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Remarks on Some Soliton Types with Certain Vector Fields

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Abstract: This paper mainly aims to investigate some soliton kinds with certain vector fields on Riemannian manifolds and gives some notable geometric results as regards such vector fields. Also, in this paper some special tensors that have an important place in Riemannian geometry are discussed and given some significant links between these tensors. Finally, an example that supports one of our results is given.

Keywords: Ricci soliton, Yamabe soliton, conformal quadratic killing tensor, $\mathcal{Z}$-curvature tensor.

1. Introduction

Over the past few years, the theory of geometric flows has become a significant tool to determine the most geometric properties of the related object of the manifolds in Riemannian geometry. Ricci flow, one of the most important geometric flows, was defined by Hamilton so that he can find a canonical metric on a smooth manifold in [12]. Another important geometric flow is the Yamabe flow that Hamilton defined as a tool in order to construct metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a smooth manifold [11]. Such flows are evolution equations for Riemannian metric. After these works, many mathematicians have studied such geometric flows and other evolution equations arising in differential geometry.

In 1988, Hamilton defined the concepts of Yamabe and Ricci solitons in Riemannian geometry [11]. The formation of singularities in the Yamabe and Ricci flows are determined by these concepts, which evolve only by diffeomorphisms and scaling. Also, the limit of the solutions of the Yamabe and Ricci flow are appeared by Yamabe and Ricci solitons, respectively. Whereas a special solution of the Yamabe flow is the Yamabe soliton, a special solution of the Ricci flow is the Ricci soliton. Ricci and Yamabe solitons are equals to each other in dimension $n = 2$. But $n > 2$, these solitons do not have such an equivalence.

Let M be a Riemannian manifold together with the Riemannian metric g and let S and r

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be the Ricci tensor and scalar curvature of M, respectively. For a vector field ξ being tangent to M, if the following

$$(\pounds_\xi g)(W,F) + 2S(W,F) + 2\lambda g(W,F) = 0 \qquad (\lambda \in \mathbb{R}) \tag{1}$$

is satisfied, then this manifold is called a Ricci soliton. Similarly, if the following

$$(\pounds_\xi g)(W,F) = 2(r-\lambda)g(W,F) \qquad (\lambda \in \mathbb{R}) \tag{2}$$

is satisfied, then this manifold is called a Yamabe soliton. Here, $\pounds_\xi g$ stands for the Lie-derivative of g with respect to ξ and $W,F \in \Gamma(TM)$. A Yamabe (Ricci) soliton with vector field ξ is denoted by (M,g,ξ,λ). If ξ is Killing or zero in (1), then the Ricci soliton becomes trivial and in such a case, the metric becomes an Einstein metric. Hence, Ricci solitons can be considered as a generalization of Einstein manifolds. Similarly, if ξ is Killing or zero in (2), then Yamabe solitons reduce to manifolds of constant scalar curvature.

Yamabe (Ricci) solitons can be categorized as steady, expanding or shrinking depending on the values of λ: $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$. In addition, Ricci (Yamabe) solitons are called gradient if ξ is the gradient $D\mu$ of a smooth function μ called potential function. Recently, several varied generalizations of Ricci and Yamabe solitons have been investigated comprehensively by many authors within the framework of the many context. For example, in 2011, Pigola et al. defined almost Ricci solitons which are a more general case of Ricci solitons by setting λ in (1) as a function [19]. Similarly, Barbosa and Ribeiro introduced an another class of Yamabe solitons evolution equation (2) by taking constant λ with a variable function and then, they called it almost Yamabe soliton in [1]. For the recent works, we refer to ([6, 7, 16, 17, 21, 26]) and references therein.

On the other hand, as it is well-known that vector fields have been used for studying differential geometry of manifolds since they characterize most geometric properties of the related object. They are widely used in several fields of differential geometry and physics. Also, they play a significant role in the study of Riemannian geometry. Therefore, many papers on a Riemannian manifold endowed with geometric vector fields so that this manifold admits a Ricci soliton or Yamabe soliton have been discussed by many mathematicians. For further readings, we refer to studies ([2–5, 14, 18, 25, 27]).

Motivated by these circumstances, we deal with Ricci solitons and Yamabe solitons, which have recently received considerable attention of many geometers, on Riemannian manifolds endowed with certain vector fields such as affine conformal, projective and concircular. Also, we investigate some geometric properties of notions of $\mathcal{Z}$-curvature tensor and conformal quadratic Killing tensor, which proves to be rich in geometrical structures. The present study is structured as follows. In Section 2, we recall some necessary and notations formulas that will be needed. The Section 3 is devoted to conclusion in which we present our results that are obtained in this paper.

2. Preliminaries

In this section, some required notions which will be used for later are recalled.

Let (M, g) be a Riemannian manifold and A be a second order symmetric tensor. For any $W, F, Z \in \Gamma(TM)$, if this tensor satisfies the following

$$(\nabla_W A)(F, Z) + (\nabla_F A)(Z, W) + (\nabla_Z A)(W, F) = 0,$$

then it is called a quadratic Killing tensor (as a generalization of a Killing vector). Likewise, if this tensor satisfies the following

$$\begin{aligned}(\nabla_W A)(F, Z) + (\nabla_F A)(Z, W) + (\nabla_Z A)(W, F) = \\ k(W)A(F, Z) + k(F)A(Z, W) + k(Z)A(W, F),\end{aligned} \tag{3}$$

then it is called a conformal quadratic Killing tensor (as a generalization of a conformal Killing vector). Here, ∇ is the Levi-Civita connection of M and k is a 1-form. For more details as regards these tensors, we refer to ([22, 23, 28]).

Let ξ be a vector field on a Riemannian manifold (M, g). The vector field ξ is named as affine conformal, projective and concircular, respectively, if the followings [5, 8, 20]

$$(\pounds_\xi \nabla)(W, F) = W(\rho)F + F(\rho)W - g(W, F)D\rho, \tag{4}$$

$$(\pounds_\xi \nabla)(W, F) = p(W)F + p(F)W \tag{5}$$

and

$$\nabla_W \xi = \mu W, \tag{6}$$

where p is a 1-form, $D\rho$ is the gradient of ρ and μ, ρ are some smooth functions on M. If variable ρ in (4) is constant, then ξ is named as an affine vector field. Also, if $p = 0$ in (5), then the vector field ξ is called affine. The vector field ξ is named as concurrent if it satisfies (6) together with $\mu = 1$.

On the other hand, the Hessian tensor H^μ of a smooth function μ on (M, g) is given by

$$H^\mu(W, F) = g(\nabla_W(D\mu), F),$$

where $D\mu$ is the gradient of μ on M [4]. Also, the Hessian tensor H^μ is symmetric in W and F.

Now, we need the following lemma for later use.

Lemma 2.1 *[4] Let μ be a function on a Riemannian manifold M. Then, the gradient $D\mu$ of μ is a concircular vector field if and only if the Hessian H^μ of μ satisfies*

$$H^\mu(W, F) = fg(W, F) \tag{7}$$

for W, F tangent to M, where f is the function on M. Moreover, in such case the function f satisfies equation $\nabla_W v = fW$ with $v = D\mu$.

Also, from Lemma 2.1 and taking $\mu = f$ in (7), it can be easily seen that the gradient $D\mu$ of μ satisfies

$$\nabla_W D\mu = \mu W$$

for any $W \in \Gamma(TM)$.

3. Main Results

In this section, our main results that we obtained in this work are given.

Proposition 3.1 *If a Riemannian manifold (M, g) admits a Yamabe soliton with vector field V, then V is an affine conformal vector field on M.*

Proof Since (M, g) is a Yamabe soliton, from (2), one has

$$(\pounds_V g)(F, Z) = 2(r - \lambda) g(F, Z) \tag{8}$$

for any $F, Z \in \Gamma(TM)$. Differentiating the equation (8) covariantly along any vector field W provides

$$(\nabla_W \pounds_V g)(F, Z) = 2W(r) g(F, Z). \tag{9}$$

On the other hand, it follows from the formula (see Yano [24, p.23]) that we have the equality

$$\begin{aligned}(\pounds_V \nabla_W g - \nabla_W \pounds_V g - \nabla_{[V,W]} g)(F, Z) &= \\ -g((\pounds_V \nabla)(W, F), Z) - g((\pounds_V \nabla)(W, Z), F).\end{aligned} \tag{10}$$

Since the Riemannian metric g is parallel with respect to ∇, namely $\nabla g = 0$, the equality (10) turns into

$$(\nabla_W \pounds_V g)(F, Z) = g((\pounds_V \nabla)(W, F), Z) + g((\pounds_V \nabla)(W, Z), F). \tag{11}$$

Also, in view of (9) and (11), we immediately have

$$2W(r) g(F, Z) = g((\pounds_V \nabla)(W, F), Z) + g((\pounds_V \nabla)(W, Z), F). \tag{12}$$

If we rearrange cyclically W, F and Z in (12), then we obtain

$$2F(r) g(Z, W) = g((\pounds_V \nabla)(F, Z), W) + g((\pounds_V \nabla)(F, W), Z) \tag{13}$$

and

$$2Z(r)g(W,F) = g((\pounds_V\nabla)(Z,W),F) + g((\pounds_V\nabla)(Z,F),W). \tag{14}$$

Due to being $\pounds_V\nabla$ symmetric tensor of type $(1,2)$, that is, $(\pounds_V\nabla)(W,F) = (\pounds_V\nabla)(F,W)$, adding (12) and (13), we get

$$\begin{aligned} 2W(r)g(F,Z) + 2F(r)g(Z,W) &= 2g((\pounds_V\nabla)(W,F),Z) \\ &\quad + g((\pounds_V\nabla)(Z,W),F) \\ &\quad + g((\pounds_V\nabla)(Z,F),W) \end{aligned} \tag{15}$$

which together with the equation (14) gives the following

$$g((\pounds_V\nabla)(W,F),Z) = W(r)g(F,Z) + F(r)g(Z,W) - Z(r)g(W,F).$$

Using the fact that $Z(r) = g(Dr,Z)$, the above last equation can be written as

$$\begin{aligned} g((\pounds_V\nabla)(W,F),Z) &= W(r)g(F,Z) + F(r)g(Z,W) \\ &\quad - g(Dr,Z)g(W,F). \end{aligned} \tag{16}$$

Removing Z from both sides in (16) gives

$$(\pounds_V\nabla)(W,F) = W(r)F + F(r)W - g(W,F)Dr \tag{17}$$

which by (4) means that the vector field V is affine conformal on M. Thus, we get the requested result. □

Lemma 3.2 *Let V be a a concircular vector field on a Riemannian manifold (M,g). Then, V is also an affine conformal vector field on M.*

Proof It follows from (6) that we have

$$\nabla_Z V = \mu Z \tag{18}$$

for vector field Z being tangent to M. It follows from (18) and the definition of the covariant derivative, we arrive at

$$\nabla_F\nabla_Z V = F(\mu)Z + \mu\nabla_F Z \tag{19}$$

for vector field F being tangent to M. If we interchange the roles of F and Z in (19), then one has

$$\nabla_Z\nabla_F V = Z(\mu)F + \mu\nabla_Z F. \tag{20}$$

Moreover, taking $[F, Z]$ instead of Z in (18), we write

$$\nabla_{[F,Z]}V = \mu\nabla_F Z - \mu\nabla_Z F. \tag{21}$$

By means of (19), (20) and (21), we find that

$$R(F,Z)V = F(\mu)Z - Z(\mu)F \tag{22}$$

which by taking inner product on both sides of (22) by W yields

$$g(R(F,Z)V,W) = F(\mu)g(Z,W) - Z(\mu)g(F,W). \tag{23}$$

This is equivalent to

$$g(R(V,W)F,Z) = F(\mu)g(Z,W) - Z(\mu)g(F,W) \tag{24}$$

from which it follows that

$$R(V,W)F = F(\mu)W - g(F,W)D\mu. \tag{25}$$

On the other hand, as is known from Yano [24, p.23], that the following identity holds

$$(\pounds_V\nabla)(W,F) = R(V,W)F - \nabla_{\nabla_W F}V + \nabla_W\nabla_F V. \tag{26}$$

Making use of (6), (25) and (26), therefore we obtain

$$(\pounds_V\nabla)(W,F) = W(\mu)F + F(\mu)W - g(W,F)D\mu$$

which is the desired result. □

Remark 3.3 *If the vector field V is concurrent, then V is also an affine vector field on M.*

Let A be a geometric/physical quantity on a Riemannian manifold M. If it satisfies

$$\pounds_\xi A = 2\Omega A, \tag{27}$$

then A inherits symmetry with respect to vector field ξ. Here, $\pounds$ stands for the Lie derivative and Ω is a function on the manifold [10]. For more details related to symmetry inheritance applications on manifolds, please see ([9, 10, 13, 28]).

The metric inheritance symmetry is one of the most basic and widely used example for which $A = g$ in (27), in this case ξ is the conformal Killing vector field such that

$$(\pounds_\xi g)(W,F) = 2\Omega g(W,F).$$

Also, if Ω is zero in above equation, then ξ is the Killing vector field.

If $A = R$ in (27), then the equation takes the form

$$\pounds_\xi R = 2\Omega R.$$

This particular symmetry is called curvature inheritance, where R is the Riemannian curvature tensor. If $\Omega = 0$, then it is called curvature collineation, which M admits a special symmetry.

In the following theorem, we discuss the role of such symmetry inheritance for the Ricci tensor field of a Riemannian manifold M.

Theorem 3.4 *Let μ be a potential function of an almost gradient Ricci soliton (M, g, V, λ), where (M, g) has symmetry inheritance and μ is the function satisfying equation (6). Then, the Ricci tensor field of M has symmetry inheritance with respect to V if the Hessian H^μ of μ satisfies*

$$H^\mu(W, F) = \mu g(W, F) \tag{28}$$

for any $W, F \in \Gamma(TM)$.

Proof Let us consider that the Hessian H^μ of μ satisfies equation (28). Then, by Lemma 2.1, the gradient $D\mu$ of μ is a concircular vector field. Since M is an almost gradient Ricci soliton and from Lemma 3.2, the concircular vector field V satisfies

$$(\pounds_V \nabla)(W, F) = W(\mu)F + F(\mu)W - g(W, F)D\mu \tag{29}$$

for vector fields W and F being tangent to M. Due to $V = D\mu$, the equation (29) transforms into

$$(\pounds_V \nabla)(W, F) = g(W, V)F + g(F, V)W - g(W, F)V. \tag{30}$$

Differentiating (30) covariantly along any vector field T and using the property of being V concircular, we obtain

$$(\nabla_T \pounds_V \nabla)(W, F) = \mu g(T, W)F + \mu g(T, F)W - \mu g(W, F)T. \tag{31}$$

Similarly, by a straight forward calculation, we also have

$$(\nabla_W \pounds_V \nabla)(T, F) = \mu g(W, T)F + \mu g(W, F)T - \mu g(T, F)W. \tag{32}$$

From [24], it is well known that

$$(\pounds_V R)(T, W)F = (\nabla_T \pounds_V \nabla)(W, F) - (\nabla_W \pounds_V \nabla)(T, F). \tag{33}$$

After inserting (31) and (32) in (33), we deduce that

$$(\pounds_V R)(T, W)F = 2\mu g(T, F)W - 2\mu g(W, F)T. \tag{34}$$

Furthermore, operating inner product with arbitrary vector field X in (34) yields

$$g((\pounds_V R)(T,W)F,X) = 2\mu g(T,F)g(W,X) - 2\mu g(W,F)g(T,X). \tag{35}$$

Setting $T = X = e_i$ in (35) and summing up over i $(i = 1,2,...,n)$, we derive that

$$(\pounds_V S)(W,F) = -2\mu(n-1)g(W,F). \tag{36}$$

Here, $\{e_i\}$ stands for the orthonormal basis of T_pM for all $p \in M$.

On the other hand, since M admits an almost gradient Ricci soliton with concircular vector field V, we find from (1) and (18) that

$$S(F,Z) = -(\lambda+\mu)g(F,Z) \tag{37}$$

for any $F,Z \in \Gamma(TM)$. Replacing F with V in (37) gives

$$S(V,Z) = -(\lambda+\mu)g(V,Z). \tag{38}$$

Owing to Lemma 3.2, putting $F = W = e_i$ in (23) and taking summation over i, we have

$$S(Z,V) = -(n-1)g(D\mu,Z). \tag{39}$$

Then, with the help of (38) and (39), one can see that $\lambda+\mu = n-1$. Using this in (37) provides

$$S(F,Z) = -(n-1)g(F,Z). \tag{40}$$

Keeping in mind (36) and from (40), we get

$$(\pounds_V S)(W,F) = 2\mu S(W,F) \tag{41}$$

which gives the conclusion. Hence, the proof is completed. □

The next result gives a necessary condition for the Ricci tensor field of M to be conformal quadratic Killing.

Theorem 3.5 *Let (M,g) be a Ricci soliton with vector field V, where V is either an affine conformal or a projective vector field. Then, the Ricci tensor field of M is conformal quadratic Killing.*

Proof Let us take into account that (M,g) is a Ricci soliton with affine conformal vector field V. Then, the equation (1) can be written as

$$(\pounds_V g)(F,Z) = -2S(F,Z) - 2\lambda g(F,Z) \tag{42}$$

for any $F, Z \in \Gamma(TM)$. Performing the properties of Lie derivative and Levi-Civita connection in (42), we infer that

$$(\nabla_W \pounds_V g)(F, Z) = -2(\nabla_W S)(F, Z) \tag{43}$$

for W being tangent to M. On the other hand, making use of the equation (4), we get

$$g((\pounds_V \nabla)(W, F), Z) + g((\pounds_V \nabla)(W, Z), Y) = 2g(W, D\rho)g(F, Z). \tag{44}$$

By virtue of the equations (11), (43) and (44), we have

$$(\nabla_W S)(F, Z) = -g(W, D\rho)g(F, Z). \tag{45}$$

If we stand for the the dual 1-form of $D\rho$ by $-\phi$, then equation (45) becomes

$$(\nabla_W S)(F, Z) = \phi(W)g(F, Z). \tag{46}$$

Also, if W, F and Z are cyclically displaced in (46), then we find that

$$(\nabla_F S)(Z, W) = \phi(F)g(Z, W) \tag{47}$$

and

$$(\nabla_Z S)(W, F) = \phi(Z)g(W, F). \tag{48}$$

By combining the equalities (46), (47) and (48), we obtain

$$(\nabla_W S)(F, Z) + (\nabla_F S)(Z, W) + (\nabla_Z S)(W, F) =$$
$$\phi(W)g(F, Z) + \phi(F)g(Z, W) + \phi(Z)g(W, F)$$

which by (3) means that the Ricci tensor field of M is a conformal quadratic Killing tensor.

When the above steps are done for the projective vector field V, it can be easily showed that the Ricci tensor field of M is conformal quadratic Killing. Hence, the proof is completed. □

In 2012, as a general concept of the Einstein gravitational tensor in General relativity, generalized $(0, 2)$ symmetric $\mathcal{Z}$ tensor was introduced by Mantica and Molina. According to them, such a tensor is defined by [15]

$$\mathcal{Z}(W, F) = S(W, F) + fg(W, F) \tag{49}$$

for f being a smooth function on M and S being the Ricci tensor field of M.

The next theorem presents an important relationship for curvature tensor $\mathcal{Z}$ and conformal quadratic Killing.

Theorem 3.6 *Let (M, g) be a Riemannian manifold admitting $\mathcal{Z}$-curvature tensor. Then, the tensor $\mathcal{Z}$ is conformal quadratic Killing if and only if the Ricci tensor field of M is conformal quadratic Killing.*

Proof Taking covariant derivative of (49) along T and using the fact that $T(f) = df(T)$, we get

$$(\nabla_T \mathcal{Z})(W, F) = (\nabla_T S)(W, F) + df(T)g(W, F) \tag{50}$$

for any $W, F, T \in \Gamma(TM)$. By a combinatorial combination, we find

$$(\nabla_W \mathcal{Z})(F, T) = (\nabla_W S)(F, T) + df(W)g(F, T) \tag{51}$$

and

$$(\nabla_F \mathcal{Z})(T, W) = (\nabla_F S)(T, W) + df(F)g(T, W). \tag{52}$$

By adding the equations (50), (51) and (52) provides

$$\begin{aligned} &(\nabla_T \mathcal{Z})(W, F) + (\nabla_W \mathcal{Z})(F, T) + (\nabla_F \mathcal{Z})(T, W) = \\ &\quad (\nabla_T S)(W, U) + (\nabla_W S)(U, T) + (\nabla_U S)(T, W) \\ &+ df(T)g(W, F) + df(W)g(F, T) + df(F)g(T, W). \end{aligned} \tag{53}$$

Now, if $\mathcal{Z}$ is conformal quadratic Killing, then we write

$$\begin{aligned} &(\nabla_T \mathcal{Z})(W, F) + (\nabla_W \mathcal{Z})(F, T) + (\nabla_F \mathcal{Z})(T, W) = \\ &\quad \alpha(T)g(W, F) + \alpha(W)g(F, T) + \alpha(F)g(T, W). \end{aligned} \tag{54}$$

Therefore, by combining (53) with (54) and using the linearity property of the 1-forms, we get

$$\begin{aligned} (\nabla_T S)(W, F) + (\nabla_W S)(F, T) + (\nabla_F S)(T, W) &= (\alpha - df)(T)g(W, F) \\ &+ (\alpha - df)(W)g(F, T) + (\alpha - df)(F)g(T, W). \end{aligned} \tag{55}$$

If we take ϕ instead of the 1-form $\alpha - df$ in (55), then one has

$$\begin{aligned} &(\nabla_T S)(W, F) + (\nabla_W S)(F, T) + (\nabla_F S)(T, W) = \\ &\quad + \phi(T)g(W, F) + \phi(W)g(F, T) + \phi(F)g(T, W). \end{aligned}$$

This, by (3), implies that the Ricci tensor field of M is a conformal quadratic Killing tensor.

Conversely, let the Ricci tensor field of M admitting $\mathcal{Z}$-curvature tensor be a conformal quadratic Killing tensor. Then, we have

$$\begin{aligned} &(\nabla_T S)(W, F) + (\nabla_W S)(F, T) + (\nabla_F S)(T, W) = \\ &\quad k(T)g(W, F) + k(W)g(F, T) + k(F)g(T, W). \end{aligned} \tag{56}$$

After substituting (56) into (53) and using the linearity of the 1-forms, we obtain

$$(\nabla_T \mathcal{Z})(W,F) + (\nabla_W \mathcal{Z})(F,T) + (\nabla_F \mathcal{Z})(T,W) = \tag{57}$$
$$\varphi(T)g(W,F) + \varphi(W)g(F,T) + \varphi(F)g(T,W),$$

where we have used $\varphi = k + df$. Consequently, by (3) the tensor $\mathcal{Z}$ is conformal quadratic Killing. Therefore, we arrive at the desired result. □

The proof of following corollary easily follows from Theorem 3.6.

Corollary 3.7 *Let (M,g) be a Riemannian manifold admitting $\mathcal{Z}-$curvature tensor. Then, we have the followings:*

i) If tensor $\mathcal{Z}$ is quadratic Killing, then the Ricci tensor field of M is conformal quadratic Killing.

ii) If the Ricci tensor field of M is quadratic Killing, then tensor $\mathcal{Z}$ is conformal quadratic Killing.

Example 3.8 *[2] We set the three-dimensional manifold as*

$$M = \{(u,v,t) \in \mathbb{R}^3, t > 0\},$$

where (u,v,t) are the Cartesian coordinates in $\mathbb{R}^3$. Take

$$g := \frac{1}{t^2}\{du \otimes du + dv \otimes dv + dt \otimes dt\},$$

$$\eta := -\frac{1}{t}dt, \qquad V := -t\frac{\partial}{\partial t}.$$

Here, η denotes the dual 1-form of the vector field V. Let E_1, E_2 and E_3 be the vector fields in $\mathbb{R}^3$ such that these vector fields are linearly independent given by:

$$E_1 = t\frac{\partial}{\partial u}, \quad E_2 = t\frac{\partial}{\partial v} \quad \text{and} \quad E_3 = -t\frac{\partial}{\partial t}.$$

Then, we have

$$\eta(E_1) = 0, \qquad \eta(E_2) = 0, \qquad \eta(E_3) = 1,$$
$$[E_2, E_1] = 0, \qquad [E_3, E_2] = -E_2, \qquad [E_1, E_3] = E_1.$$

On the other hand, we find from Koszul's formula for the Riemannian metric g:

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_2} E_1 = 0, \quad \nabla_{E_3} E_1 = 0,$$
$$\nabla_{E_1} E_2 = 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_3} E_2 = 0,$$
$$\nabla_{E_1} E_3 = E_1, \quad \nabla_{E_2} E_3 = E_2, \quad \nabla_{E_3} E_3 = 0.$$

Also, with the help of the above equations we find

$$R(E_1,E_2)E_3 = 0, \qquad R(E_3,E_1)E_2 = 0, \qquad R(E_2,E_3)E_1 = 0,$$

$$R(E_1,E_2)E_2 = -E_1, \qquad R(E_2,E_1)E_1 = -E_2, \qquad R(E_1,E_3)E_3 = -E_1,$$

$$R(E_3,E_1)E_1 = -E_3, \qquad R(E_2,E_3)E_3 = -E_2, \qquad R(E_3,E_2)E_2 = -E_3.$$

Utilizing the expressions of the curvature tensors, we obtain

$$S(E_1,E_1) = S(E_2,E_2) = S(E_3,E_3) = -2 \qquad \textit{and} \qquad S(E_i,E_j) = 0 \tag{58}$$

for all $i \neq j \ \ (i,j = 1,2,3)$*. As* $\{E_1,E_2,E_3\}$ *forms a basis of* M*, the followings can be written as*

$$W = a_1E_1 + a_2E_2 + a_3E_3,$$

$$Z = b_1E_1 + b_2E_2 + b_3E_3,$$

$$F = c_1E_1 + c_2E_2 + c_3E_3$$

for any vector field $W,Z,F \in \Gamma(TM)$*, where* $a_i, b_i, c_i \in \mathbb{R}^+$ *for* $i = 1,2,3$*. Then, by a straight forward calculation, one has*

$$\nabla_F W = -a_1c_1E_3 + a_3c_1E_1 - a_2c_2E_3 + a_3c_2E_2, \tag{59}$$

$$\nabla_F Z = -b_1c_1E_3 + b_3c_1E_1 - b_2c_2E_3 + b_3c_2E_2. \tag{60}$$

From the above equations, we find that

$$S(W,Z) = -2(a_1b_1 + a_2b_2 + a_3b_3), \tag{61}$$

$$S(\nabla_F W,Z) = 2a_1c_1b_3 - 2a_3c_1b_1 + 2a_2c_2b_3 - 2a_3c_2b_2, \tag{62}$$

$$S(W,\nabla_F Z) = 2b_1c_1a_3 - 2b_3c_1a_1 + 2b_2c_2a_3 - 2b_3c_2a_2. \tag{63}$$

Therefore, we have

$$(\nabla_F S)(W,Z) = 0. \tag{64}$$

Similarly, we obtain

$$(\nabla_W S)(Z,F), \qquad (\nabla_Z S)(F,W) = 0. \tag{65}$$

In this case, from (64) and (65), S *is a quadratic Killing tensor.*

Furthermore, using (59)-(63) in (49) gives

$$\mathcal{Z}(W,Z) = (f-2)(a_1b_1 + a_2b_2 + a_3b_3), \tag{66}$$

$$\mathcal{Z}(\nabla_F W,Z) = (2-f)a_1c_1b_3 + (f-2)a_3c_1b_1 + (2-f)2a_2c_2b_3 + (f-2)a_3c_2b_2, \tag{67}$$

$$\mathcal{Z}(W,\nabla_F Z) = (2-f)b_1c_1a_3 + (f-2)b_3c_1a_1 + (2-f)b_2c_2a_3 + (f-2)b_3c_2a_2. \tag{68}$$

From (64)-(68) and owing to the fact that $g(X,Y) = a_1b_1 + a_2b_2 + a_3b_3$*, we get*

$$(\nabla_F \mathcal{Z})(W,Z) = df(F)g(W,Z). \tag{69}$$

Likewise,

$$(\nabla_W \mathcal{Z})(Z,F) = df(W)g(Z,F), \quad (\nabla_Z \mathcal{Z})(F,W) = df(Z)g(F,W). \tag{70}$$

Thus $\mathcal{Z}$ *is a conformal quadratic Killing tensor, which verifies Corollary 3.7* (ii).

Declaration of Ethical Standards

The author declares that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Conflict of Interest

The author declares no conflicts of interest.

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