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Research Article

New Integral Inequalities for n-polynomial Exponential Type GA-Convex Functions

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Abstract: In this paper, a new type of convex function called n-polynomial exponential type GA-convex functions is introduced. Some algebraic properties of these introduced functions are determined and the new Hermite-Hadamard type inequalities are proved for n-polynomial exponential type convex functions.

n-polinomal Üstel Tip GA-Konveks Fonksiyonlar için Yeni İntegral Eşitsizlikleri

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Anahtar Kelimeler
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Öz: Bu çalışmada, n-polinomial üstel tip GA-konveks fonksiyonlar adı verilen yeni bir konveks fonksiyon tipi tanıtılmıştır. Tanıtılan bu fonksiyonların bazı cebirsel özellikleri belirlenmiştir ve n-polinomal tip konveks fonksiyonlar için bazı yeni Hermite-Hadamard tipi integral eşitsizlikleri kanıtlanmıştır.

1. Introduction

It is known that the theory of convexity has many practices in pure and applied mathematics and plays an significant and principal role in the progress of various fields of engineering, financial mathematics, economics and optimization. New classes of convex functions are still being defined by researchers in recent years Gao et al. (2020). Integral inequalities are marvelous appliance for improving many of properties of convexity. Practices of inequalities for convex functions still continue to increase today, see Rashid et al. (2020). A significant class of convex functions, called GA-convex was introduced by Niculescu (2000). Zhang et. al. (2013) obtained some Hermite-Hadamard type integral inequalities for GA-convex functions. İşcan and Turhan (2016) have gotten Hermite-Hadamard-Fejer type inequalities for GA-convex functions using fractional integrals. Khurshid et. al. (2020) also obtained new Hermite-Hadamard type integral inequalities for GA-convex functions using the conformable fractional integrals. İşcan (2020) proved the Jensen–Mercer inequality for GA-convex functions. For some other papers in the literature on this topic, see (Awan et al., 2020a; Awan et al., 2020b; Budak & Ozcelik, 2020; Butt et al., 2020a; Butt et al., 2020b; Chen et al., 2020; El-Marouf, 2018; Noor & Noor, 2020; Noor et al., 2020; Kadakal & Iscan, 2020; Toplu et al., 2020; Rashid et al, 2019).

2. Material and Methods

Niculescu (2000 and 2003), gave the new definition as follow:

Definition 1 A function $\varpi: \Omega \subseteq \Re_+ = (0,\infty) \to \Re$ is said to be GA-convex function on Ω if

$$\varpi(\gamma^\tau \delta^{1-\tau}) \leq \tau\varpi(\gamma) + (1-\tau)\varpi(\delta) \tag{1}$$

holds for all $\gamma, \delta \in \Omega$ and $\tau \in [0,1]$, where $\gamma^\tau\delta^{1-\tau}$ and $\tau\varpi(\gamma) + (1-\tau)\varpi(\delta)$ are respectively the weighted geometric mean of two positive numbers γ and δ and the weighted arithmetic mean of $\varpi(\gamma)$ and $\varpi(\delta)$.

Definition 2 A non-negative function $\varpi: \Omega \to \Re$ is called n-polynomial convex, if

$$\varpi(\tau\gamma + (1-\tau)\delta) \leq \frac{1}{n}\sum_{j=1}^{n} [1-(1-\tau)^j]\varpi(\gamma) + \frac{1}{n}\sum_{j=1}^{n} [1-\tau^j]\varpi(\delta) \tag{2}$$

holds for all $\gamma, \delta \in \Omega$, $n \in N$ and $\tau \in [0,1]$, see Gao et al. (2020).

Definition 3 A function $\varpi: \Omega \subset \Re\backslash\{0\} \to \Re$ is said to be exponentially GA-convex function, if

$$e^{\varpi(\gamma^{1-\tau}\delta^\tau)} \leq (1-\tau)e^{\varpi(\gamma)} + \tau e^{\varpi(\delta)}, \tag{3}$$

for all $\gamma, \delta \in \Omega$ and $\tau \in [0,1]$, Rashid et al. (2020).

Also note that for $\tau = \frac{1}{2}$ in this definition, Jensen type exponentially GA-convex functions are obtained.

$$e^{\varpi(\sqrt{\gamma\delta})} \leq \frac{1}{2}(e^{\varpi(\gamma)} + e^{\varpi(\delta)}), \tag{4}$$

for all $\gamma, \delta \in \Omega$.

Gao et. al. (2020) obtained a new type of Hermite-Hadamard inequality via n-polynomial harmonically exponential type convexity.

Theorem 1 Let $\varpi: [\eta, \varsigma] \to [0,\infty)$ be an n-polynomial harmonically exponential type convex function. If $\varpi \in L_1[\eta, \varsigma]$, then

$$\frac{n}{2\sum_{j=1}^{n}(\sqrt{e}-1)^j}\varpi(\frac{2\eta\varsigma}{\eta+\varsigma}) \leq \frac{\eta\varsigma}{\varsigma-\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma^2}d\gamma \leq [\frac{\varpi(\eta)+\varpi(\varsigma)}{n}]\sum_{j=1}^{n}(e-2)^j. \tag{5}$$

3. Results

Definition 4 Let $n \in N$. A function $\varpi: \Omega \subset \Re\backslash\{0\} \rightarrow \Re$ is said to be n-polynomial exponential type convex function, if

$$\varpi(\tau\gamma + (1-\tau)\delta) \leq \frac{1}{n}\sum_{j=1}^{n} (e^{\tau}-1)^{j}\varpi(\gamma) + \frac{1}{n}\sum_{j=1}^{n} (e^{1-\tau}-1)^{j}\varpi(\delta), \tag{6}$$

for all $\gamma, \delta \in \Omega$ and $\tau \in [0,1]$.

Definition 5 Let $n \in N$. A function $\varpi: \Omega \subset \Re\backslash 0 \rightarrow \Re$ is called n-polynomial exponential type GA-convex function, if

$$\varpi(\gamma^{1-\tau}\delta^{\tau}) \leq \frac{1}{n}\sum_{j=1}^{n} (e^{1-\tau}-1)^{j}\varpi(\gamma) + \frac{1}{n}\sum_{j=1}^{n} (e^{\tau}-1)^{j}\varpi(\delta), \tag{7}$$

for all $\gamma, \delta \in \Omega$ and $\tau \in [0,1]$.

Theorem 2 Let $\varpi_1, \varpi_2: \Omega \subset \Re\backslash\{0\} \rightarrow \Re$ and ϖ_1, ϖ_2 are n- polynomial exponential type GA-convex functions, then $\varpi_1 + \varpi_2$ is n-polynomial exponential type GA-convex function.

Proof Let ϖ_1 and ϖ_2 are n-polynomial exponential type GA-convex function, then

$$(\varpi_1 + \varpi_2)(\gamma^{1-\tau}\delta^{\tau}) = \varpi_1(\gamma^{1-\tau}\delta^{\tau}) + \varpi_2(\gamma^{1-\tau}\delta^{\tau})$$

$$\leq \frac{1}{n}\sum_{j=1}^{n} (e^{1-\tau}-1)^{j}\varpi_1(\gamma) + \frac{1}{n}\sum_{j=1}^{n} (e^{\tau}-1)^{j}\varpi_1(\delta)$$

$$+\frac{1}{n}\sum_{j=1}^{n} (e^{1-\tau}-1)^{j}\varpi_2(\gamma) + \frac{1}{n}\sum_{j=1}^{n} (e^{\tau}-1)^{j}\varpi_2(\delta)$$

$$= \frac{1}{n}\sum_{j=1}^{n} (e^{1-\tau}-1)^{j}[\varpi_1(\gamma) + \varpi_2(\gamma)] + \frac{1}{n}\sum_{j=1}^{n} (e^{\tau}-1)^{j}[\varpi_1(\delta) + \varpi_2(\delta)]$$

$$= \frac{1}{n}\sum_{j=1}^{n} (e^{1-\tau}-1)^{j}(\varpi_1 + \varpi_2)(\gamma) + \frac{1}{n}\sum_{j=1}^{n} (e^{\tau}-1)^{j}(\varpi_1 + \varpi_2)(\delta).$$

Theorem 3 Let $\varpi: \Omega \subset \Re\backslash\{0\} \rightarrow \Re$. For non-negative real number k, $k\varpi$ is n–polynomial exponential type GA-convex function.

Proof Let ϖ is n–polynomial exponential type GA-convex function, then

$$f(k\varpi)(\gamma^{1-\tau}\delta^{\tau}) \leq k[\frac{1}{n}\sum_{j=1}^{n} (e^{1-\tau}-1)^{j}\varpi_1(\gamma) + \frac{1}{n}\sum_{j=1}^{n} (e^{\tau}-1)^{j}\varpi_1(\delta)]$$

$$= \frac{1}{n}\sum_{j=1}^{n} (e^{1-\tau}-1)^{j}k\varpi(\gamma) + \frac{1}{n}\sum_{j=1}^{n} (e^{\tau}-1)^{j}k\varpi(\delta)$$

$$=\frac{1}{n}\sum_{j=1}^{n}\left(e^{1-\tau}-1\right)^{j}(k\varpi)(\gamma)+\frac{1}{n}\sum_{j=1}^{n}\left(e^{\tau}-1\right)^{j}(k\varpi)(\delta).$$

Theorem 4 Let $\varpi_1: \Omega \subset \Re\backslash\{0\} \rightarrow \Re$ be GA-convex function and $\varpi_2: \Re \rightarrow \Re$ is n-polynomial exponential type convex function, then $\varpi_2 \circ \varpi_1: \Omega \subset \Re\backslash\{0\} \rightarrow \Re$ is n-polynomial exponential type GA-convex function.

Proof

$$(\varpi_2 \circ \varpi_1)(\gamma^{1-\tau}\delta^{\tau})=\varpi_2(\varpi_1(\gamma^{1-\tau}\delta^{\tau}))$$

$$\leq \varpi_2((1-\tau)\varpi_1(\gamma)+\tau\varpi_1(\delta))$$

$$\leq \frac{1}{n}\sum_{j=1}^{n}\left(e^{1-\tau}-1\right)^{j}\varpi_2(\varpi_1(\gamma))+\frac{1}{n}\sum_{j=1}^{n}\left(e^{\tau}-1\right)^{j}\varpi_2(\varpi_1(\delta))$$

$$=\frac{1}{n}\sum_{j=1}^{n}\left(e^{1-\tau}-1\right)^{j}(\varpi_2\circ\varpi_1)(\gamma)+\frac{1}{n}\sum_{j=1}^{n}\left(e^{\tau}-1\right)^{j}(\varpi_2\circ\varpi_1)(\delta).$$

In this part of the paper, the aim is to obtain a new version of the Hermite-Hadamard type inequality via n-polynomial exponential type GA convexity.

Theorem 5 Let $\varpi: [\eta, \varsigma] \rightarrow \Re$ be a n-polynomial exponential type GA-convex function. If $\varpi \in L[\eta, \varsigma]$, then

$$\frac{n\varpi(\sqrt{\eta\varsigma})}{2\sum_{j=1}^{n}(\sqrt{e}-1)^{j}} \leq \frac{1}{\ln\varsigma-\ln\eta}\int_{\eta}^{\varsigma}\frac{\varpi(\gamma)}{\gamma}d\gamma \leq [\frac{\varpi(\eta)+\varpi(\varsigma)}{n}]\sum_{j=1}^{n}(e-2)^{j}. \tag{8}$$

Proof Since ϖ is n-polynomial exponential type GA-convex function, we have

$$\varpi(\gamma^{1-\tau}\delta^{\tau}) \leq \frac{1}{n}\sum_{j=1}^{n}\left(e^{1-\tau}-1\right)^{j}\varpi(\gamma)+\frac{1}{n}\sum_{j=1}^{n}\left(e^{\tau}-1\right)^{j}\varpi(\delta),$$

which lead to,

$$\varpi(\sqrt{\gamma\delta}) \leq \frac{1}{n}\sum_{j=1}^{n}(\sqrt{e}-1)^{j}\varpi(\gamma)+\frac{1}{n}\sum_{j=1}^{n}(\sqrt{e}-1)^{j}\varpi(\delta).$$

Setting $\gamma=\eta^{1-\tau}\delta^{\tau}$ and $\delta=\eta^{\tau}\delta^{1-\tau}$ in above inequality, we obtain

$$\varpi(\sqrt{\eta\varsigma}) \leq \frac{1}{n}\sum_{j=1}^{n}[(\sqrt{e}-1)^{j}][\varpi(\gamma^{1-\tau}\delta^{\tau})+\varpi(\gamma^{\tau}\delta^{1-\tau})].$$

Integrating with respect to τ on $[0,1]$, we get

$$\int_0^1 \varpi(\sqrt{\eta\varsigma})\,d\tau \le \frac{1}{n}\sum_{j=1}^{n}\left(\sqrt{e}-1\right)^j\left[\int_0^1 \varpi(\eta^{1-\tau}\varsigma^{\tau})d\tau+\int_0^1 \varpi(\eta^{\tau}\varsigma^{1-\tau})d\tau\right].$$

Setting $\eta^{1-\tau}\varsigma^{\tau}=\gamma$ and $\eta^{\tau}\varsigma^{1-\tau}=\delta$ in the right side of the above inequality, we get

$$\varpi(\sqrt{\eta\varsigma}) \le \frac{1}{n}\sum_{j=1}^{n}\left(\sqrt{e}-1\right)^j\frac{1}{\ln\varsigma-\ln\eta}\left[\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma+\int_\eta^\varsigma \frac{\varpi(\delta)}{\delta}d\delta\right],$$

$$\frac{n\varpi(\sqrt{\eta\varsigma})}{2\sum_{j=1}^{n}(\sqrt{e}-1)^j} \le \frac{1}{\ln\varsigma-\ln\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma,$$

which finishes the left side inequality. For the right side, replacement the variable of integration as $\gamma=\eta^{1-\tau}\delta^{\tau}$ and using definition of n-polynomial exponential type GA-convexity, we obtain

$$\frac{1}{\ln\varsigma-\ln\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma=\int_0^1 \varpi(\eta^{1-\tau}\varsigma^{\tau})d\tau$$

$$\le \int_0^1\left[\frac{1}{n}\sum_{j=1}^{n}(e^{1-\tau}-1)^j\varpi(\eta)+\frac{1}{n}\sum_{j=1}^{n}(e^{\tau}-1)^j\varpi(\varsigma)\right]d\tau$$

$$=\frac{\varpi(\eta)}{n}\sum_{j=1}^{n}\int_0^1 (e^{1-\tau}-1)^j d\tau+\frac{\varpi(\varsigma)}{n}\sum_{j=1}^{n}\int_0^1 (e^{\tau}-1)^j d\tau$$

$$=\left[\frac{\varpi(\eta)+\varpi(\varsigma)}{n}\right]\sum_{j=1}^{n}(e-2)^j,$$

which finishes the proof.

Corollary 1 Setting $n=1$ in Theorem-5 implies

$$\frac{\varpi(\sqrt{\eta\varsigma})}{2(\sqrt{e}-1)} \le \frac{1}{\ln\varsigma-\ln\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma \le [\varpi(\eta)+\varpi(\varsigma)](e-2). \tag{9}$$

Lemma 1 Let $\varpi:\Omega\subset\Re\backslash\{0\}\to\Re$ be a differentiable function. If $\varpi'\in L[\eta,\varsigma]$, then the equality

$$\frac{\omega\varpi(\eta)+\nu\varpi(\varsigma)}{2}+\frac{2-\nu-\omega}{2}\varpi(\sqrt{\eta\varsigma})-\frac{1}{\ln\varsigma-\ln\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma=\frac{\ln\eta-\ln\varsigma}{4}$$

$$\times\int_0^1 [(1-\nu-\tau)\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}}\varpi'(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}})+(\omega-\tau)\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}}\varpi'(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}})]d\tau, \tag{10}$$

holds for $\nu,\omega\in[0,1]$.

Proof Integrating by parts, we have

$$\int_0^1 (1-\nu-\tau)\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}}\varpi'\left(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}}\right)d\tau$$

$$=\frac{2}{\ln\eta-\ln\varsigma}\left[(1-\nu-\tau)\varpi\left(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}}\right)\Big|_0^1+\int_0^1\varpi\left(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}}\right)d\tau\right].$$

Setting $\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}}=\gamma$ in above integral, we get

$$=\frac{2}{\ln\eta-\ln\varsigma}\left[-\nu\varpi(\varsigma)-(1-\nu)\varpi(\sqrt{\eta\varsigma})\right]+\frac{4}{(\ln\eta-\ln\varsigma)^2}\int_{\sqrt{\eta\varsigma}}^{\varsigma}\frac{\varpi(\gamma)}{\gamma}d\gamma.$$

Similarly,

$$\int_0^1(\omega-\tau)\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}}\varpi'\left(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}}\right)d\tau$$

$$=\frac{2}{\ln\eta-\ln\varsigma}\left[(\omega-\tau)\varpi\left(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}}\right)\Big|_0^1+\int_0^1\varpi\left(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}}\right)d\tau\right].$$

Setting $\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}}=\gamma$ in above integral, we get

$$=\frac{2}{\ln\eta-\ln\varsigma}\left[-\omega\varpi(\eta)-(1-\omega)\varpi(\sqrt{\eta\varsigma})\right]+\frac{4}{(\ln\eta-\ln\varsigma)^2}\int_{\eta}^{\sqrt{\eta\varsigma}}\frac{\varpi(\gamma)}{\gamma}d\gamma.$$

Adding both sides of the equalities and multiplying by $\frac{-(\ln\eta-\ln\varsigma)}{4}$, we obtain the requested result.

Theorem 6 Let $\varpi:\Omega\subset\Re\backslash\{0\}\to\Re$ be a differentiable function such that $\varpi'\in L[\eta,\varsigma]$ and $\nu,\omega\in[0,1]$. If the function $|\varpi'|^q$ is an n-polynomial exponential type GA-convex, then for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|\frac{\omega\varpi(\eta)+\nu\varpi(\varsigma)}{2}+\frac{2-\nu-\omega}{2}\varpi(\sqrt{\eta\varsigma})-\frac{1}{\ln\varsigma-\ln\eta}\int_\eta^\varsigma\frac{\varpi(\gamma)}{\gamma}d\gamma\right.$$

$$\leq\frac{\ln\eta-\ln\varsigma}{4}\left[H_1^{\frac{1}{p}}(T_1|\varpi'(\eta)|^q+T_2|\varpi'(\varsigma)|^q)^{\frac{1}{q}}+H_2^{\frac{1}{p}}(T_3|\varpi'(\eta)|^q+T_4|\varpi'(\varsigma)|^q)^{\frac{1}{q}}\right], \tag{11}$$

where

$$H_1=\int_0^1|1-\nu-\tau|^p d\tau=\frac{(1-\nu)^{p+1}+\nu^{p+1}}{p+1}, \tag{12}$$

$$H_2=\int_0^1|\omega-\tau|^p d\tau=\frac{(1-\omega)^{p+1}+\omega^{p+1}}{p+1}, \tag{13}$$

$$T_1=\frac{1}{n}\sum_{j=1}^{n}\int_0^1(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}})^q(e^{\frac{1-\tau}{2}}-1)^j d\tau, \tag{14}$$

$$T_2=\frac{1}{n}\sum_{j=1}^{n}\int_0^1(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}})^q(e^{\frac{1+\tau}{2}}-1)^j d\tau, \tag{15}$$

$$T_3=\frac{1}{n}\sum_{j=1}^{n}\int_0^1(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}})^q(e^{\frac{2-\tau}{2}}-1)^j d\tau, \tag{16}$$

$$T_4=\frac{1}{n}\sum_{j=1}^{n}\int_0^1(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}})^q(e^{\frac{\tau}{2}}-1)^j d\tau. \tag{17}$$

Proof From Lemma 1, Hölder's inequality and n-polynomial exponential type GA-convexity of $|\varpi'|^q$, we have

$$|\frac{\omega\varpi(\eta)+\nu\varpi(\varsigma)}{2}+\frac{2-\nu-\omega}{2}\varpi(\sqrt{\eta\varsigma})-\frac{1}{\ln\varsigma-\ln\eta}\int_\eta^\varsigma\frac{\varpi(\gamma)}{\gamma}d\gamma|$$

$$\leq\frac{\ln\eta-\ln\varsigma}{4}[\int_0^1|(1-\nu-\tau)\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}}|\times|\varpi'(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}})|d\tau$$

$$+\int_0^1|(\omega-\tau)\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}}||\varpi'(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}})|d\tau]$$

$$\leq\frac{\ln\eta-\ln\varsigma}{4}\{(\int_0^1|(1-\nu-\tau)|^p d\tau)^{\frac{1}{p}}[\int_0^1(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}})^q$$

$$(\frac{1}{n}\sum_{j=1}^{n}(e^{\frac{1-\tau}{2}}-1)^j|\varpi'(\eta)|^q+\frac{1}{n}\sum_{j=1}^{n}(e^{\frac{1+\tau}{2}}-1)^j|\varpi'(\varsigma)|^q)d\tau]^{\frac{1}{q}}$$

$$+(\int_0^1|(\omega-\tau)|^p d\tau)^{\frac{1}{p}}[\int_0^1(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}})^q\times(\frac{1}{n}\sum_{j=1}^{n}(e^{\frac{2-\tau}{2}}-1)^j|\varpi'(\eta)|^q$$

$$+\frac{1}{n}\sum_{j=1}^{n}(e^{\frac{\tau}{2}}-1)^j|\varpi'(\varsigma)|^q)d\tau]^{\frac{1}{q}}\}$$

$$=\frac{\ln\eta-\ln\varsigma}{4}[H_1^{\frac{1}{p}}(T_1|\varpi'(\eta)|^q+T_2|\varpi'(\varsigma)|^q)^{\frac{1}{q}}+H_2^{\frac{1}{p}}(T_3|\varpi'(\eta)|^q+T_4|\varpi'(\varsigma)|^q)^{\frac{1}{q}}],$$

which finishes the proof.

Corollary 2 Setting $n=1$ in Theorem-6 implies

$$|\frac{\omega\varpi(\eta)+\nu\varpi(\varsigma)}{2}+\frac{2-\nu-\omega}{2}\varpi(\sqrt{\eta\varsigma})-\frac{1}{\ln\varsigma-\ln\eta}\int_\eta^\varsigma\frac{\varpi(\gamma)}{\gamma}d\gamma|$$

$$\leq\frac{\ln\eta-\ln\varsigma}{4}[H_1^{\frac{1}{p}}(B_1|\varpi'(\eta)|^q+B_2|\varpi'(\varsigma)|^q)^{\frac{1}{q}}+H_2^{\frac{1}{p}}(B_3|\varpi'(\eta)|^q+B_4|\varpi'(\varsigma)|^q)^{\frac{1}{q}}], \tag{18}$$

where

$$B_1 = \int_0^1 (\eta^{\frac{1-\tau}{2}} \varsigma^{\frac{1+\tau}{2}})^q (e^{\frac{1-\tau}{2}} - 1) d\tau, \tag{19}$$

$$B_2 = \int_0^1 (\eta^{\frac{1-\tau}{2}} \varsigma^{\frac{1+\tau}{2}})^q (e^{\frac{1+\tau}{2}} - 1) d\tau, \tag{20}$$

$$B_3 = \int_0^1 (\eta^{\frac{2-\tau}{2}} \varsigma^{\frac{\tau}{2}})^q (e^{\frac{2-\tau}{2}} - 1) d\tau, \tag{21}$$

$$B_4 = \int_0^1 (\eta^{\frac{2-\tau}{2}} \varsigma^{\frac{\tau}{2}})^q (e^{\frac{\tau}{2}} - 1) d\tau. \tag{22}$$

Corollary 3 Choosing $\nu = \omega$ in Theorem-6 implies

$$|\nu \frac{\varpi(\eta) + \varpi(\varsigma)}{2} + (1-\nu)\varpi(\sqrt{\eta\varsigma}) - \frac{1}{\ln\varsigma - \ln\eta} \int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma} d\gamma|$$

$$\leq \frac{\ln\eta - \ln\varsigma}{4} H^{\frac{1}{p}} [(T_1|\varpi'(\eta)|^q + T_2|\varpi'(\varsigma)|^q)^{\frac{1}{q}} + (T_3|\varpi'(\eta)|^q + T_4|\varpi'(\varsigma)|^q)^{\frac{1}{q}}], \tag{23}$$

where $H_1 = H_2 = H$.

Corollary 4 Choosing $\nu = \omega = 0$ in Theorem-6 implies

$$|\varpi(\sqrt{\eta\varsigma}) - \frac{1}{\ln\varsigma - \ln\eta} \int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma} d\gamma|$$

$$\leq \frac{\ln\eta - \ln\varsigma}{4(p+1)^{\frac{1}{p}}} [(T_1|\varpi'(\eta)|^q + T_2|\varpi'(\varsigma)|^q)^{\frac{1}{q}} + (T_3|\varpi'(\eta)|^q + T_4|\varpi'(\varsigma)|^q)^{\frac{1}{q}}]. \tag{24}$$

Corollary 5 Setting $\nu = \omega = \frac{1}{2}$ in Theorem-6 implies

$$|\frac{\varpi(\eta) + \varpi(\varsigma)}{4} + \frac{\varpi(\sqrt{\eta\varsigma})}{2} - \frac{1}{\ln\varsigma - \ln\eta} \int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma} d\gamma|$$

$$\leq \frac{\ln\eta - \ln\varsigma}{8(p+1)^{\frac{1}{p}}} [(T_1|\varpi'(\eta)|^q + T_2|\varpi'(\varsigma)|^q)^{\frac{1}{q}} + (T_3|\varpi'(\eta)|^q + T_4|\varpi'(\varsigma)|^q)^{\frac{1}{q}}]. \tag{25}$$

Corollary 6 Setting $\nu = \omega = 1$ in Theorem-6 implies

$$|\frac{\varpi(\eta) + \varpi(\varsigma)}{2} - \frac{1}{\ln\varsigma - \ln\eta} \int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma} d\gamma|$$

$$\leq \frac{\ln\eta - \ln\varsigma}{4(p+1)^{\frac{1}{p}}} [(T_1|\varpi'(\eta)|^q + T_2|\varpi'(\varsigma)|^q)^{\frac{1}{q}} + (T_3|\varpi'(\eta)|^q + T_4|\varpi'(\varsigma)|^q)^{\frac{1}{q}}]. \tag{26}$$

Theorem 7 Let $\varpi : \Omega \subset \Re\backslash\{0\} \rightarrow \Re$ be a differentiable function such that $\varpi' \in L[\eta, \varsigma]$ and $\nu, \omega \in [0,1]$. If the function $|\varpi'|^q$ is an n-polynomial exponential type GA-convex, then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$|\frac{\omega\varpi(\eta) + \nu\varpi(\varsigma)}{2} + \frac{2-\nu-\omega}{2}\varpi(\sqrt{\eta\varsigma}) - \frac{1}{\ln\varsigma - \ln\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma|$$

$$\leq \frac{\ln\eta - \ln\varsigma}{4}[D_1^{\frac{1}{p}}(T_5|\varpi'(\eta)|^q + T_6|\varpi'(\varsigma)|^q)^{\frac{1}{q}} + D_2^{\frac{1}{p}}(T_7|\varpi'(\eta)|^q + T_8|\varpi'(\varsigma)|^q)^{\frac{1}{q}}], \tag{27}$$

where

$$D_1 = \int_0^1 (\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}})^p d\tau = \frac{2}{p(\ln\varsigma - \ln\eta)}(\varsigma^p - (\sqrt{\eta\varsigma})^p), \tag{28}$$

$$D_2 = \int_0^1 (\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}})^p d\tau = \frac{2}{p(\ln\varsigma - \ln\eta)}((\sqrt{\eta\varsigma})^p - \eta^p), \tag{29}$$

$$T_5 = \frac{1}{n}\sum_{j=1}^n \int_0^1 |1-\nu-\tau|^q (e^{\frac{1-\tau}{2}} - 1)^j d\tau, \tag{30}$$

$$T_6 = \frac{1}{n}\sum_{j=1}^n \int_0^1 |1-\nu-\tau|^q (e^{\frac{1+\tau}{2}} - 1)^j d\tau, \tag{31}$$

$$T_7 = \frac{1}{n}\sum_{j=1}^n \int_0^1 |\omega-\tau|^q (e^{\frac{2-\tau}{2}} - 1)^j d\tau, \tag{32}$$

$$T_8 = \frac{1}{n}\sum_{j=1}^n \int_0^1 |\omega-\tau|^q (e^{\frac{\tau}{2}} - 1)^j d\tau. \tag{33}$$

Proof.

$$|\frac{\omega\varpi(\eta) + \nu\varpi(\varsigma)}{2} + \frac{2-\nu-\omega}{2}\varpi(\sqrt{\eta\varsigma}) - \frac{1}{\ln\varsigma - \ln\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma|$$

$$\leq \frac{\ln\eta - \ln\varsigma}{4}[\int_0^1 |(1-\nu-\tau)\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}}||\varpi'(\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}})|d\tau$$

$$+\int_0^1 |(\omega-\tau)\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}}||\varpi'(\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}})|d\tau]$$

$$\leq \frac{\ln\eta - \ln\varsigma}{4}\{(\int_0^1 (\eta^{\frac{1-\tau}{2}}\varsigma^{\frac{1+\tau}{2}})^p d\tau)^{\frac{1}{p}}[\int_0^1 |(1-\nu-\tau)|^q$$

$$\times (\frac{1}{n}\sum_{j=1}^n (e^{\frac{1-\tau}{2}} - 1)^j|\varpi'(\eta)|^q + \frac{1}{n}\sum_{j=1}^n (e^{\frac{1+\tau}{2}} - 1)^j|\varpi'(\varsigma)|^q)d\tau]^{\frac{1}{q}}$$

$$+(\int_0^1 (\eta^{\frac{2-\tau}{2}}\varsigma^{\frac{\tau}{2}})^p d\tau)^{\frac{1}{p}}[\int_0^1 |(\omega-\tau)|^q \times (\frac{1}{n}\sum_{j=1}^n (e^{\frac{2-\tau}{2}} - 1)^j|\varpi'(\eta)|^q$$

$$+\frac{1}{n}\sum_{j=1}^{n}(e^{\frac{\tau}{2}}-1)^{j}|\varpi'(\varsigma)|^{q})d\tau]^{\frac{1}{q}}\}$$

$$=\frac{\ln\eta-\ln\varsigma}{4}[D_1^{\frac{1}{p}}(T_5|\varpi'(\eta)|^q+T_6|\varpi'(\varsigma)|^q)^{\frac{1}{q}}+D_2^{\frac{1}{p}}(T_7|\varpi'(\eta)|^q+T_8|\varpi'(\varsigma)|^q)^{\frac{1}{q}}],$$

which completes the proof.

Corollary 7 Setting $n=1$ in Theorem-7 implies

$$|\frac{\omega\varpi(\eta)+\nu\varpi(\varsigma)}{2}+\frac{2-\nu-\omega}{2}\varpi(\sqrt{\eta\varsigma})-\frac{1}{\ln\varsigma-\ln\eta}\int_{\eta}^{\varsigma}\frac{\varpi(\gamma)}{\gamma}d\gamma|$$

$$\leq\frac{\ln\eta-\ln\varsigma}{4}[D_1^{\frac{1}{p}}(B_5|\varpi'(\eta)|^q+B_6|\varpi'(\varsigma)|^q)^{\frac{1}{q}}+D_2^{\frac{1}{p}}(B_7|\varpi'(\eta)|^q+B_8|\varpi'(\varsigma)|^q)^{\frac{1}{q}}],\tag{34}$$

where

$$B_5=\int_0^1|1-\nu-\tau|^q(e^{\frac{1-\tau}{2}}-1)^j d\tau,\tag{35}$$

$$B_6=\int_0^1|1-\nu-\tau|^q(e^{\frac{1+\tau}{2}}-1)^j d\tau,\tag{36}$$

$$B_7=\int_0^1|\omega-\tau|^q(e^{\frac{2-\tau}{2}}-1)^j d\tau,\tag{37}$$

$$B_8=\int_0^1|\omega-\tau|^q(e^{\frac{\tau}{2}}-1)^j d\tau.\tag{38}$$

Corollary 8 Choosing $\nu=\omega$ in Theorem-7 implies

$$|\nu\frac{\varpi(\eta)+\varpi(\varsigma)}{2}+(1-\nu)\varpi(\sqrt{\eta\varsigma})-\frac{1}{\ln\varsigma-\ln\eta}\int_{\eta}^{\varsigma}\frac{\varpi(\gamma)}{\gamma}d\gamma|$$

$$\leq\frac{\ln\eta-\ln\varsigma}{4}[D_1^{\frac{1}{p}}(C_1|\varpi'(\eta)|^q+C_2|\varpi'(\varsigma)|^q)^{\frac{1}{q}}$$

$$+D_2^{\frac{1}{p}}(C_3|\varpi'(\eta)|^q+C_4|\varpi'(\varsigma)|^q)^{\frac{1}{q}}],\tag{39}$$

where

$$C_1=\frac{1}{n}\sum_{j=1}^{n}\int_0^1|1-2\nu|^q(e^{\frac{1-\tau}{2}}-1)^j d\tau,\tag{40}$$

$$C_2=\frac{1}{n}\sum_{j=1}^{n}\int_0^1|1-2\nu|^q(e^{\frac{1+\tau}{2}}-1)^j d\tau,\tag{41}$$

$$C_3 = \frac{1}{n}\sum_{j=1}^{n}\int_0^1 |\nu-\tau|^q (e^{\frac{2-\tau}{2}}-1)^j d\tau, \quad (42)$$

$$C_4 = \frac{1}{n}\sum_{j=1}^{n}\int_0^1 |\nu-\tau|^q (e^{\frac{\tau}{2}}-1)^j d\tau. \quad (43)$$

Corollary 9 Choosing $\nu = \omega = 0$ in Theorem-7 implies

$$|\varpi(\sqrt{\eta\varsigma}) - \frac{1}{\ln\varsigma - \ln\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma|$$

$$\leq \frac{\ln\eta - \ln\varsigma}{4}[D_1^{\frac{1}{p}}(C_5|\varpi'(\eta)|^q + C_6|\varpi'(\varsigma)|^q)^{\frac{1}{q}} + D_2^{\frac{1}{p}}(C_7|\varpi^{'}(\eta)|^q + C_8|\varpi^{'}(\varsigma)|^q)^{\frac{1}{q}}], \quad (44)$$

where

$$C_5 = \frac{1}{n}\sum_{j=1}^{n}\int_0^1 (e^{\frac{1-\tau}{2}}-1)^j d\tau, \quad (45)$$

$$C_6 = \frac{1}{n}\sum_{j=1}^{n}\int_0^1 (e^{\frac{1+\tau}{2}}-1)^j d\tau, \quad (46)$$

$$C_7 = \frac{1}{n}\sum_{j=1}^{n}\int_0^1 \tau^q (e^{\frac{2-\tau}{2}}-1)^j d\tau, \quad (47)$$

$$C_8 = \frac{1}{n}\sum_{j=1}^{n}\int_0^1 \tau^q (e^{\frac{\tau}{2}}-1)^j d\tau. \quad (48)$$

Corollary 10 Choosing $\nu = \omega = \frac{1}{2}$ in Theorem-7 implies

$$|\frac{\varpi(\eta)+\varpi(\varsigma)}{4} + \frac{\varpi(\sqrt{\eta\varsigma})}{2} - \frac{1}{\ln\varsigma - \ln\eta}\int_\eta^\varsigma \frac{\varpi(\gamma)}{\gamma}d\gamma|$$

$$\leq \frac{\ln\eta - \ln\varsigma}{4}[D_2^{\frac{1}{p}}(C_9|\varpi'(\eta)|^q + C_{10}|\varpi'(\varsigma)|^q)^{\frac{1}{q}}], \quad (49)$$

where

$$C_9 = \frac{1}{2^q n}\sum_{j=1}^{n}\int_0^1 |1-2\tau|^q (e^{\frac{2-\tau}{2}}-1)^j d\tau, \quad (50)$$

$$C_{10} = \frac{1}{2^q n}\sum_{j=1}^{n}\int_0^1 |1-2\tau|^q (e^{\frac{\tau}{2}}-1)^j d\tau. \quad (51)$$

Corollary 11 Choosing $\nu = \omega = 1$ in Theorem-7 implies

$$\left|\frac{\varpi(\eta)+\varpi(\varsigma)}{2}-\frac{1}{\ln\varsigma-\ln\eta}\int_{\eta}^{\varsigma}\frac{\varpi(\gamma)}{\gamma}d\gamma\right|$$

$$\leq\frac{\ln\eta-\ln\varsigma}{4}\left[D_1^{\frac{1}{p}}\left(C_5|\varpi'(\eta)|^q+C_6|\varpi'(\varsigma)|^q\right)^{\frac{1}{q}}+D_2^{\frac{1}{p}}\left(C_{11}|\varpi'(\eta)|^q+C_{12}|\varpi'(\varsigma)|^q\right)^{\frac{1}{q}}\right], \quad (52)$$

where

$$C_{11}=\frac{1}{n}\sum_{j=1}^{n}\int_0^1|1-\tau|^q\left(e^{\frac{2-\tau}{2}}-1\right)^j d\tau, \quad (53)$$

$$C_{12}=\frac{1}{n}\sum_{j=1}^{n}\int_0^1|1-\tau|^q\left(e^{\frac{\tau}{2}}-1\right)^j d\tau, \quad (54)$$

and C_5, C_6 is defined in Corollary 9.

4. Discussion and Conclusion

A new kind of convex function called the n-polynomial exponential type GA-convex function is introduced and studied. It has been shown that the new type of n -polynomial exponential type GA-convex function is more comprehensive than the known type of functions such as n -polynomial convex. A new version of the Hermite-Hadamard type integral inequality is obtained for these functions. It is aimed that these studies will continue in this way in the future.

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