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On the Bi-Periodic Mersenne Sequence

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Abstract

In this paper, the bi-periodic Mersenne sequence, which is a generalization of the Mersenne sequence, is defined. The characteristic function, generating function and Binet's formula for this sequence are obtained. Also, by using Binet's formula, some important identities and properties for the bi-periodic Mersenne sequence are presented.

1. Introduction

There are some studies about integer sequences in the literature [1]-[4]. Especially Mersenne primes are an active field in the number theory and computer science [5]. They are popular research objects because of their interesting representation in the binary system properties as $(1)_2$, $(11)_2$, $(111)_2$, $(1111)_2$,.... The Mersenne numbers can also be defined as [6]

$$M_n + 2 = 3M_{n+1} - 2M_n,$$

with the initial conditions $M_0 = 0$ and $M_1 = 1$.

The roots of the respective characteristic equation $r^2 - 3r + 2 = 0$ are $r_1 = 2$ and $r_2 = 1$ and we easily get the Binet formula

$$M_n = 2^n - 1.$$

The first few terms of the Mersenne sequence are

$$M_n = \{0, 1, 3, 7, 15, 31, 63, 127, 255, 511, ...\} \quad [7].$$

In many studies, generalizations of the integer sequences have been examined [8]-[15]. The bi-periodic Fibonacci sequence, which was introduced by Edson and Yayenie, have made an important contribution to the literature [16]. Inspired by this study, many new generalized sequences have been described [17]-[20].

The main purpose of this paper is to first define bi-periodic Mersenne sequence, to find the generating function and Binet formula, and then to present some identities that include the bi-periodic Mersenne sequence as a result of the corresponding Binet formula.

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2. Bi-periodic Mersenne sequence

Definition 2.1. *The bi-periodic Mersenne sequence* $\{m_n\}_{n=0}^{\infty}$ *is defined by*

$$m_0=0,\quad m_1=1,\quad m_n=\begin{cases} 3am_{n-1}-2m_{n-2}, & \text{if } n \text{ is even;} \\ 3bm_{n-1}-2m_{n-2}, & \text{if } n \text{ is odd.}\end{cases},\ n\geq 2.$$

where a and b are any two non-zero real numbers.
By setting $a=b=1$, we get the classic Mersenne numbers.
The quadratic equation for the bi-periodic Mersenne sequence is defined as

$$x^2-3abx+2ab=0$$

with the roots

$$\alpha_1=\frac{3ab+\sqrt{9a^2b^2-8ab}}{2}\quad\text{and}\quad\alpha_2=\frac{3ab-\sqrt{9a^2b^2-8ab}}{2}. \tag{2.1}$$

Lemma 2.2. *The bi-periodic Mersenne sequence satisfies the following properties:*

$$\begin{aligned} m_{2n}&=(9ab-4)\,m_{2n-2}-4m_{2n-4},\\ m_{2n+1}&=(9ab-4)\,m_{2n-1}-4m_{2n-3}.\end{aligned}$$

Proof. By using the recurrence relation for bi-periodic Mersenne sequence, we obtain m_{2n} and m_{2n+1} as follows:

$$\begin{aligned} m_{2n} &= 3am_{2n-1}-2m_{2n-2}\\ &= 3a(3bm_{2n-2}-2m_{2n-3})-2m_{2n-2}\\ &= (9ab-2)m_{2n-2}-6am_{2n-3}\\ &= (9ab-2)m_{2n-2}-(2m_{2n-2}+4m_{2n-4})\\ &= (9ab-4)\,m_{2n-2}-4m_{2n-4}\end{aligned}$$

and

$$\begin{aligned} m_{2n+1} &= 3bm_{2n}-2m_{2n-1}\\ &= 3b(3am_{2n-1}-2m_{2n-2})-2m_{2n-1}\\ &= (9ab-2)m_{2n-1}-6bm_{2n-2}\\ &= (9ab-2)m_{2n-1}-(2m_{2n-1}+4m_{2n-3})\\ &= (9ab-4)m_{2n-1}-4m_{2n-3}.\end{aligned}$$

□

Lemma 2.3. *The roots* α_1 *and* α_2 *defined in (2.1) satisfy the following properties:*

$$\begin{aligned} \alpha_1\alpha_2 &= 2ab,\\ \alpha_{1+}\alpha_2 &= 3ab,\\ 3\alpha_1-2 &= \frac{\alpha_1^2}{ab},\\ 3\alpha_2-2 &= \frac{\alpha_2^2}{ab},\\ (3\alpha_1-2)(3\alpha_2-2) &= 4.\end{aligned}$$

Proof. By using the definitions of α_1 and α_2, the proof can be easily obtained. □

Theorem 2.4. *The generating function for the bi-periodic Mersenne sequence is*

$$M(x)=\frac{x(1+2x^2+3ax)}{1-(9ab-4)x^2+4x^4}.$$

Proof. $M(x) = m_0 + m_1x + m_2x^2 + ... + m_sx^s + ... = \sum_{k=0}^{\infty} m_kx^k$ is the formal power series representation of the generating function for $\{m_n\}_{n=0}^{\infty}$. If this series is multiplied by $3bx$ and $2x^2$, then we get

$$3bxM(x) = 3bm_0x + 3bm_1x^2 + ... = 3\sum_{k=0}^{\infty} bm_kx^{k+1} = 3\sum_{k=1}^{\infty} bm_{k-1}x^k$$

and

$$2x^2M(x) = 2m_0x^2 + 2m_1x^3 + ... = 2\sum_{k=0}^{\infty} m_kx^{k+2} = 2\sum_{k=2}^{\infty} m_{k-2}x^k.$$

So,

$$\left(1-3bx+2x^2\right)M(x) = m_0 + m_1x - 3bm_0x + \sum_{k=2}^{\infty}\left(m_k - 3bm_{k-1} + 2m_{k-2}\right)x^k. \tag{2.2}$$

Since $m_{2k+1} = 3bm_{2k} - 2m_{2k-1}$ and $m_0 = 0$, $m_1 = 1$, we have

$$\left(1-3bx+2x^2\right)M(x) = x + \sum_{k=1}^{\infty}(m_{2k} - 3bm_{2k-1} + 2m_{2k-2})x^{2k}.$$

$m_{2k} = 3am_{2k-1} - 2m_{2k-2}$ implies that

$$\left(1-3bx+2x^2\right)M(x) = x + 3(a-b)x\sum_{k=1}^{\infty} m_{2k-1}x^{2k-1}.$$

Now, we let

$$m(x) = \sum_{k=1}^{\infty} m_{2k-1}x^{2k-1}.$$

Then,

$$\begin{aligned}\left(1-(9ab-4)x^2+4x^4\right)m(x) &= \sum_{k=1}^{\infty} m_{2k-1}x^{2k-1} - (9ab-4)\sum_{k=2}^{\infty} m_{2k-3}x^{2k-1} + 4\sum_{k=3}^{\infty} m_{2k-5}x^{2k-1}\\ &= m_1x + m_3x^3 - (9ab-4)m_1x^3 + \sum_{k=3}^{\infty}\left(m_{2k-1} - (9ab-4)m_{2k-3} + 4m_{2k-5}\right)x^{2k-1}.\end{aligned}$$

From Lemma 2.2, we have $m_{2n-1} = (9ab-4)m_{2n-3} - 4m_{2n-5}$. By substituting this in the expression above, we get

$$\left(1-(9ab-4)x^2+4x^4\right)m(x) = x + (9ab-2)x^3 - (9ab-4)x^3 = x + 2x^3.$$

Therefore,

$$m(x) = \frac{x+2x^3}{(1-(9ab-4)x^2+4x^4)}.$$

Substituting $m(x)$ in $M(x)$ gives

$$(1-3bx+2x^2)M(x) = x + \left(3(a-b)x\frac{x+2x^3}{(1-(9ab-4)x^2+4x^4)}\right).$$

After simplifying the above expression, we get the desired result

$$M(x) = \frac{x(1+2x^2+3ax)}{1-(9ab-4)x^2+4x^4}.$$

□

Theorem 2.5. *The terms of the bi-periodic Mersenne are given by*

$$m_n = \frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right),$$

where α_1 and α_2 are as in (2.1), $\lfloor b \rfloor$ is the floor function of b and $\xi(n)$ is the parity function.

Proof. Using the partial fraction decomposition, we can write the generating function for the bi-periodic Mersenne sequence $M(x)$ as

$$M(x)=\frac{1}{\alpha_1-\alpha_2}\left[\frac{\alpha_1 x+a(\frac{3\alpha_1-2}{2})}{2x^2-(\frac{3\alpha_1-2}{2})}-\frac{\alpha_2 x+a(\frac{3\alpha_2-2}{2})}{2x^2-(\frac{3\alpha_2-2}{2})}\right].$$

The Maclaurin series expansion of the function $\frac{A-Bz}{z^2-C}$ is expressed in the form

$$\frac{A-Bz}{z^2-C}=\sum_{n=0}^{\infty}BC^{-n-1}z^{2n+1}-\sum_{n=0}^{\infty}AC^{-n-1}z^{2n}.$$

So, $M(x)$ can be written as

$$\begin{aligned}M(x) &= \frac{1}{2(\alpha_1-\alpha_2)}\left[\sum_{n=0}^{\infty}\frac{(-\alpha_1)(\frac{3\alpha_2-2}{4})^{n+1}+\alpha_2(\frac{3\alpha_1-2}{4})^{n+1}}{(\frac{3\alpha_1-2}{4})^{n+1}(\frac{3\alpha_2-2}{4})^{n+1}}x^{2n+1}\right]\\ &\quad +\frac{1}{2(\alpha_1-\alpha_2)}\left[\sum_{n=0}^{\infty}\frac{(2a)(\frac{3\alpha_2-2}{4})^{n}+(2a)(\frac{3\alpha_1-2}{4})^{n}}{(\frac{3\alpha_1-2}{4})^{n}(\frac{3\alpha_2-2}{4})^{n}}x^{2n}\right].\end{aligned}$$

By using Lemma 2.3, we obtain

$$\begin{aligned}M(x) &= \frac{1}{2(\alpha_1-\alpha_2)}\left(\sum_{n=0}^{\infty}\frac{\frac{(-\alpha_1)(\alpha_2)^{2n+2}}{(4ab)^{n+1}}+\frac{\alpha_2(\alpha_1)^{2n+2}}{(4ab)^{n+1}}}{\frac{1}{4^{n+1}}}\right)x^{2n+1}\\ &\quad -\frac{1}{2(\alpha_1-\alpha_2)}\left(\sum_{n=0}^{\infty}\frac{\frac{(2a)(\alpha_2)^{2n}}{(4ab)^{n}}-\frac{(2a)(\alpha_1)^{2n}}{(4ab)^{n}}}{\frac{1}{4^{n}}}\right)x^{2n}\\ &= \sum_{n=0}^{\infty}\frac{1}{(ab)^n}\frac{(\alpha_1)^{2n+1}-(\alpha_2)^{2n+1}}{\alpha_1-\alpha_2}x^{2n+1}+\sum_{n=0}^{\infty}\frac{a}{(ab)^n}\frac{(\alpha_1)^{2n}-(\alpha_2)^{2n}}{\alpha_1-\alpha_2}x^{2n}.\end{aligned}$$

By the help of the parity function $\xi(n)$, it follows that

$$M(x)=\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right)x^n.$$

Therefore, for all $n\geq 0$, we have

$$m_n=\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right).$$

□

Theorem 2.6. *(Catalan's Identity) For any two nonnegative integer n and r, with $r\leq n$, we get*

$$a^{\xi(n-r)}b^{1-\xi(n-r)}m_{n-r}m_{n+r}-a^{\xi(n)}b^{1-\xi(n)}m_n^2=-\left(2^{n-r}\right)a^{\xi(r)}b^{1-\xi(r)}m_r^2.$$

Proof. Using the Binet's formula, we obtain

$$\begin{aligned}a^{\xi(n+r)}b^{1-\xi(n-r)}m_{n-r}m_{n+r} &= a^{\xi(n+r)}b^{1-\xi(n-r)}\frac{a^{1-\xi(n-r)}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{\alpha_1^{n-r}-\alpha_2^{n-r}}{\alpha_1-\alpha_2}\right)\frac{a^{1-\xi(n+r)}}{(ab)^{\lfloor\frac{n+r}{2}\rfloor}}\left(\frac{\alpha_1^{n+r}-\alpha_2^{n+r}}{\alpha_1-\alpha_2}\right)\\ &= \left(\frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{n-\xi(n-r)}}\right)\left(\frac{\alpha_1^{n-r}-\alpha_2^{n-r}}{\alpha_1-\alpha_2}\right)\left(\frac{\alpha_1^{n+r}-\alpha_2^{n+r}}{\alpha_1-\alpha_2}\right)\\ &= \left(\frac{a}{(ab)^{n-1}}\right)\left(\frac{\alpha_1^{2n}-(\alpha_1\alpha_2)^{n-r}(\alpha_1^{2r}+\alpha_2^{2r})+\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2}\right)\end{aligned}$$

and

$$\begin{aligned}a^{\xi(n)}b^{1-\xi(n)}m_n^2 &= a^{\xi(n)}b^{1-\xi(n)}\left(\frac{a^{2-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{\alpha_1^{2n}-2(\alpha_1\alpha_2)^n+\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2}\right)\\ &= \left(\frac{a}{(ab)^{2\lfloor\frac{n}{2}\rfloor+\xi(n)-1}}\right)\left(\frac{\alpha_1^{2n}-2(\alpha_1\alpha_2)^n+\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2}\right)\\ &= \left(\frac{a}{(ab)^{n-1}}\right)\left(\frac{\alpha_1^{2n}-2(\alpha_1\alpha_2)^n+\alpha_2^{2n}}{(\alpha_1-\alpha_2)^2}\right).\end{aligned}$$

So,

$$\begin{aligned}
a^{\xi(n+r)}b^{1-\xi(n-r)}m_{n-r}m_{n+r}-a^{\xi(n)}b^{1-\xi(n)}m_n^2 &= \left(\frac{a}{(ab)^{n-1}}\right)\left(\frac{2(\alpha_1\alpha_2)^n-(\alpha_1\alpha_2)^{n-r}(\alpha_1^{2r}+\alpha_2^{2r})}{(\alpha_1-\alpha_2)^2}\right)\\
&= \left(\frac{a}{(ab)^{n-1}}\right)\left(\frac{(\alpha_1\alpha_2)^{n-r}(2(\alpha_1\alpha_2)^r-\alpha_1^{2r}-\alpha_2^{2r}))}{(\alpha_1-\alpha_2)^2}\right)\\
&= \left(\frac{-a}{(ab)^{n-1}}\right)(\alpha_1\alpha_2)^{n-r}\left(\frac{\alpha_1^{2r}+\alpha_2^{2r}-2(\alpha_1\alpha_2)^r)}{(\alpha_1-\alpha_2)^2}\right)\\
&= \left(\frac{-a}{(ab)^{n-1}}\right)(2ab)^{n-r}\left(\frac{\alpha_1^r-\alpha_2^r}{\alpha_1-\alpha_2}\right)^2\\
&\quad -(a)^{2\xi(r)-1}(ab)^{1-\xi(r)}\left(2^{n-r}\right)m_r^2\\
&= -\left(2^{n-r}\right)a^{\xi(r)}b^{1-\xi(r)}m_r^2.
\end{aligned}$$

□

Theorem 2.7. *(Cassini's Identity) The following equality holds*

$$a^{1-\xi(n)}b^{\xi(n)}m_{n-1}m_{n+1}-a^{\xi(n)}b^{1-\xi(n)}m_n^2=-a\left(2^{n-1}\right),$$

where n is any nonnegative integer.

Proof. In Catalan's identity, if we take $r=1$, we get Cassini's identity. So, the proof can be obtained from the relevant identity. □

Theorem 2.8. *(d'Ocagne's Identity) For any two nonnegative integer n and r, with $r\leq n$, we have*

$$a^{\xi(nr+n)}b^{\xi(nr+r)}m_nm_{r+1}-a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n+1}m_r=2^ra^{\xi(n-r)}m_{n-r}.$$

Proof. There are such equations

$$\xi(n+1)+\xi(r)-2\xi(nr+r)=\xi(n)+\xi(r+1)-2\xi(nr+n)=1-\xi(n-r) \tag{2.3}$$

and

$$\xi(n-r)=\xi(nr+n)+\xi(nr+r) \tag{2.4}$$

for the floor function defined as $\xi(n)=n-2\left\lfloor\frac{n}{2}\right\rfloor$.

Using the Binet's formula, (2.3) and (2.4), it follows that

$$a^{\xi(nr+n)}b^{\xi(nr+r)}m_nm_{r+1}=\frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{\alpha_1^{n+r+1}+\alpha_2^{n+r+1}-(\alpha_1\alpha_2)^r(\alpha_1^{n-r}\alpha_2+\alpha_1\alpha_2^{n-r})}{(\alpha_1-\alpha_2)^2}$$

and

$$a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n+1}m_r=\frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{\alpha_1^{n+r+1}+\alpha_2^{n+r+1}-(\alpha_1\alpha_2)^r(\alpha_1^{n-r+1}+\alpha_2^{n-r+1})}{(\alpha_1-\alpha_2)^2}.$$

So,

$$\begin{aligned}
a^{\xi(nr+n)}b^{\xi(nr+r)}m_nm_{r+1}-a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n+1}m_r &= \frac{a}{(ab)^{(n+r-\xi(n-r))/2}}(\alpha_1\alpha_2)^r\frac{(\alpha_1^{n-r+1}+\alpha_2^{n-r+1}+\alpha_1^{n-r}\alpha_2-\alpha_1\alpha_2^{n-r})}{(\alpha_1-\alpha_2)^2}\\
&= \frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{(\alpha_1^{n-r}-\alpha_2^{n-r})(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)^2}\\
&= 2^ra^{\xi(n-r)}m_{n-r}.
\end{aligned}$$

□

Theorem 2.9. *(Honsberger Identity) For any two nonnegative integer n and r, with $r \leq n$, we have*

$$\left(a^{\xi(nr+n)}b^{\xi(nr+r)} - \frac{1}{\alpha_1\alpha_2}\frac{a^{\xi(r+1)-\xi(n)-1}}{(ab)^{\xi(nr+n)}}\right)m_n m_{r+1} + \left(a^{\xi(nr+r)}b^{\xi(nr+n)} - (\alpha_1\alpha_2)\frac{a^{\xi(r)+\xi(n-1)-1}}{(ab)^{\xi(nr+r)+1}}\right)m_{n-1}m_r = \frac{(\alpha_1^{-1}+\alpha_1)(-\alpha_2^{-1}-\alpha_2)}{\alpha_1-\alpha_2}a^{\xi(n+r)}m_{n+r}.$$

Proof. Using the Binet's formula, (2.3) and (2.4), we obtain

$$a^{\xi(nr+n)}b^{\xi(nr+r)}m_n m_{r+1} = \frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{\left(\alpha_1^{n+r+1}+\alpha_2^{n+r+1}-\alpha_1^{r+1}\alpha_2^{n}-\alpha_1^{n}\alpha_2^{r+1}\right)}{(\alpha_1-\alpha_2)^2}$$

and

$$a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n-1}m_r = \frac{a}{(ab)^{(n+r-\xi(n-r))/2}}\frac{\left(\alpha_1^{n+r-1}+\alpha_2^{n+r-1}-\alpha_1^{r}\alpha_2^{n-1}-\alpha_1^{n-1}\alpha_2^{r}\right)}{(\alpha_1-\alpha_2)^2}.$$

So, we get

$$\begin{aligned}
a^{\xi(nr+n)}b^{\xi(nr+r)}m_n m_{r+1} + a^{\xi(nr+r)}b^{\xi(nr+n)}m_{n-1}m_r &= \frac{a}{(ab)^{\frac{n+r-\xi(n-r)}{2}}}\frac{(\alpha_1^{n+r}-\alpha_2^{n+r})(-\alpha_2^{-1}-\alpha_2)(\alpha_1^{-1}+\alpha_1)}{(\alpha_1-\alpha_2)^2}\\
&+\frac{\alpha_1^{n+r}(\alpha_2^{-1}+\alpha_2)+\alpha_2^{n+r}(\alpha_1^{-1}+\alpha_1)-(1+\alpha_1\alpha_2)(\alpha_1^{r}\alpha_2^{n-1}+\alpha_2^{r}\alpha_1^{n-1})}{(\alpha_1-\alpha_2)^2}\\
&=\frac{a}{(ab)^{\frac{n+r-\xi(n-r)}{2}}}\frac{(-\alpha_2^{-1}-\alpha_2)(\alpha_1^{-1}+\alpha_1)}{\alpha_1-\alpha_2}\frac{(ab)^{\lfloor\frac{n+r}{2}\rfloor}}{a^{1-\xi(n+r)}}m_{n+r}\\
&+\frac{(\alpha_1^{r}\alpha_2^{-1}-\alpha_1^{-1}\alpha_2^{r})}{\alpha_1-\alpha_2}\frac{(ab)^{\lfloor\frac{n}{2}\rfloor}}{a^{1-\xi(n)}}m_n+\frac{(\alpha_1^{r+1}\alpha_2-\alpha_1\alpha_2^{r+1})}{\alpha_1-\alpha_2}\frac{(ab)^{\lfloor\frac{n-1}{2}\rfloor}}{a^{1-\xi(n-1)}}m_{n-1}\\
&=\left(\frac{(-\alpha_2^{-1}-\alpha_2)(\alpha_1^{-1}+\alpha_1)}{\alpha_1-\alpha_2}\right)a^{\xi(n+r)}m_{n+r}+\frac{1}{\alpha_1\alpha_2}\frac{a^{\xi(r+1)-\xi(n)-1}}{(ab)^{\xi(nr+n)}}m_n m_{r+1}\\
&+(\alpha_1\alpha_2)\frac{a^{\xi(r)+\xi(n-1)-1}}{(ab)^{\xi(nr+r)+1}})m_{n-1}m_r.
\end{aligned}$$

Hence, we obtain

$$\left(a^{\xi(nr+n)}b^{\xi(nr+r)} - \frac{1}{\alpha_1\alpha_2}\frac{a^{\xi(r+1)-\xi(n)-1}}{(ab)^{\xi(nr+n)}}\right)m_n m_{r+1} + \left(a^{\xi(nr+r)}b^{\xi(nr+n)} - (\alpha_1\alpha_2)\frac{a^{\xi(r)+\xi(n-1)-1}}{(ab)^{\xi(nr+r)+1}}\right)m_{n-1}m_r = \frac{(\alpha_1^{-1}+\alpha_1)(-\alpha_2^{-1}-\alpha_2)}{\alpha_1-\alpha_2}a^{\xi(n+r)}m_{n+r}.$$ □

Theorem 2.10. *(Sums Involving Binomial Coefficient) For any nonnegative integer r, we have*

$$\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)(ab)^{\lfloor\frac{s}{2}\rfloor}a^{\xi(s)}m_s = m_{2r}$$

and

$$\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)(ab)^{\lfloor\frac{s+1}{2}\rfloor}a^{\xi(s+1)-1}m_{s+1} = m_{2r+1}.$$

Proof. For any integer s, we have

$$(ab)^{\lfloor\frac{s}{2}\rfloor}a^{\xi(s)}m_s = a\left(\frac{\alpha_1^s-\alpha_2^s}{\alpha_1-\alpha_2}\right).$$

By using this equality above, we get

$$\begin{aligned}
\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)(ab)^{\left\lfloor \frac{s}{2}\right\rfloor}a^{\xi(s)}m_s &= \sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)a\left(\frac{\alpha_1^s-\alpha_2^s}{\alpha_1-\alpha_2}\right)\\
&= \frac{a}{\alpha_1-\alpha_2}\left[\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3\alpha_1)^s-\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3\alpha_2)^s\right]\\
&= \frac{a}{\alpha_1-\alpha_2}\left[(3\alpha_1-2)^r-(3\alpha_2-2)^r\right]\\
&= \frac{a}{\alpha_1-\alpha_2}\left[\left(\frac{\alpha_1^2}{ab}\right)^r-\left(\frac{\alpha_2^2}{ab}\right)^r\right]\\
&= \frac{a}{(ab)^r}\left(\frac{\alpha_1^{2r}-\alpha_2^{2r}}{\alpha_1-\alpha_2}\right)\\
&= m_{2r}.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)(ab)^{\left\lfloor \frac{s+1}{2}\right\rfloor}a^{\xi(s+1)-1}m_{s+1} &= \sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3^s)a\left(\frac{\alpha_1^{s+1}-\alpha_2^{s+1}}{\alpha_1-\alpha_2}\right)\\
&= \frac{1}{\alpha_1-\alpha_2}\left[\alpha_1\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3\alpha_1)^s-\alpha_2\sum_{s=0}^{r}\binom{r}{s}(-2)^{r-s}(3\alpha_2)^s\right]\\
&= \frac{1}{\alpha_1-\alpha_2}\left[\alpha_1(3\alpha_1-2)^r-\alpha_2(3\alpha_2-2)^r\right]\\
&= \frac{1}{\alpha_1-\alpha_2}\left[\alpha_1\left(\frac{\alpha_1^2}{ab}\right)^r-\alpha_2\left(\frac{\alpha_2^2}{ab}\right)^r\right]\\
&= \frac{1}{(ab)^r}\left(\frac{\alpha_1^{2r+1}-\alpha_2^{2r+1}}{\alpha_1-\alpha_2}\right)\\
&= m_{2r+1}.
\end{aligned}$$

□

Theorem 2.11. *The nonnegative terms of the bi-periodic Mersenne sequence are defined in terms of the positive terms as*

$$m_{-n}=-2^{-n}m_n.$$

Proof. By using the Binet's formula, we obtain

$$\begin{aligned}
m_{-n} &= \frac{a^{1-\xi(-n)}}{(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\left(\frac{\alpha_1^{-n}-\alpha_2^{-n}}{\alpha_1-\alpha_2}\right)\\
&= \frac{a^{1-\xi(-n)}}{(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\left(\frac{\alpha_2^{n}-\alpha_1^{n}}{(\alpha_1\alpha_2)^n(\alpha_1-\alpha_2)}\right)\\
&= \frac{a^{1-\xi(n)}}{2^n(ab)^n(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\left(\frac{\alpha_2^{n}-\alpha_1^{n}}{\alpha_1-\alpha_2}\right)\\
&= \frac{-a^{1-\xi(n)}}{2^n(ab)^{\left\lfloor \frac{-n}{2}\right\rfloor}}\left(\frac{\alpha_1^{n}-\alpha_2^{n}}{\alpha_1-\alpha_2}\right)\\
&= -2^{-n}m_n.
\end{aligned}$$

□

3. Conclusion

In this paper, we define bi-periodic Mersenne sequence, which is called bi-periodic Mersenne sequence. We obtain some properties for this sequence such as Binet formula, generating function, Catalan, Cassini, d'Ocagne and Honsberger identities.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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