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Research Article

New results over Zappa-Szép products via a recent semigroup

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Abstract

In [16], the authors established a new semigroup $\mathcal{N}$ as an extension of both the Rees matrix and completely zero-simple semigroups. In this paper, by taking into account the Zappa-Szép product obtained by special subsemigroups of $\mathcal{N}$, we will expose some new distinguishing theoretical results on this product.

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1. Introduction and preliminaries

It is known Zappa-Szép products have been studied widely ([2–4, 15, 17, 18, 20]) which are also referred to as bilateral semidirect products ([6]), general products ([7]) or knit products ([1, 11]). Unlikely semi-direct products, non of the factors are normal in the Zappa product of any two groups. In other words, for a group G with subgroups A and B that satisfy $A \cap B = \{1_G\}$ and $G = AB$, we know that each element $g \in G$ is expressible (uniquely) as $g = ab$ with $a \in A$ and $b \in B$. Now to reserve certain products, let us consider an element $ba \in G$. In fact there must be unique elements $b' \in B$ and $a' \in A$ such that $ba = a'b'$. Define $a' = b.a$ and $b' = b^a$. Associativity in G implies that the actions $(a, b) \mapsto b.a$ and $(a, b) \mapsto b^a$ satisfy axioms first formulated by Zappa [20]. In fact there are two cases in Zappa-Szép products under the names of internal and external which are equivalent in groups. For a semigroup S (or a monoid M) cases, the above reminder corresponds to the internal Zappa-Szép products. The detailed introductory material, examples and different types of studies on this product can be found in [2–4, 8, 9, 17, 18].

Now, by considering any two semigroups A and B, let us recall the following material as depicted in [4] which will be needed for the next section. Suppose that we are given

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two functions $B \times A \to A$, $(b,a) \mapsto b.a$ and $B \times A \to B$, $(b,a) \mapsto b^a$ satisfy the following Zappa axioms [20] for all $a, a' \in A$ and $b, b' \in B$:

$$b.\left(aa'\right) = (b.a)(b^a.a') \tag{1.1}$$

$$\left(b^a\right)^{a'} = b^{aa'} \tag{1.2}$$

$$bb'.a = b.\left(b'.a\right) \tag{1.3}$$

$$\left(bb'\right)^a = b^{b'.a}b'^a \,. \tag{1.4}$$

After that the set $A \times B$ defines the Zappa-Szép product of semigroups A and B with the product $(a,b)(a',b') = \left(a\left(b.a'\right), b^{a'}b'\right)$. In fact, hese above axioms define mutual left semigroup actions of B on A and A on B.

The other key point of this paper is the semigroup $\mathcal{N}$ which was built as an extension of Reesmatrix semigroups by completely zero-simple semigroups. (The detailed studies and propertieson this semigroup see [16, 19]). Below, we will briefly recall only the definition of $\mathcal{N}$.

Suppose that the Rees matrix semigroup $M^0[S^0; I, J; P]$ which is defined on the set $I \times S^0 \times J$ and completely 0-simple semigroup $M^0[G^0; I, J; P']$ which is defined on the set $I \times G^0 \times J$ are denoted by the notations M_R and M_C, respectively. According to their original definitions, without causing any confusion, in these last statements and throughout the rest of the paper the semigroup S andthe group G are actually assumed to be S^0 and G^0, respectively (i.e. including zero). For arbitrary elements $(a,b,c), (k,l,m) \in M_R$ and $(d,e,f), (x,y,z) \in M_C$, let us consider the mapping $\gamma : (M_R \times M_C) \odot (M_R \times M_C) \longrightarrow (M_R \times M_C)$ such that the binary operation on γ is defined by

$$[(a,b,c),(d,e,f)] \odot [(k,l,m),(x,y,z)] =$$

$$= \begin{cases} ((a, bp_{ck}l, m), 0) & \text{if } p_{ck} \neq 0 \text{ and } p'_{fx} = 0 \\ (0, (d, ep'_{fx}y, z)) & \text{if } p_{ck} = 0 \text{ and } p'_{fx} \neq 0 \\ ((a, bp_{ck}l, m), (d, ep'_{fx}y, z)) & \text{if } p_{ck} \neq 0 \text{ and } p'_{fx} \neq 0 \\ (0_R, 0_C) & \text{if } p_{ck} = 0 \text{ and } p'_{fx} = 0 \end{cases} . \tag{1.5}$$

The first line in (1.5) corresponds to the Rees matrix semigroup while the second corresponds to the completely 0-simple semigroup. On the other hand the third line defines the general situation since the operation defined in (1.5) is a generalization for both semigroups M_R and M_C which implies it must also contains both of them. Therefore, by taking into account the operation defined in (1.5) on the set $M_R \times M_C$, we obtain a semigroup $M^0[S, G; M_R, M_C; P, P']$ which is denoted shortly by $\mathcal{N}$. In fact $\mathcal{N}$ can be thought as the form

$$\left[(I \times S \times J), (I \times G \times J)\right],$$

where P and P' are the matrices entire by the elements of S and G, respectively, having both dimensions are $I \times J$.

The main target of this paper is to expose and classify some new results for the semigroup obtained by the Zappa-Szép product of two subsemigroups of $\mathcal{N}$.

2. Internal Zappa-Szép product for $\mathcal{N}$

Kunze [6] showed that the Zappa-Szép product can be obtained by considering Rees matrix semigroups $(I \times G \times J)$, where I, J are index sets and G is any group. In [9, Theorem 3.3], Lawson et al. proved that a left Rees monoid $X^* \times G$, where X^* is the submonoid generated by a transversal of the generators of the maximal proper principal right ideals X, is actually the Zappa-Szép product of X^*.

In particular, one may briefly summarize what Kunze in [6] obtained about Rees matrix semigroups as follows. By extending the index sets of the Rees matrix semigroup $M^0[G;I,\wedge;P]$ (and so the size of the matrix of it is also extended), it has been first considered a larger Rees matrix semigroup, say $R' = M^0[G;I',\wedge';P']$, where $I' = I \cup \{*\}$, $\wedge' = \wedge \cup \{*\}$ and the elements of P' are defined by

$$p'_{\lambda i} = \begin{cases} p_{\lambda i} & ; \text{ if } \lambda \in \wedge,\, i \in I \\ 1 & ; \text{ otherwise} \end{cases} .$$

Thus $R'_1 = \{(i,1,*) : i \in I'\}$ and $R'_2 = \{(*,a,\lambda) : a \in G, \lambda \in \wedge'\}$ are two Rees matrix subsemigroups of R' since $p_{*\lambda} = 1$. Now if I' is a left zero semigroup while $\wedge'$ is a right zero semigroup, then clearly $R'_1 \cong I'$ and $R'_2 \cong G \times \wedge'$ since $p_{\lambda *} = 1$. Furthermore every element $(i,a,\lambda) \in R' \setminus \{0\}$ is uniquely represented as a product in $R'_1.R'_2$. Thus R' is a Zappa-Szép product of R'_1 and R'_2 with actions of R'_1 on R'_2 and R'_2 on R'_1 are defined by

$$(a,\lambda)^i = (ap_{\lambda i}, *) \quad \text{and} \quad (a,\lambda).i = *\,,$$

respectively. In this case the Zappa-Szép product of R'_1 and R'_2 is contained in R'.

In the light of this above approximation in a quite similar manner, in the following section, we will define and investigate the (internal) Zappa-Szép product over $\mathcal{N}$ will defined and classified. We recall that, by omitting zeros, the semigroup $\mathcal{N}$can be symbolized by $[(I \times S \times J)\,,(I \times G \times J)]$ or shortly $M[S,G;I,J;P,P']$ (see Section 1 or [16]). In fact, similarly as in [8] (or [9, Theorem 3.3]), to achieve some distinguishing theoretical results on Zappa-Szép products one may assume some additional properties for the semigroup S inside of $\mathcal{N}$. Therefore unless stated otherwise S will denote a primitive right zero semigroup in this paper.

Now, by extending the index sets as $I' = I \cup \{*\}$ and $J' = J \cup \{*\}$, we can obtain a larger semigroup $\mathcal{N}'$ with the form $[(I' \times S \times J')\,,(I' \times G \times J')]$, where G is a group and S is primitive right zero semigroup. Then we also have two new matrices $\overline{P}$ and $\overline{P}'$ formed by the elements of S and G, respectively, such that the dimensions of both of them are $I' \times J'$. We should note that the meaning of dimension $I' \times J'$ corresponds to $(I+1) \times (J+1)$. So we may specify the elements of the matrices $\overline{P}$ and $\overline{P}'$ as

$$\overline{p}_{i\lambda} = \begin{cases} p_{i\lambda} & ; \text{ if } i \in I,\, \lambda \in J \\ e & ; \text{ otherwise} \end{cases} \quad \text{and} \quad \overline{p}'_{i\lambda} = \begin{cases} p'_{i\lambda} & ; \text{ if } i \in I,\, \lambda \in J \\ 1 & ; \text{ otherwise} \end{cases} , \tag{2.1}$$

where e is the primitive idempotent of S. Now let us choose any two subsets

$$\mathcal{N}'_1 = \{[(i_1,e,*)\,,(i_2,1,*)] : i_1,i_2 \in I',\, e \text{ is the primitive idempotent of } S\} \subseteq \mathcal{N}' \tag{2.2}$$

and

$$\mathcal{N}'_2 = \{[(*,s,j_1)\,,(*,g,j_2)] : j_1,j_2 \in J',\, s \in S,\, g \in G\} \subseteq \mathcal{N}' \tag{2.3}$$

Lemma 2.1. *Subsets $\mathcal{N}'_1$ and $\mathcal{N}'_2$ given in* (2.2) *and* (2.3)*, respectively, are actually subsemigroups of $\mathcal{N}'$.*

Proof. To achieve the truthfulness of the lemma, it is enough to see that $\overline{p}_{*\lambda} = e$ and $\overline{p}'_{*\lambda} = 1$ in the matrices defined in (2.1). Since $* \notin I$, by the second part of the first matrix in (2.1), we get $\overline{p}_{*\lambda} = e$. Similarly, since $* \notin I$, by the second part of the second matrix in (2.1), we reach $\overline{p}'_{*\lambda} = 1$, as required. □

Lemma 2.2. *If I' is a left zero semigroup and J' is a right zero semigroup, then $\mathcal{N}'_1 \cong I' \times I'$ and $\mathcal{N}'_2 \cong [(S \times J')\,,(G \times J')]$.*

Proof. We will apply a similar proof as in Lemma 2.1. So to reach the result, it is enough to see that $\overline{p}_{i*} = e$ and $\overline{p}'_{i*} = 1$ in the matrices presented in (2.1). These two hold since $* \notin J$ and so by the second part of the first matrix in (2.1), we get $\overline{p}_{i*} = e$ and by the second part of the second matrix in (2.1), we obtain $\overline{p}'_{i*} = 1$. □

Proposition 2.3. *For all $i_1, i_2 \in I'$, $j_1, j_2 \in J'$, $s \in S$ and $g \in G$, every element $[(i_1, s, j_1), (i_2, g, j_2)] \in \mathcal{N}' \setminus \{0\}$ is uniquely representable as an element in $\mathcal{N}'_1 \odot \mathcal{N}'_2$, where $\odot$ is the operation given in* (1.5).

Proof. For arbitrary elements $[(i_1, e, *), (i_2, 1, *)]$ and $[(*, s, j_1), (*, g, j_2)]$ from the semigroups $\mathcal{N}'_1$ and $\mathcal{N}'_2$, respectively, if we apply the operation (1.5), then we obtain

$$\begin{aligned}
[(i_1, e, *), (i_2, 1, *)] \odot [(*, s, j_1), (*, g, j_2)] &= [(i_1, e\overline{p}_{**}s, j_1), (i_2, 1\overline{p}'_{**}g, j_2)] \\
&= [(i_1, ees, j_1), (i_2, 1g, j_2)] \\
&\quad \text{since } \overline{p}_{**} = e \text{ and } \overline{p}'_{**} = 1 \text{ by the} \\
&\quad \text{matrices in (2.1)} \\
&= [(i_1, es, j_1), (i_2, g, j_2)] \\
&\quad \text{since } ees = es \text{ by the meaning of } e \\
&= [(i_1, s, j_1), (i_2, g, j_2)] \\
&\quad \text{since } S \text{ is a right zero semigroup,} \\
&\quad \text{and so } es = s\,.
\end{aligned}$$

Hence the result. □

By the proof of Proposition 2.3, it can be clearly seen the reasons why we assumed S as a right zero semigroup and e as a primitive idempotent in S. We note that if e was not chosen as a primitive, then we would have to deal with different idempotents.

Keeping in our minds of Lemmas 2.1, 2.2 and also the operation given in (1.5), let us define two actions $\mathcal{N}'_1 \times \mathcal{N}'_2 \to \mathcal{N}'_1$ and $\mathcal{N}'_1 \times \mathcal{N}'_2 \to \mathcal{N}'_2$ as

$$((i_1, i_2), [(e, j_1), (g, j_2)]) \mapsto [(e, j_1), (g, j_2)]\,.(i_1, i_2) = (*, *) \quad \text{and} \tag{2.4}$$

$$((i_1, i_2), [(e, j_1), (g, j_2)]) \mapsto [(e, j_1), (g, j_2)]^{(i_1, i_2)} = \left[\left(s\overline{p}_{j_1 i_1} e, *\right), \left(g\overline{p}'_{j_2 i_2}, *\right)\right], \tag{2.5}$$

respectively, where $i_1, i_2 \in I'$, $j_1, j_2 \in J'$, $s \in S$ and $g \in G$. Note that since S is a right zero semigroup, in action (2.5), we actually have $s\overline{p}_{j_1 i_1} e = e$ and so the element $\left[\left(s\overline{p}_{j_1 i_1} e, *\right), \left(g\overline{p}'_{j_2 i_2}, *\right)\right]$ can be thought as $\left[(e, *), \left(g\overline{p}'_{j_2 i_2}, *\right)\right]$.

Proposition 2.4. *Actions defined in* (2.4) *and* (2.5) *satisfy the Zappa axioms.*

Proof. In the proof, we will only apply the elements in (2.4) and (2.5) into the axioms (1.1), (1.2), (1.3) and (1.4). To do that, the elements a, b, a' and b' in these four axioms will be represented by (i_1, i_2), $[(e, j_1), (g, j_2)]$, $(*, *)$ and $\left[(s\overline{p}_{j_1 i_1} e, *), (g\overline{p}'_{j_2 i_2}, *)\right]$, respectively.

- *Axiom* (1.1) *holds*: The LHS is

$$\begin{aligned}
[(e, j_1), (g, j_2)]\,.(i_1 *, i_2 *) &= [(e, j_1), (g, j_2)]\,.(i_1, i_2) \\
&\quad \text{since } I' \text{ is a left zero semigroup by Lemma 2.2} \\
&= (*, *)
\end{aligned}$$

whereas the RHS is written as

$$([(e, j_1), (g, j_2)]\,.(i_1, i_2)) \left([(e, j_1), (g, j_2)]^{(i_1, i_2)}\,.(*, *)\right)$$

and this is equal to

$$(*, *)\left[\left((s\overline{p}_{j_1 i_1} e, *), (g\overline{p}'_{j_2 i_2}, *)\right).(*, *)\right] = (*, *)(*, *) = (**, **) = (*, *)\,.$$

• *Axiom* (1.2) *holds*: The LHS of (1.2) is

$$\begin{aligned}
[(e,j_1),(g,j_2)]^{(i_1,i_2)(*,*)} &= [(e,j_1),(g,j_2)]^{(i_1*,i_2*)} = [(e,j_1),(g,j_2)]^{(i_1,i_2)} \\
&\quad \text{since } I' \text{ is a left zero semigroup by Lemma 2.2} \\
&= \left[\left((s\overline{p}_{j_1i_1}e,*),(g\overline{p}'_{j_2i_2},*)\right)\right] = \left[\left((e,*),(g\overline{p}'_{j_2i_2},*)\right)\right] \\
&\quad \text{since } S \text{ is a right zero semigroup}
\end{aligned}$$

and also RHS of it is

$$\begin{aligned}
&\left([(e,j_1),(g,j_2)]^{(i_1,i_2)}\right)^{(*,*)} = \left[(s\overline{p}_{j_1i_1}e,*),(g\overline{p}'_{j_2i_2},*)\right]^{(*,*)} = \\
&= \left[(e,*),(g\overline{p}'_{j_2i_2},*)\right]^{(*,*)} = \left[(s\overline{p}_{**}e,*),(g\overline{p}'_{j_2i_2}\overline{p}'_{**},*)\right] = \left[(e,*),(g\overline{p}'_{j_2i_2},*)\right]
\end{aligned}$$

since $\overline{p}'_{**}=1$ and S is a right zero semigroup, as required.

• *Axiom* (1.3) *holds*: Similarly as above, by considering the action 2.4, LHS of (1.3) is

$$\begin{aligned}
&\left([(e,j_1),(g,j_2)]\left[(s\overline{p}_{j_1i_1}e,*),(g\overline{p}'_{j_2i_2},*)\right]\right).(i_1,i_2) \\
&= \left[(es\overline{p}_{j_1i_1}e,j_1*),(gg\overline{p}'_{j_2i_2},j_2*)\right].(i_1,i_2) = (*,*)
\end{aligned} \tag{2.6}$$

and RHS is

$$[(e,j_1),(g,j_2)].\left(\left[(s\overline{p}_{j_1i_1}e,*),(g\overline{p}'_{j_2i_2},*)\right].(i_1,i_2)\right) = [(e,j_1),(g,j_2)].(*,*) = (*,*)\,.$$

Therefore both parts are equal to each other.

• *Axiom* (1.4) *holds*: As in the above three processes, the LHS of (1.4) is obtained by considering

$$\begin{aligned}
\left([(e,j_1),(g,j_2)]\left[(s\overline{p}_{j_1i_1}e,*),(g\overline{p}'_{j_2i_2},*)\right]\right)^{(i_1,i_2)} &= \left[(es\overline{p}_{j_1i_1}e,j_1*),(gg\overline{p}'_{j_2i_2},j_2*)\right]^{(i_1,i_2)} \\
&= \left[(e,*),(gg\overline{p}'_{j_2i_2},*)\right]^{(i_1,i_2)} \\
&\quad \text{since } S \text{ and } J' \text{ are right zero} \\
&\quad \text{semigroups by Lemma2.2} \\
&= \left[(s\overline{p}_{*i_1}e,*),(gg\overline{p}'_{j_2i_2}\overline{p}'_{*i_2},*)\right] \\
&= \left[(e,*),(gg\overline{p}'_{j_2i_2},*)\right] \\
&\quad \text{since } \overline{p}_{*i_1}=e \text{ and } \overline{p}'_{*i_2}=1\,.
\end{aligned}$$

Furthermore, by taking into account that S and J' are right zero semigroups, the RHS is obtained by

$$\begin{aligned}
&[(e,j_1),(g,j_2)]^{\left[(s\overline{p}_{j_1i_1}e,*),(g\overline{p}'_{j_2i_2},*)\right].(i_1,i_2)}\left[(s\overline{p}_{j_1i_1}e,*),(g\overline{p}'_{j_2i_2},*)\right]^{(i_1,i_2)} = \\
&= [(e,j_1),(g,j_2)]^{(*,*)}\left[(s\overline{p}_{j_1i_1}e,*),(g\overline{p}'_{j_2i_2},*)\right]^{(i_1,i_2)} \\
&= \left[(s\overline{p}_{j_1*}e,*),(g\overline{p}'_{j_2*},*)\right]\left[(s\overline{p}_{*,i_1}e,*),(g\overline{p}'_{j_2i_2}\overline{p}'_{*i_2},*)\right] \\
&= \left[(ee,**),(g\overline{p}'_{j_2*}g\overline{p}'_{j_2i_2}\overline{p}'_{*i_2},**)\right] \\
&= \left[(e,*),(gg\overline{p}'_{j_2i_2},*)\right] \text{ since } \overline{p}_{j_2*}=1 \text{ and } \overline{p}'_{*i_2}=1\,.
\end{aligned}$$

Hence the result. □

After all, by considering Lemmas 2.1, 2.2 and Propositions 2.3, 2.4, we obtain the following main result of this section.

Theorem 2.5. *We have a Zappa-Szép product $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ based on Rees matrix semigroups, where $\mathcal{N}_1'$ and $\mathcal{N}_2'$ are defined by Eq. (2.2) and Eq. (2.3) respectively. Morever $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is contained in the semigroup $\mathcal{N}$ and so are contained in $\mathcal{N}'$.*

Remark 2.6. Theorem *2.5* is a generalization of the result presented by Kunze *[6]*. Althought it was only considered Rees matrix semigroups of the form $(I \times G \times J)$, where I, J are index sets and G is any group in *[6]*, in this paper it is considered the form $[(I \times S \times J), (I \times G \times J)]$, where I, J are index sets, G is a group and S is a (right zero) semigroup (see Theorem *2.5*).

Remark 2.7. By considering the product $(a,b)(a',b') = \left[a\,(b.a')\,, b^{a'}b'\right]$ (which was reminded in the motivationpart), it would be easy to see that the elements of Zappa-Szép product obtained in Theorem *2.5* are the form of

$$[(i_1, e, *)\,, (i_2, g, *)]$$

where e is the primitive idempotent element of S and g is an element of the group G.

3. Some theoretical results on Zappa-Szép products via $\mathcal{N}$

Throughout this section same notations and assumptions will be used as in Section 2. Therefore S will denote a primitive right zero semigroup, the elements of $\mathcal{N}_1$, $\mathcal{N}_2$ and their Zappa-Szép product are the form given in (2.2), (2.3) and Remark 2.7, respectively. Finally, as depicted in Lemma 2.2, I' and J' will denote the extended index sets which are actually left and right zero semigroups, respectively.

3.1. Part I

In previous studies, for instance [4, 5, 17], authors mostly presented some results about the topics *Green's relation on the Zappa-Szép product, regularity of Zappa-Szép products obtained by regular semigroups or algebraic properties over Zappa-Szép products obtained by a semigroup and a band.*

In a different manner from these above topics, in this section, we will state and prove some theoretical results on the Zappa-Szép product of two semigroups, namely to be a *groupbound*, to be *disjoint union of groups* and to be a *semilattice of groups*. In fact our results can be thought of as a basement to obtain a generalization of [10][Theorem 2.1].

A semigroup S is said to have *periodic condition on principal left* (and *right*) *ideals*, notated by P_L (and P_R), if for each $x \in S$ there exists a positive integer n such that $Sx^n = Sx^{n+t}$ (and $x^nS = x^{n+t}S$) for all $t \in \mathbb{N}$. Furthermore, a semigroup S is said to be *left separative* if $x^2 = xy$ and $y^2 = yx$ imply $x = y$ for all $x, y \in S$. *Right separativity* is defined dually and so S is called *separative* if both left and right separativity hold. We note that all details and properties of these definitions can be found in [10].

The following lemmas will be needed in our results.

Lemma 3.1 ([10], Theorem 1.1). *A semigroup S satisfies both P_L and P_R at the same time if and only if S is groupbound.*

Lemma 3.2 ([10], Theorem 1.2). *A left separative semigroup satisfying P_R is a disjoint union of groups.*

Lemma 3.3 ([10], Corollary 1.3). *If S is a separative semigroup satisfying P_R, then S is a semilattice of groups.*

Lemma 3.4 ([10], Theorem 1.4). *A semigroup is periodic if and only if all its subsemigroups satisfy P_L.*

Now we can state and prove the following proposition which is the first result of this section.

Proposition 3.5. *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ satisfies the condition P_L if and only if I' contains a single element.*

Proof. *The necessity part*: By the definition of P_L, we have $(\mathcal{N}_1' \bowtie \mathcal{N}_2')\, x^n = (\mathcal{N}_1' \bowtie \mathcal{N}_2')\, x^{n+t}$ for each $x \in \mathcal{N}_1' \bowtie \mathcal{N}_2'$. By Remark 2.7, take an element $x = [(i_1, e, *)\,, (i_2, g, *)]$ from the semigroup $\mathcal{N}_1' \bowtie \mathcal{N}_2'$. Let us consider the case $t = 1$. Then

$$\begin{aligned} & [(I', e, *)\,, (I', G, *)]\ [(i_1, e, *)\,, (i_2, g, *)]^n \\ = \ & [(I', e, *)\,, (I', G, *)]\ [(i_1, e\overline{p}_{*i_1} e\overline{p}_{*i_1} e \ldots e, *)\,, (i_2, g\overline{p}'_{*i_2} g\overline{p}'_{*i_2} g \ldots g, *)]\ , \end{aligned}$$

where each $\overline{p}_{*i_1} = e$ and $\overline{p}'_{*i_2} = 1$. So we obtain

$$\begin{aligned} & = [(I', e, *)\,, (I', G, *)]\ [(i_1, e, *)\,, (i_2, g^n, *)] \\ & = [(I', e\overline{p}_{*i_1} e, *)\,, (I', G\overline{p}'_{*i_2} g^n, *)] \\ & = [(I', e, *)\,, (I', Gg^n, *)] \qquad (3.1) \\ & \quad \text{since } \overline{p}_{*i_1} = e \text{ and } \overline{p}'_{*i_2} = 1\,. \end{aligned}$$

On the other hand

$$\begin{aligned} & [(I', e, *)\,, (I', G, *)]\ [(i_1, e, *)\,, (i_2, g, *)]^{n+1} \qquad (3.2) \\ = \ & [(I', e, *)\,, (I', G, *)]\ \left[(i_1, e, *)\,, \left(i_2, g^{n+1}, *\right)\right] \\ = \ & \left[(I', e\overline{p}_{*i_1} e, *)\,, \left(I', G\overline{p}'_{*i_2} g^{n+1}, *\right)\right] \\ = \ & \left[(I', e, *)\,, \left(I', Gg^{n+1}, *\right)\right] \qquad (3.3) \end{aligned}$$

Let us assume that I' contains a single element. Hence a simple calculation implies that Equations (3.1) and (3.3) are equal to each other. Until now we proceeded with the case t=1 for the equality $(\mathcal{N}_1' \bowtie \mathcal{N}_2')\, x^n = (\mathcal{N}_1' \bowtie \mathcal{N}_2')\, x^{n+t}$. However the inductive proceses gives truth to this equality for all $t \in \mathbb{N}$.

The *sufficiency part*: Similarly as in the end of necessity part, by assuming I' contains a single element, we see that $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ satisfies the condition P_L.

Hence the result. □

Proposition 3.6. *Every elements of $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ satisfy the condition P_R.*

Proof. Our aim is to show that $x^n\, (\mathcal{N}_1' \bowtie \mathcal{N}_2') = x^{n+t}\, (\mathcal{N}_1' \bowtie \mathcal{N}_2')$ for all $x \in \mathcal{N}_1' \bowtie \mathcal{N}_2'$.

As in the proof of Proposition 3.5, let us take an element $x = [(i_1, e, *)\,, (i_2, g, *)]$ from the semigroup $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ and consider the case $t = 1$. Therefore, the LHS is equal to

$$\begin{aligned} [(i_1, e, *)\,, (i_2, g, *)]^n\ [(I', e, *)\,, (I', G, *)] \ &= \ [(i_1, e\overline{p}_{*i_1} e, *)\,, (i_2, g^n \overline{p}'_{*i_2} G, *)] \\ &= \ [(i_1, e, *)\,, (i_2, g^n G, *)] \end{aligned}$$

since $\overline{p}_{*i_1} = e$ and $\overline{p}'_{*i_2} = 1$. Furthermore, the RHS is

$$[(i_1, e, *)\,, (i_2, g, *)]^{n+1}\ [(I', e, *)\,, (I', G, *)] = \left[(i_1, e, *)\,, \left(i_2, g^{n+1} G, *\right)\right]\,.$$

In addition, for any group G, since $g^n G = g^{n+1} G \implies g^n G = g^n g G \implies G = gG$, we clearly achieve

$$[(i_1, e, *)\,, (i_2, g^n G, *)] = \left[(i_1, e, *)\,, \left(i_2, g^{n+1} G, *\right)\right]\,,$$

as required.

□

Thus the proof of the following main result of this section is clear by Lemma 3.1, Proposition 3.5 and Proposition 3.6.

Theorem 3.7. *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is groupbound if and only if I' contains a single element.*

In the following, we will focus our attention on the separativity property for the semigroup $\mathcal{N}_1' \bowtie \mathcal{N}_2'$. To reach it, let us first prove some lemmas.

Lemma 3.8. *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is left separative if and only if I' contains a single element.*

Proof. Assume that $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is left separative. Thus, for any two elements $x = [(i_1, e, *), (i_2, g, *)]$, $y = [(a, e, *), (b, g_1, *)] \in \mathcal{N}_1' \bowtie \mathcal{N}_2'$, we certainly have $x = y$ while $x^2 = xy$ and $y^2 = yx$. In other words,

$$\begin{aligned} x^2 = xy \quad &\Longrightarrow \quad [(i_1, e, *), (i_2, g, *)]^2 = [(i_1, e, *), (i_2, g, *)]\ [(a, e, *), (b, g_1, *)] \\ &\Longrightarrow \quad [(i_1, e\overline{p}_{*i_1}e, *), (i_2, g\overline{p}'_{*i_2}g, *)] = [(i_1, e\overline{p}_{*a}e, *), (i_2, g\overline{p}'_{*b}g_1, *)] \\ &\Longrightarrow \quad \left[(i_1, e, *), \left(i_2, g^2, *\right)\right] = [(i_1, e, *), (i_2, gg_1, *)] \\ &\qquad\ \ \text{since } \overline{p}_{*i_1} = e = \overline{p}_{*a} \text{ and } \overline{p}'_{*i_2} = 1 = \overline{p}'_{*b}\,. \end{aligned}$$

So, we get $gg = gg_1$ from the last equation.

On the other hand,

$$\begin{aligned} y^2 = yx \quad &\Longrightarrow \quad [(a, e, *), (b, g_1, *)]^2 = [(a, e, *), (b, g_1, *)]\ [(i_1, e, *), (i_2, g, *)] \\ &\Longrightarrow \quad [(a, e\overline{p}_{*a}e, *), (b, g_1\overline{p}'_{*b}g_1, *)] = [(a, e\overline{p}_{*i_1}e, *), (b, g_1\overline{p}'_{*i_2}g, *)] \\ &\Longrightarrow \quad [(a, e, *), (b, g_1g_1, *)] = [(a, e, *), (b, g_1g_1, *)] \\ &\qquad\ \ \text{since } \overline{p}_{*a} = e = \overline{p}_{*i_1} \text{ and } \overline{p}'_{*b} = 1 = \overline{p}'_{*i_2} \end{aligned}$$

which implies $g_1g_1 = g_1g$. Since $gg = gg_1$ and $g_1g_1 = g_1g$, we have $g = g_1$. Therefore, by the assumption $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is left separative, we must have $i_1 = a$, $i_2 = b$ and $g = g_1$. So the index set I' must contains a single element.

Now, for the sufficiency part, if we assumeI' contains a single element, then it must be $i_1 = a$ and $i_2 = b$ must be held in the above calculations. Then, by applying a similar approach, the truthfulness of the other way can be seen clearly. □

Lemma 3.9. *Every elements of $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ are right separative.*

Proof. We will follow quite similar path as in the proof of Lemma 3.8. Therefore we have to show that if $x^2 = yx$ and $y^2 = xy$, then $x = y$ holds for any two elements $x = [(i_1, e, *), (i_2, g, *)]$, $y = [(a, e, *), (b, g_1, *)] \in \mathcal{N}_1' \bowtie \mathcal{N}_2'$. So, let us first assume $x^2 = yx$. Replacing the elements x and y, we obtain

$$[(i_1, e, *), (i_2, gg, *)] = [(a, e, *), (b, g_1g, *)]$$

which implies $i_1 = a$, $i_2 = b$ and $gg = g_1g$. On the other hand, by assuming $y^2 = xy$ and then by replacing x and y as above, we get

$$[(a, e, *), (b, g_1g, *)] = [(i_1, e, *), (i_2, gg_1, *)]$$

which implies $a = i_1$, $b = i_2$ and $g_1g_1 = gg_1$. In fact these two equalities give $i_1 = a$, $i_2 = b$ and $g = g_1$ and so $x = y$, as required. □

Therefore other main results of this section are the following.

Theorem 3.10. *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is disjoint union of groups if and only if I' contains a single element.*

Proof. The proof of the theorem is clear by Lemmas 3.2, 3.8 and Proposition 3.6. □

Theorem 3.11. *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is a semilattice of groups if and only if I' contains a single element.*

Proof. The proof follows by Lemmas 3.3, 3.8, 3.9 and Proposition 3.6. □

By considering Lemma 3.4 and Proposition 3.5, we finally obtain the next theorem.

Theorem 3.12. *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is periodic if and only if I' contains a single element.*

Proof. We will follow a similar way as in [10, Theorem 1.4]. It is known that a semigroup is said to be periodic if and only if all its subsemigroups satisfy the condition P_L. According to this definition, if the index set I' has a single element then the sufficiency part is clear.

Conversely, let us take an element $a \in \mathcal{N}_1' \bowtie \mathcal{N}_2'$. Then any subsemigroup, say T, that is generated by a satisfies the condition P_L by the assumption of necessity part. Hence there exists a natural number $n \in \mathbb{N}$ such that $a^nT = a^{n+1}T$. Thus there must exists $t \in \mathbb{N}$ such that $a^{n+1} = a^{n+t}$. Consequently, $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is periodic. Since T satisfies P_L, by Proposition 3.5, we know that I' contains just one element. □

Until now, by taking into account Propositions 3.5, 3.6 and Lemmas 3.8, 3.9, we have proved four theorems, namely Theorems 3.7, 3.10, 3.11 and 3.12 in above. In fact, since these theorems are exposing a generalization of [10, Theorem 2.1], the main parts of this paper are built up on them. As a consequence, one may summarize the statements of these results as in the following corollary.

Corollary 3.13. *Let the index set I' contains a single element. Then the following are equaivalent.*

i) *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is groupbound.*

ii) *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is disjoint union of groups.*

iii) *$\mathcal{N}_1' \bowtie \mathcal{N}_2'$ is semilattice of groups.*

3.2. Part II

Remember that, in Section 2, we obtained a Zappa-Szép product $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ which is based on Rees matrix semigroups and is contained in both $\mathcal{N}$ and $\mathcal{N}'$. Also we mentioned that $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ contains an a semigroup and a group at the same time, and so this fact implies the main difference between some results in [6, 8] and the results in Section 2. Thus we indicated that our results pointed out an extended version of these previous studies. After that, in Section 3.1, we exposed some theoretical results as a generalization of [10, Theorem 2.1] on Zappa-Szép products which those had not been obtained until now.

In this final section, we continuing our investigation on $\mathcal{N}_1' \bowtie \mathcal{N}_2'$, we will obtain some recent results which have not been touched in previous studies (see, for instance, [6, 8, 9, 16, 17]). In detail, we will obtain some new results on $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ in terms of different Greens relations. We remind that to satisfy the periodicity of $\mathcal{N}_1' \bowtie \mathcal{N}_2'$, the index set I' must contain a single element (see Theorem 3.11). Hence, unless stated otherwise, it will be assumed that the index I' contains a single element in the following results.

In [14], Sedlock defined some different Green's relations for periodic semigroups. Here, we will follow this reference to build up these relationships. For each element x of a periodic semigroup S, there exist some power of x which is an idempotent, which leads to defining an equivalence relation $\mathcal{K}$ on S as follows.

Definition 3.14 ([14])**.** For $a, b \in S$ and $m, n \in \mathbb{N}$, we have

$$a\mathcal{K}b \Leftrightarrow \text{there exists an idempotent } e \text{ such that } a^n = b^m = e\,.$$

The $\mathcal{K}$-classes of S will be denoted by $\mathcal{K}^e$ for each idempotent e.

For each $\mathcal{K}$-class in $\mathcal{K}^e$, the notions of maximal subgroup G^e and maximal semigroups contained in $\mathcal{K}^e$ were first studied by Schwarz (see [12, 13]). As a consequence of these works, the following facts on an arbitrary S are obtained.

Lemma 3.15 ([14])**.** *Let S be a semigroup. Then the following hold.*

(i) *For each idempotent e, it always true $ex = xe$ where $x \in \mathcal{K}^e$. Moreover*

$$e\mathcal{K}^e = \mathcal{K}^e e = G^e = \{x \in \mathcal{K}^e \,:\, ex = xe = x\}$$

is the maximal subgroup of S containing e.

(ii) *S is a union of groups*

- *if and only if each element of S has index one,*
- *if and only if $\mathcal{K}^e = G^e$ for each idempotent e.*

The first result of this section is the following.

Proposition 3.16. *Every element of $\mathcal{N}_1'$ is an idempotent for the semigroup $\mathcal{N}_1' \bowtie \mathcal{N}_2'$.*

Proof. To do that we will only apply the definition of an idempotent element. For a sample element $x = [(a, e, *)\,, (b, g, *)]$ in $\mathcal{N}_1' \bowtie \mathcal{N}_2'$, if we have

$$[(a, e, *)\,, (b, g, *)]^2 = [(a, e, *)\,, (b, g, *)]\ , \tag{3.4}$$

then it leads the equality

$$[(a, e\overline{p}_{*a}e, *)\,, (b, g\overline{p}'_{*b}g, *)] \;\;=\;\; [(a, e, *)\,, (b, gg, *)]$$

since $\overline{p}_{*a} = e$ and $\overline{p}'_{*b} = 1$. According to (3.4), we must have

$$[(a, e, *)\,, (b, gg, *)] = [(a, e, *)\,, (b, g, *)]$$

which implies $g^2 = g \Rightarrow g = 1$. Therefore the idempotent element of $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ must be the form of $[(a, e, *)\,, (b, 1, *)]$ which is actually the same form of elements in $\mathcal{N}_1'$.

Hence the result. □

Next, we will find out the form of elements of $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ in $\mathcal{K}^e$ classes. First of all, by Definition 3.14, it is known that the property

$$x\mathcal{K}y \Leftrightarrow \text{there exists an idempotent } e \text{ such that } x^n = y^m = e$$

must be satisfied for $x, y \in S$ and $m, n \in \mathbb{N}$. Therefore, for choosen elements

$$x = [(a_1, e, *)\,, (a_2, g_1, *)] \qquad \text{and} \qquad y = [(b_1, e, *)\,, (b_2, g_2, *)]\ ,$$

where $a_1, a_2, b_1, b_2 \in I'$, $g_1, g_2 \in G$ and e is the primitive idempotent, we have

$$\begin{aligned} x^n &= [(a_1, e, *)\,, (a_2, g_1, *)]^n = [(a_1, e, *)\,, (a_2, 1, *)] \Rightarrow \\ &\Rightarrow [(a_1, e, *)\,, (a_2, g_1, *)] = [(a_1, e, *)\,, (a_2, 1, *)] \end{aligned}$$

which leads g_1^n must be equal to 1 for the element x. Similarly, we will get $g_2^m = 1$ for the element y. However, the equality $x^n = y^m$ gives $g_1^n = g_2^m = 1$ that means $G = \langle g_1 \rangle = \langle g_2 \rangle$.

This above process implies the following lemma.

Lemma 3.17. *The elements of $\mathcal{K}^e$ classes are the form of $[(I', e, *)\,, (I', G, *)]$ whenever G is an unique finite cyclic (sub)group.*

On the other hand, the form of elements of the maximal subgroup G^e in $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ can be obtained as in the following. By Lemma 3.18-(i), it is known that the elements of a general G^e are the form of

$$e\mathcal{K}^e = \mathcal{K}^e e = G^e = \{x \in \mathcal{K}^e \,:\, ex = xe = x\}$$

for an idempotent e. Furthermore Lemma 3.17 states how the form of elements of $\mathcal{K}^e$ classes need to be. Now, by considering an idempotent element $e = [(a_1, e, *)\,, (a_2, 1, *)]$ from the semigroup $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ and by Equation (1.5), we clearly get

$$\begin{aligned} e\mathcal{K}^e &= [(a_1, e, *)\,, (a_2, 1, *)]\ [(b_1, e, *)\,, (b_2, g, *)] \\ &= [(a_1, e, *)\,, (a_2, g, *)] \end{aligned}$$

as well as

$$\begin{aligned} \mathcal{K}^e e &= [(b_1, e, *), (b_2, g, *)]\,[(a_1, e, *), (a_2, 1, *)] \\ &= [(b_1, e, *), (b_2, g, *)]\,, \end{aligned}$$

where $a_1, a_2, b_1, b_2 \in I'$, g is an element of the cyclic sub(group G (by Lemma 3.17) and e is the primitive idempotent. Therefore to equality $e\mathcal{K}^e = \mathcal{K}^e e$ be hold, we must have

$$a_1 = b_1 \quad \text{and} \quad a_2 = b_2\,.$$

Regarding the above processes, we obtain the following lemma.

Lemma 3.18. *The elements of G^e are the form of $[(I', e, *), (I', G, *)]$ whenever G is an unique finite cyclic (sub)group and I' contains at most two elements.*

Remark 3.19. The case I' contains a single element can be thought as a special case of Lemma *3.18* since the equality $e\mathcal{K}^e = \mathcal{K}^e e$ always holds for any $I' = \{a\}$.

By Definition 3.14, Lemmas 3.15, 3.17, 3.18-(i),(ii) and Proposition 3.16, we obtain the following final result of this paper.

Theorem 3.20. *Suppoe that G is a unique finite cyclic (sub)group and I' contains at most two elements. Then the following are equivalent.*
(i) $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ *is a union of groups.*
(ii) *Each element of* $\mathcal{N}_1' \bowtie \mathcal{N}_2'$ *has index one.*
(iii) $\mathcal{K}^e = G^e$ *for each idempotent element e.*

4. Some future problems

The scope of this paper is to expose some new results on a new type of semigroup, namely the Zappa-Szép product $\mathcal{N}_1' \bowtie \mathcal{N}_2'$. However there would still be some problems that could be studied in the near future. We may summarize some of them as follows:

We assumed the semigroup S as a primitive right zero in Section 3.2. The first possible problem would be considering the situation for a general semigroup S and checking whether the results still hold. Additionally, omitting the method presented in [6] and thus without using the subsemigroups $\mathcal{N}_1'$ and $\mathcal{N}_2'$, one may define a new Zappa-Szép product regarding the semigroup $\mathcal{N}$, and thus check related similar results on this product. We truly believe that most of the theories would be invalid without special assumptions on $\mathcal{N}$ as we have done in the theorems of this paper.

Instead of the studies presented in Section 3, by defining a presentation for the semigroup $\mathcal{N}_1' \bowtie \mathcal{N}_2'$, one may study Gröbner-Shirshov bases and the solvability of the word problem.

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