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On An Existential Question for Strictly Decreasing Convergent Sequences

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Abstract

In this paper, we study an existential question for strictly decreasing convergent sequences. Applying Du's existence theorem, our question will be answered affirmatively.

Keywords: Strictly decreasing sequence, convergent sequence, existential question, Du's existence theorem.
2010 MSC: 26E40, 54C30.

1. Question and Answer

In this paper, we study the following interesting question:

Question: *Does there exist a strictly decreasing sequence* $\{a_n\}_{n\in\mathbb{N}}$ *of positive real numbers such that* $\lim_{n\to\infty} a_n = 0$ *and*

$$\exp\left(2022\,(a_{n+1})^3\right)\sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin a_{n+1}\right)\right)\right)\right)\right)\right)\right)\right) < a_n \quad \text{for all } n\in\mathbb{N}?$$

In fact, this existential question is not easy to answer. In this article, we will apply the following known existence theorem, proved by Du [2], to slove this question. We give the proof of Du's existence theorem here for the sake of completeness and the readers convenience.

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Theorem 1. (*see [2, Lemma 3.1]*) *Let $\beta \in \mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\lim_{x\to\beta^+} \tau(x) = \beta$. Then there exists a strictly decreasing $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\tau(\beta + \lambda_{n+1}) < \beta + \lambda_n$ for all $n \in \mathbb{N}$ and $\lambda_n \downarrow 0$ as $n \to \infty$.*

Proof. If $\tau(x) = \beta$ is a constant function, then we can choose a positive real number a and finish the proof by taking $\lambda_n = \frac{a}{n}$ for all $n \in \mathbb{N}$. Suppose that τ is not a constant function. For any $\epsilon > 0$, since $\lim_{x\to\beta^+} \tau(x) = \beta$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\beta < x < \beta + \delta \quad \text{implies} \quad \tau(x) < \beta + \epsilon.$$

Given $\lambda_1 > 0$. Then there is $\delta_1 > 0$ such that

$$\beta < x < \beta + \delta_1 \quad \text{implies} \quad \tau(x) < \beta + \lambda_1.$$

Let $\lambda_2 = \min\left\{\frac{\delta_1}{2}, \frac{\lambda_1}{2}\right\}$. Then $\beta < \beta + \lambda_2 < \beta + \delta_1$ and $\lambda_2 < \lambda_1$. So we have from the last inequality that

$$\tau(\beta + \lambda_2) < \beta + \lambda_1.$$

For λ_2, it must exist $\delta_2 > 0$ such that

$$\beta < x < \beta + \delta_2 \quad \text{implies} \quad \tau(x) < \beta + \lambda_2.$$

Put $\lambda_3 = \min\left\{\frac{\delta_2}{2}, \frac{\lambda_2}{2}\right\}$. Thus $\beta < \beta + \lambda_3 < \beta + \delta_2$ and $\lambda_3 < \lambda_2$. The last inequality deduces

$$\tau(\beta + \lambda_3) < \beta + \lambda_2.$$

Continuing this process, for λ_k, $k \in \mathbb{N}$ with $k \geq 2$, it must exist $\delta_k > 0$ such that

$$\beta < x < \beta + \delta_k \quad \text{implies} \quad \tau(x) < \beta + \lambda_k.$$

Take

$$\lambda_{k+1} = \min\left\{\frac{\delta_k}{2}, \frac{\lambda_k}{2}\right\}.$$

Then we get $\lambda_{k+1} < \lambda_k$ and $\tau(\beta + \lambda_{k+1}) < \beta + \lambda_k$. So, we can construct a strictly decreasing sequences $\{\lambda_n\}$ of positive real numbers such that

$$\tau(\beta + \lambda_{n+1}) < \beta + \lambda_n \quad \text{for all } n \in \mathbb{N}.$$

By the definition of λ_n, we have $0 < \lambda_{n+1} \leq \frac{\lambda_1}{2^n}$ for $n \in \mathbb{N}$, which yields $\lambda_n \downarrow 0$ as $n \to \infty$. The proof is completed. □

Take $\beta = 0$ in Theorem 1, we can obtain the following result immediately.

Corollary 2. (*see [1, Corollary 2]*) *Let $\tau : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\lim_{x\to 0^+} \tau(x) = 0$. Then there exists a strictly decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\tau(\lambda_{n+1}) < \lambda_n$ for all $n \in \mathbb{N}$ and $\lambda_n \downarrow 0$ as $n \to \infty$.*

By Corollary 2 (or Theorem 1), our question will be answered affirmatively.

Solution: The answer is **Yes**. Indeed, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \exp\left(2022x^3\right) \sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin\left(\sin x\right)\right)\right)\right)\right)\right)\right)\right)\right).$$

It is easy to see that $\lim_{x\to 0^+} f(x) = 0$. Hence, by applying Corollary 2, there exists a strictly decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\to\infty} a_n = 0$ and

$$\begin{aligned}\exp\left(2022\,(a_{n+1})^3\right)\sin(\sin(\sin(\sin(\sin(\sin(\sin(\sin(\sin(\sin a_{n+1}))))))))) &= f(a_{n+1})\\ &< a_n\end{aligned}$$

for all $n \in \mathbb{N}$.

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References

[1] W.-S. Du, Existence and uniqueness of zeros for vector-valued functions with K-adjustability convexity and their applications, Mathematics 2019, 7(9), 809.

[2] W.-S. Du, Minimization of functionals with adjustability quasiconvexity and its applications to eigenvector problem and fixed point theory, Journal of Nonlinear and Convex Analysis 22(7) (2021) 1389-1398.