Research *Göcen/Kırklareli University Journal of Engineering and Science 8-1 (2022) 1-14*

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

THE EXACT SOLUTIONS OF SOME DIFFERENCE EQUATIONS ASSOCIATED WITH ADJUSTED JACOBSTHAL-PADOVAN NUMBERS

Melih GÖCEN[1*]

[1]*Zonguldak Bülent Ecevit University, Faculty of Arts and Sciences, Department of Mathematics, Zonguldak.*

Abstract

In this paper, we obtain the form of the solutions of some rational difference equations via adjusted Jacobsthal-Padovan numbers. We find a relation between the exact solutions and the adjusted Jacobsthal-Padovan numbers. Apart from the literature, we give the closed form of the solutions associated with these well-known integer sequence using exponential functions. Furthermore, we investigate the asymptotic behavior of the equilibrium point of the solutions of these difference equations.

Keywords: Difference equation, Form of the solutions, Stability, Jacobsthal-Padovan numbers.

BAZI FARK DENKLEMLERİNİN AYARLANMIŞ JACOBSTHAL-PADOVAN SAYILARI İLE İLİŞKİLİ TAM ÇÖZÜMLERİ

Öz

Bu makalede, bazı rasyonel fark denklemlerinin ayarlanmış Jacobsthal-Padovan sayıları ile çözümlerinin formunu elde ediyoruz. Kesin çözümler ile ayarlanmış Jacobsthal-Padovan sayıları arasında bir ilişki buluyoruz. Literatürün dışında, çözümlerin bu iyi bilinen diziler ile ilişkili kapalı formunu üstel fonksiyonlar kullanarak veriyoruz. Ayrıca, bu fark denklemlerinin denge noktasının asimptotik davranışını inceliyoruz.

Anahtar Kelimeler: Fark denklemi, Çözümlerin formu, Karalılık, Jacobsthal-Padovan sayıları.

[*]*Sorumlu Yazar:Melih GÖCEN, gocenm@hotmail.com*

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

1. INTRODUCTION

Recently there has been a rapid increase in studying the theory and applications of difference equations, for related studies, see [3,4,5,19,20]. Especially, the studies that concern the relation between the special sequences and the closed form of the solutions of difference equations or systems have been received great interest by many researchers for a long time. Because, some of the solution forms of these equations are even expressible in terms of well-known integer sequences. As it can be seen from the references, there are many papers on such these studies from several authors, see [6,7,8,9,10,11,12,13,21].

In [15], the author studied on the dynamical behaviours of the following difference equation

$$x_{n+1} = \frac{\alpha}{x_n x_{n-1} - 1}, \quad n = 0, 1, \dots \tag{1}$$

where only $\alpha > 0$. But they did not give the exact solution of the Eq. (1). In this work, we present the exact solutions for the case $\alpha = 2$ and $\alpha = -2$ via adjusted Jacobsthal-Padovan numbers for the first time in the literature.

Firstly, we give the closed form of the following rational difference equations

$$x_{n+1} = \frac{2}{x_n x_{n-1} - 1}, \quad n = 0, 1, \dots \tag{2}$$

$$x_{n+1} = \frac{-2}{x_n x_{n-1} - 1}, \quad n = 0, 1, \dots \tag{3}$$

in terms of adjusted Jacobsthal-Padovan numbers and investigate the asymptotic behavior of solutions of these equations. Also, we find a relation between the exact solutions of Eq. (2), Eq. (3) and the adjusted Jacobsthal-Padovan numbers.

Let us consider the following definitions which will be needed for the results throughout the paper. Firstly, we give some information about the generalized Jacobsthal-Padovan sequence. A generalized Jacobsthal-Padovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations,

DOI: 10.34186/klujes.1078836 Geliş Tarihi:24.02.2022 Kabul Tarihi:30.06.2022

$$V_n = V_{n-2} + 2V_{n-3} \quad (4)$$

with initial conditions $V_0 = c_{0,}\, V_1 = c_1, V_2 = c_2$ not all being zero. This sequence can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} + \frac{1}{2}V_{-(n-3)} \quad (5)$$

for $= 1, 2, 3, \ldots$. Therefore, recurrences (4) and (5) hold for all integer n.

Here, α, β and γ are the roots of the cubic equation

$$x^3 - x - 2 = 0 \quad (6)$$

Moreover

$$\alpha = (1 + \frac{\sqrt{78}}{9})^{1/3} + (1 - \frac{\sqrt{78}}{9})^{1/3} \quad (7)$$

$$\beta = \omega(1 + \frac{\sqrt{78}}{9})^{1/3} + \omega^2(1 - \frac{\sqrt{78}}{9})^{1/3}$$

$$\gamma = \omega^2(1 + \frac{\sqrt{78}}{9})^{1/3} + \omega(1 - \frac{\sqrt{78}}{9})^{1/3}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(\frac{2\pi i}{3})$$

is a primitive cube root of unity. Note that

$$\alpha + \beta + \gamma = 0,$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -1,$$

and

$$\alpha\beta\gamma = 2.$$

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

Now, we define a special case of the generalized Jacobsthal-Padovan sequence $\{V_n\}$. The adjusted Jacobsthal-Padovan sequence $\{K_n\}_{n\geq 0}$ is defined by the following third order recurrence relation

$$K_{n+3} = K_{n+1} + 2K_n, \qquad K_0 = 0,\ K_1 = 1,\ K_2 = 0.$$

Also, it can be extended to the negative subscripts as

$$K_{-n} = -\frac{1}{2}K_{-(n-1)} + \frac{1}{2}K_{-(n-3)}$$

for $= 1, 2, 3, \dots$. So, these recurrences hold for all integer n.

For more information, see [1,2,17,18].

Next, we present the first few values of the adjusted Jacobsthal-Padovan numbers with positive and negative subscripts:

Table 1. The first few values of the adjusted Jacobsthal-Padovan numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
K_n	0	1	0	1	2	1	4	5	6	13	16	25	42	57
K_{-n}	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$-\frac{7}{32}$	$\frac{11}{64}$	$\frac{1}{128}$	$-\frac{29}{256}$	$\frac{73}{512}$	$-\frac{69}{1024}$	$-\frac{47}{2048}$	$\frac{339}{4096}$

2. PRELIMINAIRES

We begin by introducing some basic definitions and some theorems needed in the sequel. For details, see [14].

Let I be some interval of real numbers and jet $f: I^{k+1} \to I$ be continuously differentiable function. A difference equation of order (k+1) is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots \tag{9}$$

A solution of Eq. (9) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that is satisfies Eq. (9) for all $n \geq -k$.

DOI: 10.34186/klujes.1078836 Geliş Tarihi:24.02.2022 Kabul Tarihi:30.06.2022

Definition 1. A solution of Eq. (9) that is constant for all $n \geq -k$ is called an equilibrium solution of Eq. (9). If

$$x_n = \bar{x}, for\ all\ n \geq -k$$

is an equilibrium solution of Eq. (9), then $\bar{x}$ is called an equilibrium point, or simply an equilibrium of Eq. (9).

Suppose that the function f is contiuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \dots, \bar{x}), \qquad i = 0, 1, \dots, k$$

denote the partial derivative of $f(u_0, u_1, \dots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(9).

Definition 2. The equation

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + \cdots + q_k y_{n-k}, \qquad n = 0, 1, \dots \tag{10}$$

is called the linearized equation of Eq. (9) about the equilibrium point $\bar{x}$, and the equation

$$\lambda^{k+1} - q_0 \lambda^k - \cdots - q_{k+1}\lambda - q_k = 0 \tag{11}$$

is called the characteristic equation of Eq. (10) about $\bar{x}$.

Let us consider the following theorem which will be needed for the stability results in this study.

Theorem 3. (The Linearized Stability Theorem) Assume that the function f is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$. Then the following statements are true:

(a) If all roots of Eq. (11) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq. (9) is locally asymptotically stable.

(b) If at least one root of Eq.(11) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(9) is unstable.

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

Furthermore, the equilibrium point $\bar{x}$ of Eq. (9) is called *hyperbolic* if no root of Eq. (11) has absolute value equal to one. If there exists a root of Eq. (11) with absolute value equal to one, then the equilibrium $\bar{x}$ is called *nonhyperbolic.*

An equilibrium point $\bar{x}$ of Eq. (9) is called a *repeller* if all roots of Eq. (11) have absolute value greater than one.

3. MAIN RESULTS

In this section, we prove the main results by using the method of induction. Firstly, we obtain the form of the solutions of Eq. (2) and show that this solution is related to well-known adjusted Jacobsthal-Padovan numbers. Furthermore, we investigate the asymptotic behavior of the real equilibrium point of the Eq. (2). Then, we deal with the closed form of solutions and stability of the Eq. (3) in a similar way.

3.1 The First Equation

Theorem 4. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of the following difference equation

$$x_{n+1} = \frac{2}{x_n x_{n-1} - 1}, \quad n = 0, 1, \ldots,$$

Then, for $n = 0, 1, 2, \ldots$, the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by

$$x_n = \frac{2^n K_{-n} x_{-1} x_0 + 2^n K_{-(n-1)} x_0 + 2^{n+1} K_{-(n+1)}}{2^n K_{-(n+1)} x_{-1} x_0 + 2^n K_{-n} x_0 + 2^{n+1} K_{-(n+2)}}, \tag{12}$$

where K_n is the nth adjusted Jacobsthal-Padovan number and the initial conditions $x_{-1}, x_0 \in \mathbb{R} - F$, with F is the forbidden set of Eq.(2) given by

$$F = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0): 2^n K_{-(n+1)} x_{-1} x_0 + 2^n K_{-n} x_0 + 2^{n+1} K_{-(n+2)} = 0\right\}.$$

Proof. We prove by mathematical induction on k. For $k = 0$, from Eq. (2),

$$x_1 = \frac{2}{x_{-1} x_0 - 1} = \frac{2K_{-1} x_{-1} x_0 + 2K_0 x_0 + 2^2 K_{-2}}{2K_{-2} x_{-1} x_0 + 2K_{-1} x_0 + 2^2 K_{-3}}$$

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

Assume that the equality holds for $n = k$, that is

$$x_k = \frac{2^k K_{-k} x_{-1} x_0 + 2^k K_{-(k-1)} x_0 + 2^{k+1} K_{-(k+1)}}{2^k K_{-(k+1)} x_{-1} x_0 + 2^k K_{-k} x_0 + 2^{k+1} K_{-(k+2)}}. \tag{13}$$

Thus, we have to prove that it is true for $k+1$. Taking into account (8) and (13), it follows that

$$x_{k+1} = \frac{2}{x_k x_{k-1} - 1}$$

$$= \frac{2}{\left(\frac{2^k K_{-k} x_{-1} x_0 + 2^k K_{-(k-1)} x_0 + 2^{k+1} K_{-(k+1)}}{2^k K_{-(k+1)} x_{-1} x_0 + 2^k K_{-k} x_0 + 2^{k+1} K_{-(k+2)}}\right)\left(\frac{2^{k-1} K_{-(k-1)} x_{-1} x_0 + 2^{k-1} K_{-(k-2)} x_0 + 2^k K_{-k}}{2^{k-1} K_{-k} x_{-1} x_0 + 2^{k-1} K_{-(k-1)} x_0 + 2^k K_{-(k+1)}}\right) - 1}$$

$$= \frac{2(x_0 K_{-k} + 2K_{-k-2} + x_0 K_{-k-1} x_{-1})}{-(x_0 K_{-k} - 2K_{-k} - x_0 K_{2-k} + 2K_{-k-2} + x_0 K_{-k-1} x_{-1} - x_0 K_{1-k} x_{-1})}$$

$$= \frac{2K_{-(k+1)} x_{-1} x_0 + 2K_{-k} x_0 + 2^2 K_{-(k+2)}}{\left(-K_{-(k+1)} + K_{-(k-1)}\right) x_{-1} x_0 + \left(-K_{-k} + K_{-(k-2)}\right) x_0 + \left(-2K_{-(k+2)} + 2K_{-k}\right)}$$

Let us recall that

$$2K_{-n} = -K_{-(n-1)} + K_{-(n-3)}$$

and multiplying both the numerator and denominator by 2^k, we have

$$\frac{2^{k+1} K_{-(k+1)} x_{-1} x_0 + 2^{k+1} K_{-k} x_0 + 2^{k+2} K_{-(k+2)}}{2^{k+1} K_{-(k+2)} x_{-1} x_0 + 2^{k+1} K_{-(k+1)} x_0 + 2^{k+2} K_{-(k+3)}}$$

which ends the induction and the proof.

3.2 The Second Equation

Theorem 5. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of the following difference equation

$$x_{n+1} = \frac{-2}{x_n x_{n-1} - 1}, \qquad n = 0, 1, \ldots,$$

Then, for $n = 0, 1, 2, \ldots$, the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by

DOI: 10.34186/klujes.1078836 Geliş Tarihi:24.02.2022 Kabul Tarihi:30.06.2022

$$x_n = \frac{2^n K_{-n} x_{-1} x_0 - 2^n K_{-(n-1)} x_0 + 2^{n+1} K_{-(n+1)}}{-2^n K_{-(n+1)} x_{-1} x_0 + 2^n K_{-n} x_0 - 2^{n+1} K_{-(n+2)}},$$

where K_n is the nth adjusted Jacobsthal-Padovan number and the initial conditions $x_{-1}, x_0 \in \mathbb{R} - F$, with F is the forbidden set of Eq. (3) given by

$$\mathrm{F} = \bigcup_{n=-1}^{\infty} \{(x_{-1}, x_0): -2^n K_{-(n+1)} x_{-1} x_0 + 2^n K_{-n} x_0 - 2^{n+1} K_{-(n+2)} = 0\}.$$

Proof. We start to prove the theorem, by using the mathematical induction as in Theorem 4. For $k = 0$, from Eq. (3), we have

$$x_1 = \frac{-2}{x_0 x_{-1} - 1} = \frac{2K_{-1} x_{-1} x_0 - 2K_0 x_0 + 2^2 K_{-2}}{-2K_{-2} x_{-1} x_0 + 2K_{-1} x_0 - 2^2 K_{-3}}$$

Now suppose that our assumption holds for all $1 \leq n \leq k$. That is,

$$x_k = \frac{2^k K_{-k} x_{-1} x_0 - 2^k K_{-(k-1)} x_0 + 2^{k+1} K_{-(k+1)}}{-2^k K_{-(k+1)} x_{-1} x_0 + 2^k K_{-k} x_0 - 2^{k+1} K_{-(k+2)}}. \tag{15}$$

Hence, we must prove that it is true for $k + 1$. Taking into account (8) and (15), as in Theorem 4, similarly, we have

$$x_{k+1} = \frac{-2}{x_k x_{k-1} - 1}$$
$$= \frac{-2}{\left(\frac{2^k K_{-k} x_{-1} x_0 - 2^k K_{-(k-1)} x_0 + 2^{k+1} K_{-(k+1)}}{-2^k K_{-(k+1)} x_{-1} x_0 + 2^k K_{-k} x_0 - 2^{k+1} K_{-(k+2)}}\right)\left(\frac{2^{k-1} K_{-(k-1)} x_{-1} x_0 - 2^{k-1} K_{-(k-2)} x_0 + 2^k K_{-k}}{-2^{k-1} K_{-k} x_{-1} x_0 + 2^{k-1} K_{-(k-1)} x_0 - 2^k K_{-(k+1)}}\right) - 1}$$
$$= \frac{2^{k+1} K_{-(k+1)} x_{-1} x_0 - 2^{k+1} K_{-k} x_0 + 2^{k+2} K_{-(k+2)}}{-2^{k+1} K_{-(k+2)} x_{-1} x_0 + 2^{k+1} K_{-(k+1)} x_0 - 2^{k+2} K_{-(k+3)}}$$

and the proof is complete.

3.3 Stability of Eq. (2) and Eq. (3)

In this section, we deal with the asymptotic behavior of the equilibrium point of Eq. (2) and Eq. (3), respectively. The real equilibrium point of Eq. (2) is the solution of the following equation

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

$$\bar{x} = \frac{2}{\bar{x}(\bar{x}) - 1}$$

After simplification, one can easily obtain the unique real equilibrium point of Eq. (2) is $\bar{x} = \alpha$, where α is given with Eq. (7) and numerically as $\alpha \approx 1.5214$.

Now, we show that the equilibrium point of Eq. (2) is repeller.

Theorem 6. The unique real equilibrium point of Eq. (2) is repeller.

Proof. Let I be an interval of real numbers and

$$f: I^2 \to I$$

be a continuous function defined by

$$f(x_n, x_{n-1}) = \frac{2}{x_n x_{n-1} - 1}$$

Therefore, it follows that

$$\frac{\partial f(x_n, x_{n-1})}{\partial x_n} = \frac{-2x_{n-1}}{(x_n x_{n-1} - 1)^2},$$

$$\frac{\partial f(x_n, x_{n-1})}{\partial x_{n-1}} = \frac{-2x_n}{(x_n x_{n-1} - 1)^2}.$$

Then,

$$\frac{\partial f(\bar{x}, \bar{x})}{\partial x_n} = \frac{-2\bar{x}}{(\overline{xx} - 1)^2}$$

$$= \frac{-2\bar{x}}{(\bar{x}^2 - 1)^2}$$

$$= \frac{-2\bar{x}}{\left(\frac{2}{\bar{x}}\right)^2}$$

$$= -\frac{\bar{x}^3}{2}$$

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

$$= -\frac{\alpha^3}{2}$$

$$\frac{\partial f(\bar{x},\bar{x})}{\partial x_{n-1}} = \frac{-2\bar{x}}{(\bar{x}^2-1)^2}$$

$$= \frac{-2\bar{x}}{\left(\frac{2}{\bar{x}}\right)^2}$$

$$= -\frac{\bar{x}^3}{2}$$

$$= -\frac{\alpha^3}{2}$$

and the linearized equation of Eq. (2) about $\bar{x} = \alpha$ is

$$y_{n+1} = \left(-\frac{\alpha^3}{2}\right) y_n + \left(-\frac{\alpha^3}{2}\right) y_{n-1}$$

or equivalently

$$y_{n+1} + \frac{\alpha^3}{2} y_n + \frac{\alpha^3}{2} y_{n-1} = 0.$$

Therefore, the corresponding characteristic equation is

$$\lambda^2 + \frac{\alpha^3}{2}\lambda + \frac{\alpha^3}{2} = 0.$$

Then, it is clearly seen that

$$\lambda_{1,2} = -\frac{1}{4}\alpha^3 \pm \frac{1}{4}\sqrt{\alpha^3(\alpha^3-8)}$$

and we have

$$|\lambda_1| = |\lambda_2| \approx 1.32.69 > 1.$$

So, from Theorem (3), the equilibrium point $\bar{x}$ is repeller point and this completes the proof.

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

Now, let us consider the difference equation Eq. (3) and show that the equilibrium point of Eq. (3) is repeller.

The real equilibrium point of Eq. (3) is the solutions of the following equation

$$\bar{x} = \frac{-2}{\bar{x}(\bar{x}) - 1}$$

After simplification, one can easily obtain the equilibrium point of Eq. (3) is $\bar{x} = -\alpha$, and numerically as -1.5214.

Theorem 7. The unique real equilibrium point of Eq. (3) is repeller.

Proof. Let *I* be an interval of real numbers and

$$f: I^2 \to I$$

be a continuous function defined by

$$f(x_n, x_{n-1}) = \frac{-2}{x_n x_{n-1} - 1}$$

Therefore, it follows that

$$\frac{\partial f(x_n, x_{n-1})}{\partial x_n} = \frac{2x_{n-1}}{(x_n x_{n-1} - 1)^2},$$

$$\frac{\partial f(x_n, x_{n-1})}{\partial x_{n-1}} = \frac{2x_n}{(x_n x_{n-1} - 1)^2}.$$

Then

$$\frac{\partial f(\bar{x}, \bar{x})}{\partial x_n} = \frac{2\bar{x}}{(\overline{xx} - 1)^2}$$

$$= \frac{2\bar{x}}{(\bar{x}^2 - 1)^2}$$

$$= \frac{2\bar{x}}{\left(\frac{2}{\bar{\bar{x}}}\right)^2}$$

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

$$= \frac{\alpha^3}{2}$$

$$\frac{\partial f(\bar{x},\bar{x})}{\partial x_{n-1}} = \frac{2\bar{x}}{(\bar{x}^2 - 1)^2}$$

$$= \frac{2\bar{x}}{\left(\frac{2}{\bar{\bar{x}}}\right)^2}$$

$$= \frac{\alpha^3}{2}$$

and the linearized equation of Eq. (3) about $\bar{x}$ is

$$y_{n+1} = \left(\frac{\alpha^3}{2}\right) y_n + \left(\frac{\alpha^3}{2}\right) y_{n-1}$$

or equivalently

$$y_{n+1} - \frac{\alpha^3}{2} y_n - \frac{\alpha^3}{2} y_{n-1} = 0$$

Therefore, the corresponding characteristic equation is

$$\lambda^2 - \frac{\alpha^3}{2}\lambda - \frac{\alpha^3}{2} = 0.$$

Then, it is clearly seen that

$$\lambda_{1,2} = \frac{1}{4}\alpha^3 \pm \frac{1}{4}\sqrt{\alpha^3(\alpha^3 + 8}$$

and we have

$$|\lambda_1| = |\lambda_2| \approx 1{,}3269 > 1.$$

So, from Theorem (3), the equilibrium point $\bar{x}$ is repeller point and this completes the proof.

DOI: 10.34186/klujes.1078836 Geliş Tarihi:24.02.2022 Kabul Tarihi:30.06.2022

4. CONCLUSION

In recent years, there has been a great interest in studying some special rational difference equations. Because some of the solution forms of these equations are even expressible in terms of well-known integer sequences. We find a relation between the exact solution of Eq.(2), Eq.(3) and the adjusted Jacobsthal-Padovan sequence.

Moreover, one can obtain the form of the solutions of the following difference equations

$$x_{n+1} = \frac{\pm 2}{x_n x_{n-1} + 1}, \qquad n = 0, 1, \dots,$$

and these specific forms also contain some special sequences A089977 and A122946 numbers, for more information about these sequences, see [16]. Furthermore, the unique real equilibrium points of these equations are locally asymptotically stable.

5. REFERENCES

[1] Deveci, Ö., The Pell-Padovan sequences and the Jacobsthal-Padovan sequences in finite groups, Util. Math., 98, 257-270, 2015.

[2] Deveci, Ö., The Jacobsthal-Padovan p-sequences and their applications, Proc. Rom. Acad. Ser. A, 20(3), 215-224, 2019.

[3] Göcen, M., Cebeci, A., On the perodie solutions of some systems of higher order difference equation, Rocky Mt. J. Math., 48(3), 845-858, 2018.

[4] Göcen, M., Güneysu, M., The global attractivity of some rational difference equations, J. Comput. Anal. Appl., 25(7), 1233-1243, 2018.

[5] Haddad N., Tonafek, N., Rabago, J. F. T., Solution form of a higher-oreder system of difference equations and dynamical behavior of its special case, Math. Methods Appl. Sci., 40(10), 3599-3607, 2017.

[6] Halim, Y., Bayram, M., On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences. Mathematical Methods in the Applied Seciences, 39, 2974-2982, 2016.

[7] Halim, Y., A System of Difference Equations with Solutions Associated to Fibonacci Numbers. International Journal of Difference Equations, 11(1), 65-77, 2016.

DOI: 10.34186/klujes.1078836 *Geliş Tarihi:24.02.2022* *Kabul Tarihi:30.06.2022*

[8] Halim, Y., Rabago, J. F. T., On the Some Solvable Systems of Difference Equations with Solutions Associated to Fibonacci numbers. Electronic Journal of Mathematical Analysis and Applications, 5(1), 166-178, 2017.

[9] Halim, Y., On the Solutions of a Second-Order Difference Equation in terms of Generalized Padovan Sequences. Mathematica Slovaca, 68(3), 625-638, 2018.

[10] Okumuş, İ., Soykan, Y., On the Solutions of Four Rational Defference Equations Associated to Tribonacci Numbers, preprints.org, doi. 10.20944/preprints201906.0266.v1., 2019.

[11] Kara, M., Yazlik, Y., On a Solvabe Three-Dimensional System of Difference Equations, Filomat, 34(4), 1167-1186, 2020.

[12] Khelifa, A., Halim, Y., Bouchair A., Berkal M., On a system of three difference equations of higher order solved in terms of Lucas and Fibonacci Numbers. Math. Slovaca, 70(3), 641-656, 2020.

[13] Khelifa, A., Halim Y., General solutions to system of difference equations and some of their representations. J. Appl. Math. Comput., 67(3), 439-453, 2021.

[14] Kulenovic M. R. S., Ladas, G., Dynamic of second order rational difference equations: With Open Problems and Conjectures, Chapman Hall/ CRC, Boca Raton, FL, 2002.

[15] R. Abo-Zeid, Behavior of solutions of a second order rational difference equation, Math. Morav., 23, 11-25, 2019.

[16] N. J. A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/.

[17] Soykan, Y., A Study on Generalized Jacobsthal-Padovan Numbers, Earthline Journal of Mathematical Seciences, 4(2), 227-251, 2020.

[18] Soykan, Y., A Study On Generalized (r; s; t)- Numbers. MathLAB Journal, 7, 101-129, 2020.

[19] Taşdemir, E. Global dynamics of a higher order difference equation with a quadratic term. J. Appl. Math. Comput., 67(1-2), 423-437, 2021.

[20] Yacine, H., Form and periodicity of solutions of some system of higherorder difference equation, Math. Sci. Lett., 5(1), 79-84, 2016.

[21] Yazlik, Y., Tollu, D. T., Taskara, N., On the Solutions of Difference Equation System with Padovan Numbers. Applied Mathematics, 4, 15-20. 2013.