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Research Article

I-Statistical Convergent Sequence Spaces of Fuzzy Star–Shaped Numbers

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Abstract

In this study, we acquire I-statistical convergence of sequences of fuzzy star–shaped numbers. We examine topological and algebraic features of the obtained new sequence spaces. We put forward to significant examples of these new notions.

Keywords: I-convergence, Fuzzy star-shaped numbers, L_p-space.

Bulanık Yıldız-Şekilli Sayıların I-İstatistiksel Yakınsak Dizi Uzayları

Öz

Bu çalışmada, bulanık yıldız-şekilli sayıların I-istatistiksel yakınsaklığını elde ettik. Elde edilen yeni dizi uzaylarının bazı topolojik ve cebirsel özelliklerini inceledik. Bu yeni kavramların önemli örneklerini ortaya koyduk.

Anahtar Kelimeler: I-yakınsaklık, Bulanık yıldız-şekilli sayılar, L_p-uzayı.

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1. Introduction

Kostyrko et al. [7] proposed ideal convergence and examined significant features of this convergence concept. Then, ideal convergence of fuzzy numbers was presented by Kumar and Kumar [8]. Some implementations of ideal convergence can be seen in [6,9]. I-statistical convergence was investigated by Savaş and Das [10]. Theory of fuzzy was firstly originated by Zadeh [11]. Zadeh primarily studied the convexity feature of fuzzy sets. Some applications of fuzzy sets can be found in [11]. As a result of the significance of the star-shapedness and convexity that can be examined as a natural extension to this feature, it can be investigated in various ways ([2,3]). Diamond [1] presented the formulation of the fuzzy star-shaped numbers and examined the features of L_p-metric for $p \geq 1$ on the same study.

Throughout the study, we denote the set of all sequences $t = (t_k)$ of fuzzy star-shaped numbers in $\mathbb{R}^n$ by $w^*(S^n)$. Significant definitions and notations which are used in present paper can be found in [4,5,10,12].

2. Material and Method

With the description in the introduction, it can be observed that this study is qualitative with grounded theory method. Papers [1] and [12] put forward to concept of fuzzy star-shaped numbers and also [4], [5] provide a fundamental survey of the convergence concepts of fuzzy star-shaped numbers.

By utilizing the notions of statistical convergence, ideal and fuzzy star-shaped numbers, we acquire new class of I-statistical convergence of sequences of fuzzy star–shaped numbers.

3. Results and Discussion

Now, we aim to present the sequence spaces $c^{S(I)}(S^n), c_0^{S(I)}(S^n)$ and $l_\infty^{S(I)}(S^n)$ of fuzzy star-shaped numbers with regards to the L_p-metric. We identify

$$c^{S(I)}(S^n) = \left\{t = (t_k) \in w^*(S^n): \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t_k, t_0) \geq \xi\right| \geq \zeta\right\} \in I \text{ for some } \xi > 0 \text{ and some } t_0 \in S^n\right\};$$

$$c_0^{S(I)}(S^n) = \left\{t = (t_k) \in w^*(S^n): \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t_k, \bar{0}) \geq \xi\right| \geq \zeta\right\} \in I \text{ for some } \xi > 0 \text{ and some } \bar{0} \in S^n\right\};$$

$$l_\infty^{S(I)}(S^n) = \left\{t = (t_k) \in w^*(S^n): \exists H > 0 \text{ such that } \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t_k, \bar{0}) \geq H\right| \geq \zeta\right\} \in I;\right\}$$

$m^{S(I)}(S^n) = c^{S(I)}(S^n) \cap l_\infty^{S(I)}(S^n)$ and $m_0^{S(I)}(S^n) = c_0^{S(I)}(S^n) \cap l_\infty^{S(I)}(S^n)$.

Definition 3.1. *A sequence $t = (t_k)$ is named to be I-statistically Cauchy if for each $\xi, \zeta > 0$,*

$$\left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t_k, t_j) \geq \xi\right| \geq \zeta\right\} \in I.$$

Theorem 3.1. *The spaces $c^{S(I)}(S^n), c_0^{S(I)}(S^n)$ and $l_\infty^{S(I)}(S^n)$ are linear.*

Proof. Assume $t = (t_k)$ and $r = (r_k)$ be sequences of $c^{S(I)}(S^n)$ which convergence to t_0 and r_0 respectively and α, β be scalars. Then

$$K\{\xi, \zeta\} = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t_k, t_0) \geq \frac{\xi}{2}\right| \geq \zeta\right\} \in I,$$

$$L\{\xi, \zeta\} = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(r_k, r_0) \geq \frac{\xi}{2}\right| \geq \zeta\right\} \in I.$$

$$\begin{aligned}
\rho_p(\alpha t + \beta r, \alpha t_0 + \beta r_0) &= \left(\int_0^1 \rho_H([\alpha t_k + \beta r_k]^\sigma, [\alpha t_0 + \beta r_0]^\sigma)^p \, d\sigma\right)^{\frac{1}{p}} \\
&= \left(\int_0^1 \rho_H(\alpha[t_k]^\sigma + \beta[r_k]^\sigma, \alpha[t_0]^\sigma + \beta[r_0]^\sigma)^p \, d\sigma\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^1 \rho_H([\alpha t_k]^\sigma + [\alpha t_0]^\sigma)^p \, d\sigma\right)^{\frac{1}{p}} + \left(\int_0^1 \rho_H([\beta r_k]^\sigma + [\beta r_0]^\sigma)^p \, d\sigma\right)^{\frac{1}{q}} \\
&= |\alpha|\left(\int_0^1 \rho_H([t_k]^\sigma + [t_0]^\sigma)^p \, d\sigma\right)^{\frac{1}{q}} + |\beta|\left(\int_0^1 \rho_H([r_k]^\sigma + [r_0]^\sigma)^p \, d\sigma\right)^{\frac{1}{q}} \\
&= |\alpha|\rho_p(t, t_0) + |\beta|\rho_p(r, r_0).
\end{aligned}$$

Now

$$\begin{aligned}
M\{\xi, \zeta\} &= \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(\alpha t + \beta r, \alpha t_0 + \beta r_0) \geq \xi\right| \geq \zeta\right\} \\
&\subseteq \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: |\alpha|\rho_p(t, t_0) \geq \frac{\xi}{2}\right| \geq \zeta\right\} \cup \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: |\beta|\rho_p(r, r_0) \geq \frac{\xi}{2}\right| \geq \zeta\right\} \\
&= \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t, t_0) \geq \frac{\xi}{2|\alpha|}\right| \geq \zeta\right\} \cup \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(r, r_0) \geq \frac{\xi}{2|\beta|}\right| \geq \zeta\right\} \\
&\subseteq \left\{K\left\{\frac{\xi}{2|\alpha|}, \zeta\right\} \cup L\left\{\frac{\xi}{2|\beta|}, \zeta\right\}\right\} \in I.
\end{aligned}$$

This gives that $(\alpha t+\beta r)\in c^{S(I)}(S^n)$. As a result, $c^{S(I)}(S^n)$ is a linear space.

Theorem 3.2. *The inclusions* $c_0^{S(I)}(S^n)\subset c^{S(I)}(S^n)\subset l_\infty^{S(I)}(S^n)$ *are strict.*

Proof. Obviously $c_0^{S(I)}(S^n)\subset c^{S(I)}(S^n)$. Now, to indicate that $c_0^{S(I)}(S^n)$ is a proper subset of $c^{S(I)}(S^n)$, consider $t=(t_k)\in w^*(S^n)$ as

$$t_k(s)=\begin{cases} s, & 0\le s<2\\ 3-s, & 2\le s\le 3\\ 0, & \text{otherwise.}\end{cases}$$

Obviously the sequence $(t_k)\in c^{S(I)}(S^n)$ but $(t_k)\notin c_0^{S(I)}(S^n)$, that is $(t_k)\in c^{S(I)}(S^n)/c_0^{S(I)}(S^n)$. Now, contemplate a sequence $t=(t_k)\in c^{S(I)}(S^n)$. Then, there is a $t_0\in S^n$ such that $I-stlimt_k=t_0$, that is,

$$\left\{k\in\mathbb{N}:\frac{1}{k}\left|n\le k:\rho_p(t_k,t_0)\ge\xi\right|\ge\zeta\right\}\in I.$$

We get

$$\rho_p(t_k,\bar{0})\le\rho_p(t_k,t_0)+\rho_p(t_0,\bar{0}).$$

This denotes that (t_k) have to belongs to $l_\infty^{S(I)}(S^n)$. Subsequent is an example to demonstrate the strictness of the inclusion $c^{S(I)}(S^n)\subset l_\infty^{S(I)}(S^n)$.

Example2.1. Contemplate the subsequent sequence:

$$t_k(s)=\begin{cases} \frac{1+2s}{2}, & \text{for } \frac{-1}{2}\le s\le\frac{1}{2}\\ 2(1-s), & \text{for } \frac{1}{2}\le s\le 1\\ 0, & \text{otherwise.}\end{cases}$$

Take I as a non maximal ideal. Determine a sequence $r=(r_k)$ as

$$r_k=\begin{cases} t_k, & k\in K\\ 0, & \text{otherwise.}\end{cases}$$

We acquire $(r_k)\in l_\infty^{S(I)}(S^n)$ but $(r_k)\notin c^{S(I)}(S^n)$.

Also, assume the sequence (t_k) be identified as

$$t_k(s)=\begin{cases} 1-ks, & 0\le s\le\frac{1}{k}\\ 1+ks, & \frac{-1}{k}\le s\le 0 \quad \text{for } k=2m\\ 0, & \text{otherwise.}\end{cases}$$

Otherwise

$$t_k(s)=\begin{cases} s+5, & -5\le s\le 0\\ 1, & 0\le s\le 2\\ -s+5, & 2\le s\le 5\\ 0, & \text{otherwise.}\end{cases}$$

Consequently, $(t_k)\in l_\infty^{S(I)}(S^n)$ but $(t_k)\notin c^{S(I)}(S^n)$. Hence, the inclusions $c_0^{S(I)}(S^n)\subset c^{S(I)}(S^n)\subset l_\infty^{S(I)}(S^n)$ are strict.

Theorem 3.3. *If I is not maximal then* $c^{S(I)}(S^n)$ *is neither normal nor monotone.*

Proof. We examine the subsequent example. Think a sequence $t=(t_k)\in w^*(S^n)$

$$t_k(s)=\begin{cases} 2s, & \text{if } 0\le s\le\frac{1}{2}\\ 1, & \text{if } \frac{1}{2}\le s\le\frac{3}{2}\\ -2(s-2), & \text{if } \frac{3}{2}\le s\le 2\\ 0, & \text{otherwise.}\end{cases}$$

Then, $(t_k)\in c^{S(I)}(S^n)$. As I is not maximal, we identify a sequence $r=(r_k)$ as

$$r_k=\begin{cases} t_k, & k\in K\\ 0, & \text{otherwise.}\end{cases}$$

such that $r=(r_k)$ exists in the canonical pre-image of (t_k) of K-step spaces of $c^{S(I)}(S^n)$. But $(r_k)\notin c^{S(I)}(S^n)$. Hence, $c^{S(I)}(S^n)$ is not monotone, so it is not normal.

Theorem 3.4. *The spaces* $c_0^{S(I)}(S^n), c^{S(I)}(S^n), l_\infty^{S(I)}(S^n)$ *are sequence algebra.*

Proof. When K and L are fuzzy star-shaped numbers then, their product is determined as

$$\mu_{K.L}(y)=\sup_{y=z.x}\min\big(\mu_K(z),\mu_L(x)\big)$$

for every $y \in \mathbb{R}$. Assume t_0 be $I - stlimt_k$ and r_0 be $I - stlimr_k$. For $\sigma \in [0,1]$ and $\alpha, \beta \geq 0$.

$$\rho_H([t_k]^\sigma [r_k]^\sigma, [t_0]^\sigma [r_0]^\sigma) \leq \alpha \rho_p([t_k]^\sigma, [t_0]^\sigma) + \beta \rho_p([r_k]^\sigma, [r_0]^\sigma).$$

Thus, we obtain

$$\rho_H(t_k r_k, t_0 r_0) \leq \alpha \rho_p(t_k, t_0) + \beta \rho_p(r_k, r_0).$$

Let $\xi, \zeta > 0$ be taken. Then

$$K\left\{\frac{\xi}{2}, \zeta\right\} = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(\mathrm{t_k}, \mathrm{t_0}) \geq \frac{\xi}{2}\right| \geq \zeta\right\} \in I,$$
$$L\left\{\frac{\xi}{2}, \zeta\right\} = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(r_k, r_0) \geq \frac{\xi}{2}\right| \geq \zeta\right\} \in I.$$

Think the set

$$M\{\xi, \zeta\} = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t_k r_k, t_0 r_0) \geq \xi\right| \geq \zeta\right\}.$$

It suffices to denote that $M\{\xi, \zeta\} \subseteq K\{\xi, \zeta\} \cup L\{\xi, \zeta\}$. Then

$$\left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t_k r_k, t_0 r_0) \geq \xi\right| \geq \zeta\right\} \subseteq \alpha\left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(\mathrm{t_k}, \mathrm{t_0}) \geq \frac{\xi}{2}\right| \geq \zeta\right\} \cup \beta\left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(\mathrm{r_k}, \mathrm{r_0}) \geq \frac{\xi}{2}\right| \geq \zeta\right\}.$$

Since

$$M\{\xi, \zeta\} \subseteq \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(\mathrm{t_k}, \mathrm{t_0}) \geq \frac{\xi}{2\alpha}\right| \geq \zeta\right\} \cup \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(\mathrm{r_k}, \mathrm{r_0}) \geq \frac{\xi}{2\beta}\right| \geq \zeta\right\}.$$

As a result $M\{\xi, \zeta\} \subseteq K\{\xi, \zeta\} \cup L\{\xi, \zeta\}$.

Theorem 3.5. *The function $h: m^{S(I)}(S^n) \to \mathbb{R}$ given by $h(p) = I - stlimp$ is a Lipschitz function and so uniformly continuous.*

Proof. Assume $t, r \in m^{S(I)}(S^n)$ with $p \neq r$ such that $h(t) = I - stlimt$ and $h(r) = I - stlimr$. Then

$$K_p = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t, h(t)) \geq \|t - r\|\right| \geq \zeta\right\} \in I,$$

$$L_p = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(r, h(r)) \geq \|t - r\|\right| \geq \zeta\right\} \in I.$$

Therefore

$$K_p^c = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t, h(t)) \geq \|t - r\|\right| < \zeta\right\} \in F(I),$$

$$L_p^c = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(r, h(r)) \geq \|t - r\|\right| < \zeta\right\} \in F(I).$$

So $M_p^c = K_p^c \cap L_p^c \in F(I)$. Namely $M_p^c \neq \emptyset$. Let $k \in M_p^c$ such that

$$\rho_p(h(t), h(r)) \leq \rho_p(h(t), t) + \rho_p(t, r) + \rho_p(r, h(r)) \leq 3\|t - r\|.$$

As a result, *h* is Lipschitz continuous.

Theorem 3.6. *When $t, r \in m^{S(I)}(S^n)$, then $(t.r) \in m^{S(I)}(S^n)$ and $h(tr) = h(t)h(r)$.*

Proof. As $t, r \in m^{S(I)}(S^n)$, for $\xi, \zeta > 0$ the subsequent conditions supplies

$$K_p = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(t, h(t)) \geq \xi\right| < \frac{\zeta}{2M}\right\} \in F(I),$$

$$L_r = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(r, h(r)) \geq \xi\right| < \frac{\zeta}{2N}\right\} \in F(I),$$

where $M, N > 0$ where $\rho_p(t, \bar{0}) < M$ and $\rho_p(r, \bar{0}) < N$. Think the set

$$R = \left\{k \in \mathbb{N}: \frac{1}{k}\left|n \leq k: \rho_p(tr, h(t)h(r)) \geq \xi\right| < \zeta\right\}$$

and let $k \in K_p \cap L_r$.

Now

$$\rho_p(tr, h(t)h(r)) \leq \rho_p(tr, th(r)) \leq \rho_p(th(r), h(t)h(r)) \leq \rho_p(t, 0)\rho_p(r, h(r)) + \rho_p(h(r), 0)\rho_p(t, h(t)) \leq \frac{\zeta}{2M}M + \frac{\zeta}{2N}N = \zeta.$$

Hence, $K_p \cap L_r \in R$, so that $R \in F(I)$. So $(t.r) \in m^{S(I)}(S^n)$ and $h(tr) = h(t)h(r)$.

4. Conclusions and Recommendations

In this study, we investigate I-statistical convergence of sequences of fuzzy star–shaped numbers. We put forward to topological and algebraic features of the obtained new sequence spaces. We examine significant examples of these new notions.

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