrow{b}), \tag{41}$$

where v is angle between the isotropic vectors $\overleftarrow{n}$ and $\overleftarrow{A_1}$. Clearly,

$$\Gamma^1(w,v) = (2\eta\cos\tau w + A_1\cos v)\overleftarrow{n}$$
$$+(-2\eta\sin\tau w + A_1\sin v)\overleftarrow{b}. \tag{42}$$

Recall, we can write

$$\Gamma_w^1 = (-\tau A_1\sin v)\overleftarrow{n} + \tau A_1\cos v\overleftarrow{b} = \tau A_1 N_w;$$

$$\Gamma_v^1 = A_1(-\sin v\overleftarrow{n} + \cos v\overleftarrow{b}) = A_1 N_v.$$

Therefore, we have $\Gamma_w^1 \times \Gamma_{v_1}^1 = 0$, that means there is no such surface in G_3.

Theorem 4. The general equations of geodesics on the tube surface Γ^2 formed by the normal curve in G_3, for the special parameters $w = \int \cos\theta ds($or $2\int E(w,v)\partial w = c_2 s + c_3)$ and $v = \arccos\left(\frac{\tau\tau_w A^2+\rho\rho_w}{A\gamma_3(\tau_w\rho+\tau\rho_w)}\right)$, are given by

$$\frac{dv}{dw} = \frac{c_{11}E(w,v)}{A}\sqrt{L - \frac{c_{10}}{E(w,v)}};$$

$$\frac{dv}{dw} = \frac{1}{A\cos\theta}\sqrt{L - E(w,v)\cos^2\theta}, \tag{43}$$

where $c_i \in \mathbb{R}_0$.

Proof. We consider the Euler-Lagrange equations in (4.13) for the general equation of geodesics. For $w = \int \cos\theta ds$ or $2\int E(w,v)\partial w = c_2 s + c_3$, we explain the equation of geodesic, solving the equation in $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w} = 0$, we obtain

$$\dot{w} = \frac{c_2}{2E(w,v)}; \dot{w} = \cos\theta.$$

If we put the value of $\dot{w}$ at the Lagrange equation,

$$E(w,v)\left(\frac{dw}{ds}\right)^2 + A^2\left(\frac{dv}{dw}\frac{dw}{ds}\right)^2 = L.$$

Hence, we obtain the general equation of geodesics on Γ^2 as

$$\frac{dv}{dw} = \frac{c_{11}E(w,v)}{A}\sqrt{L - \frac{c_{10}}{E(w,v)}}.$$

Furthermore, according to the parameters

$$w = \int \cos\theta ds, v = \arccos\left(\frac{\tau\tau_w A^2 + \rho\rho_w}{A\gamma_3(\tau_w\rho + \tau\rho_w)}\right),$$

the geodesic equation on G_3 is given as

$$\frac{dv}{dw} = \frac{1}{A\cos\theta}\sqrt{L - E(w,v)\cos^2\theta},$$

where $c_i \in \mathbb{R}$.

5. A Physical Study on the Special Tube Surface with Normal Curve in G₃

In this article, we have carried out experiments to explain why the specific kinetic energy and angular momentum of particles, following a path called the trajectory of the particle.

Let $\Gamma^2(w(s),v(s))$ be a curve on the surface and we can write the position vector of this curve as

$$\Gamma^2(w(s),v(s)) = (\gamma_3\xi^1(w(s)) + A\cos v(s))\overleftarrow{n}$$
$$+(\tfrac{\gamma_3}{\tau(w(s))}\xi^3(w(s)) + A\sin v(s))\overleftarrow{b}.$$

Calculating the derivative of this tangent vector along the curve on Γ^2 and using the product and chain rules. Thus, the tangent vector is obtained by

$$\frac{d\Gamma^2(w(s),v(s))}{ds} = \frac{dw(s)}{ds}\Gamma_w^2 + \frac{dv(s)}{ds}\Gamma_v^2; \tag{44}$$

$$\dot{\varrho}_2 = N_w\cos\theta + N_v\sin\theta$$
$$= \dot{w}\Gamma_w^2 + \dot{v}\Gamma_v^2 = \dot{w}N_w + \dot{v}AN_v. \tag{45}$$

The tangent vector of the geodesic curve is said to be as the velocity given as

$$\overleftarrow{W} = \frac{d\Gamma^2(w(s),v(s))}{ds} = W^w\Gamma_w^2 + W^v\Gamma_v^2 \tag{46}$$

and the norm of W is called as the speed. Take into account that $W^{w^*} = \sqrt{E(w,v)}\ .W^w = W\cos\theta$ is the radial velocity and W^v is the horizontal angular velocity. Hence, $W^{v^*} = AW^v = W\sin\theta$ is the horizontal component of the velocity vector.

Physically, the role of the radial variable in this velocity plane can be explained by the speed: the angle θ is expressed the side of the velocity in accordance with $\Gamma_{w^*}^2$, and the physical properties such as energy and momentum that have the mass as a proportional element can be taken instead of the specific statements obtained by partitioning out the mass. So, we can write the specific kinetic energy E_{energy} as

$$E_{energy} = \frac{\left(\sqrt{2E_{energy}}\cos\theta\right)^2 + \left(\sqrt{2E_{energy}}\sin\theta\right)^2}{2} = \frac{W^2}{2} = \frac{1}{2}(W^2\cos^2\theta + W^2\sin^2\theta)$$

$$= \frac{1}{2}E(w,v)\left(\frac{dw}{ds}\right)^2 + \frac{1}{2}A^2\left(\frac{dv}{ds}\right)^2, \tag{47}$$

from the right side of equ. (5.4) the specific energy and speed are constant along geodesic. Physically, the specific energy is constant on account of attribute of its motion, it is thought perpendicular to the surface. Therefore, the specific energy E_{energy} and the speed $W = \sqrt{2E_{energy}}$ have to be constant along a geodesic. Therefore, we can give following theorems in respect to previous expressions that we explain.

Theorem 5. Let $\Gamma^2(w,v)$ be the tube surface generated by the normal curve. Then, for the parameters

$$v = \arccos\left(\frac{\tau\tau_w A^2 + \rho\rho_w}{A\gamma_3(\tau_w\rho + \tau\rho_w)}\right),$$

$$w = \int \cos\theta ds (or \int E(w,v)\partial w = c_2 s + c_3)$$

and the equation $A^2\ddot{v} + 2\tau(w)A\gamma_3\sin v\dot{w}^2 = 0,$ the specific angular momentum ℓ is given by

$$\ell = \sqrt{\tau^2A^2 + \gamma_3^2\rho^2 + 2\gamma_3\tau\rho A\cos v}W\cos\theta,$$

where $\rho = \tau(e^f\cosh g) + \frac{d}{dw}\left(\frac{(e^f\cosh g)_w}{\tau}\right)$ and the specific energy E_{energy} is given by

$$E_{energy} = \frac{1}{2}\left(\frac{\ell^2}{\tau^2A^2 + \gamma_3^2\rho^2 + 2\gamma_3\tau\rho A\cos v} + \frac{\gamma_3^2\rho^2\cos^2 v}{\tau^2}\left(\frac{dv}{ds}\right)^2\right),$$

$$E_{energy} = \frac{\ell}{\sqrt{2}\cos\theta},$$

where the curve $\varrho_2(w)$ is a geodesic on the surface Γ^2.

Proof. For the equations

$$v = \arccos\left(\frac{\tau\tau_w A^2 + \rho\rho_w}{A\gamma_3(\tau_w\rho + \tau\rho_w)}\right),$$

$$w = \int \cos\theta ds \text{ (or } 2\int E(w,v)\partial w = c_2 s + c_3)$$

and

$$A^2\ddot{v} + 2\tau(w)A\gamma_3\sin v\dot{w}^2 = 0,$$

we can write

$$2\sqrt{E(w,v)}\dot{w} = 2\sqrt{E(w,v)}\cos\theta \tag{48}$$

being constant along $\varrho_2(w)$. Also, we may consider as in the case of circular movement round an axis with radius $\|\overleftarrow{R}\| = \sqrt{E(w,v)}$ or $\overleftarrow{R} = -\sqrt{E(w,v)}\overleftarrow{e_2}$, and the velocity $W^{w^*} = \sqrt{E(w,v)}W^w = W\cos\theta = \sqrt{E(w,v)}\frac{dw}{ds} = \sqrt{2E_{energy}}\cos\theta$ in the angular direction multiplied by the radius $\sqrt{E(w,v)}$ of the circle. From the first geodesic equation ℓ is constant along geodesic and the specific angular momentum ℓ can be taken down as following equation

$$\ell = \overleftarrow{e_3}.\left(\overleftarrow{R} \times_{G_3} \overleftarrow{W}\right) = \sqrt{E(w,v)}W\cos\theta, \tag{49}$$

where $E(w,v) = \tau^2A^2 + \gamma_3^2\rho^2 + 2\gamma_3\tau\rho A\cos v$ and $\rho = \tau(e^f\cosh g) + \frac{d}{dw}\left(\frac{(e^f\cosh g)_w}{\tau}\right)$ and since $\sqrt{E(w,v)}\frac{dw}{ds} = W\cos\theta$, we can write $E(w,v)\frac{dw}{ds} = \sqrt{E(w,v)}W\cos\theta$, and ℓ is constant along geodesic. Hence, one gets

$$\ell = E(w,v)\frac{dw}{ds} \Rightarrow \frac{dw}{ds} = \frac{\ell}{E(w,v)}$$

$$\text{or } \ell = \sqrt{E(w,v)}\sqrt{2E_{energy}}\cos\theta. \tag{50}$$

Hence, using (50) from the radial motion and another constant of the motion the specific energy E_{energy} can be written by

$$E_{energy}=\frac{1}{2}\left(E(w,v)\left(\frac{dw}{ds}\right)^2+A^2\left(\frac{dv}{ds}\right)^2\right);$$

$$E_{energy}=\frac{1}{2}\left(\frac{\ell^2}{\tau^2(w)A^2+\gamma_3^2\rho^2(w)+2\gamma_3\tau(w)\rho(w)A\cos v}+\frac{\gamma_3^2\rho^2(w)\cos^2 v_2}{\tau^2(w)}\left(\frac{dv}{ds}\right)^2\right) \quad (51)$$

and from $\ell=\sqrt{E(w,v)}\sqrt{2E_{energy}}\cos\theta$, we find

$$\frac{\ell^2}{2E(w,v)\cos^2\theta}=E_{energy}.$$

6. Conclusion

In this study, the special tube surface formed by a normal curve is investigated, and certain results of describing geodesics on the tube surface are expressed. One important conclusion of our analysis is that the specific energy and the specific angular momentum on free particles of the tube surfaces can be considered in Galilean 3-space. We have carried out to research explain the conditions of being a geodesic normal curve.

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Contributions of the authors

FA put forward the first idea on the stated title and wrote, analyzed and commented on data. MAK analyzed the data and reinterpreted it in a well-organized for. All authors read and approved the final manuscript.

Conflict of Interest Statement

The study is complied with research and publication Ethics

Statement of Research and Publication Ethics

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