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Bicomplex Numbers with respect to the Geometric Calculus and Some Inequalities

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Abstract: In this paper, we deal with complex and bicomplex numbers with respect to the geometric calculus, and we obtain the set of complex numbers with respect to the geometric calculus $\mathbb{C}(GC)$ is a field and the set of bicomplex numbers with respect to the geometric calculus $\mathbb{BC}(GC)$ is a vector space on the field $\mathbb{C}(GC)$ by defining addition and multiplication operations on the sets of such numbers. Also, we give the concepts of norm, metric, sequence, convergence of a sequence, Cauchy sequence and completeness in the settings $\mathbb{C}(GC)$ and $\mathbb{BC}(GC)$. Moreover, we discuss bicomplex versions with respect to geometric calculus of some well-known inequalities. This paper is a new and important addition to the current literature thanks to its applications in different areas and the obtained results unify, private and complement the corresponding results.

Geometrik Kalkülüse göre Bikompleks Sayılar ve Bazı Eşitsizlikler

Anahtar Kelimeler
Geometrik Kalkülüs,
Non-Newtonian Bikompleks Sayı,
Non-Newtonian Kompleks Sayı,
Eşitsizlikler

Öz: Bu makalede, geometrik kalkülüse göre kompleks sayıları ve bikompleks sayıları ele aldık ve böyle sayılardan oluşan kümeler üzerinde toplama ve çarpma işlemlerini tanımlayarak geometrik kalkülüse göre $\mathbb{C}(GC)$ kompleks sayılar kümesinin bir cisim olduğunu ve geometrik kalkülüse göre $\mathbb{BC}(GC)$ bikompleks sayılar kümesinin $\mathbb{C}(GC)$ cismi üzerinde bir vektör uzayı olduğunu elde ettik. Ayrıca $\mathbb{C}(GC)$ ve $\mathbb{BC}(GC)$ kurulumlarında norm, metrik, dizi, dizinin yakınsaklığı, Cauchy dizisi ve tamlık kavramlarını verdik. Diğer yandan, bazı iyi bilinen eşitsizliklerin geometrik kalkülüse göre bikompleks versiyonlarını tartıştık. Bu makale, farklı alanlardaki uygulamaları ve elde edilen sonuçların birleştirilmesi, özelleştirilmesi ve ilgili sonuçları tamamlaması sayesinde mevcut literatüre yeni ve önemli bir katkıdır.

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1. Introduction

Corrado Segre [1] presented the concept of a bicomplex number in 1892. After that, Price [2] published a book on bicomplex numbers and bicomplex functions. Hereupon, Alpay et al. in [3] gave a clear and general survey of bicomplex functional analysis and additionally put forward some new ideas and results.

In 1972, Grossman and Katz [4] laid the foundations of non-newtonian calculus which modify the calculi initiated by Gottfried Wilhelm Leibnitz and Isaac Newton in the 17th century. A generator is a one-to-one function $\alpha:\mathbb{R}\rightarrow A\subset\mathbb{R}$. The set $\alpha(\mathbb{R})$ is denoted by $\mathbb{R}(N)$ or $\mathbb{R}_{\alpha}$ and is called non-Newtonian real line. For example, the identity function I generates classical arithmetic and the exponential function $\exp$ generates geometric arithmetic. So, the results obtained with respect to non-Newtonian calculus are stronger than those of classical calculus.

Each choice of specific isomorphisms for the generators α and β creates a $*-$calculus [5]. Geometric calculus obtained by choosing I instead of the generator α and $\exp$ instead of the generator β; that is, $\alpha(u)=u$ and $\beta(u)=e^u$ for all $u\in\mathbb{R}$ is one of the most popular $*-$calculi and has some attractive applications. In this situation, $\mathbb{R}_\alpha$ turns into $\mathbb{R}$ and $\mathbb{R}_\beta$ turns into $\mathbb{R}_{\exp}=\{e^u: u\in\mathbb{R}\}$. Geometric calculus has huge applications in problems related growth, price elasticity, economy and numerical approximations problems.

Many investigators have published some papers on extensions and generalizations in different ways on non-Newtonian calculus. In the literature, there are great contributions to non-Newtonian calculus and their applications some of which can be seen in Stanley [6], C´ordova-Lepe [7], Bashirov et al. [8,9], Uzer [10], Bashirov and Rıza [11], Mısırlı and Gurefe [12], Çakmak and Başar [13,14], Tekin and Başar [15], Kadak and Efe [16], Duyar et al. [17], Boruah and Hazarika [18,19,20], Güngör [21] etc.

Our first study on non-Newtonian bicomplex analysis is [22] in which we discussed non-Newtonian bicomplex numbers and non-Newtonian bicomplex versions of some well-known inequalities. Also, Sager and Sağır [23] constructed vector spaces $l_p(\mathbb{BC}(N))$ and showed that these vector spaces are Banach with the $*-$norm $\ddot{\|}.\ddot{\|}_{2,l_p(\mathbb{BC}(N))}$. Besides, in [24], we derived some elementary topological and geometric properties of $l_p(\mathbb{BC}(N))$.

Motivated by above studies, the focus of this study is giving complex and bicomplex numbers with respect to geometric calculus and examining some inequalities for such numbers. In Section 2, we give some required definitions and fundamental facts. In Section 3, we introduce the notion of a complex number with recpect to the geometric calculus and then obtain that the set of these numbers is a Banach space according to the norm $\|.\|_1^{GC}$. Also, we investigate some properties and inequalities by defining bicomplex numbers according to the geometric calculus.

2. Material and Method

Now, we briefly mention several known concepts on non-Newtonian calculus. The details can be found in [4,5,25]. Let α and β be arbitrarily determined generators which map the set $\mathbb{R}$ to A and B respectively and $*-$calculus also be the ordered pair of arithmetics ($\alpha-$arithmetic, $\beta-$arithmetic). We will use the following symbols and operations:

	$\alpha-$arithmetic	$\beta-$arithmetic
Realm	$A\left(=\mathbb{R}_\alpha=\mathbb{R}(N)_\alpha=\{\alpha(s): s\in\mathbb{R}\}\right)$	$B\left(=\mathbb{R}_\beta=\mathbb{R}(N)_\beta\right)$
Summation	$s\dot{+}t=\alpha\{\alpha^{-1}(s)+\alpha^{-1}(t)\}$	$\ddot{+}$
Subtraction	$s\dot{-}t=\alpha\{\alpha^{-1}(s)-\alpha^{-1}(t)\}$	$\ddot{-}$
Multiplication	$s\dot{\times}t=\alpha\{\alpha^{-1}(s)\times\alpha^{-1}(t)\}$	$\ddot{\times}$
Division	$s\dot{/}t=\frac{s}{t}\alpha=\alpha\left\{\frac{\alpha^{-1}(s)}{\alpha^{-1}(t)}\right\}\left(t\neq\dot{0}\right)$	$\ddot{/}$
Ordering	$s\dot{\leq}t\Leftrightarrow\alpha^{-1}(s)\leq\alpha^{-1}(t)$	$\ddot{\leq}$

$\alpha-$absolute value of $s\in\mathbb{R}(N)_\alpha$ is characterized by

$$|s|_{\alpha}=\alpha\left(\left|\alpha^{-1}(s)\right|\right)=\begin{cases} s & if \quad s\dot{>}\dot{0} \\ \dot{0} & if \quad s=\dot{0} \\ \dot{0}\dot{-}s & if \quad s\dot{<}\dot{0} \end{cases}.$$

If α and β are chosen as one of I and $\exp$, the following special calculuses are obtained.

Calculus	α (arguments)	β (values)
Classic	I	I
Geometric	I	$\exp$
Anageometric	$\exp$	I
Bigeometric	$\exp$	$\exp$

In geometric calculus, the operations in $\mathbb{R}_{\beta}=\mathbb{R}_{\exp}$ are as follows:

Geometric addition	$e^x +_{\exp} e^y = e^{\ln e^x+\ln e^y} = e^{x+y}$
Geometric subtraction	$e^x -_{\exp} e^y = e^{\ln e^x-\ln e^y} = e^{x-y}$
Geometric multiplicaiton	$e^x \times_{\exp} e^y = e^{\ln e^x.\ln e^y} = e^{xy}$
Geometric division $\left(y\neq e^0\right)$	$\dfrac{e^x}{e^y}\exp = e^{\frac{\ln e^x}{\ln e^y}} = e^{\frac{x}{y}}$
Geometric ordering	$e^x <_{\exp} e^y \Leftrightarrow \ln e^x < \ln e^y \Leftrightarrow x<y$

Also, $\exp-$ absolute value of a number $x\in\mathbb{R}_{\exp}$ is as follows:

$$|x|_{\exp}=e^{|\ln x|}=\begin{cases} x & if \quad x>_{\exp} e^0 \\ e^0 & if \quad x=e^0 \\ e^0 -_{\exp} x & if \quad x<_{\exp} e^0 \end{cases}=\begin{cases} x & if \quad x>1 \\ 1 & if \quad x=1. \\ \dfrac{1}{x} & if \quad x<1 \end{cases}$$

The isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic is the unique function ι (iota) and $\iota:A\rightarrow B$ has the following three properties:

1. ι is injective,
2. ι is surjective,
3. For all $s,t\in A$,

$$\iota(s\dot{+}t)=\iota(s)\ddot{+}\iota(t),$$
$$\iota(s\dot{-}t)=\iota(s)\ddot{-}\iota(t),$$
$$\iota(s\dot{\times}t)=\iota(s)\ddot{\times}\iota(t),$$
$$\iota(s\dot{/}t)=\iota(s)\ddot{/}\iota(t),\ t\neq\dot{0}$$
$$s\dot{<}t\Leftrightarrow\iota(s)\ddot{<}\iota(t).$$

It turns out that $\iota(s)=\beta\left\{\alpha^{-1}(s)\right\}$ for every number $s\in A$.

Based on the definitions above, the concept of a non-Newtonian complex number is defined by Tekin and Başar in [15] as follows:

Let $\dot{a}\in\left(A,\dot{+},\dot{-},\dot{\times},\dot{/},\dot{\leq}\right)$ and $\ddot{b}\in\left(B,\ddot{+},\ddot{-},\ddot{\times},\ddot{/},\ddot{\leq}\right)$ be arbitrarily chosen elements from corresponding arithmetics. Then, the ordered pair $\left(\dot{a},\ddot{b}\right)$ is called as a $*-$ complex numbers (non-Newtonian complex numbers) and is denoted by $\mathbb{C}^*$ or $\mathbb{C}(N)$.

In the rest of the study, when necessary we will use the abbreviations "w.r.t." and "w.r.t.g.c." for the statements "with respect to" and "with respect to the geometric calculus", respectively.

The set $\mathbb{C}(N)$ forms a field w.r.t. $\oplus_1$ and $\otimes_1$ for all $z_1^*=\left(\dot{a}_1,\ddot{b}_1\right)$, $z_2^*=\left(\dot{a}_2,\ddot{b}_2\right)\in\mathbb{C}(N)$ defined as

$$\oplus_1:\mathbb{C}(N)\times\mathbb{C}(N)\to\mathbb{C}(N),$$

$$\left(z_1^*,z_2^*\right)\to z_1^*\oplus_1 z_2^*=\left(\dot{a}_1,\ddot{b}_1\right)\oplus_1\left(\dot{a}_2,\ddot{b}_2\right)=\left(\dot{a}_1\dot{+}\dot{a}_2,\ddot{b}_1\ddot{+}\ddot{b}_2\right),$$

$$\otimes_1:\mathbb{C}(N)\times\mathbb{C}(N)\to\mathbb{C}(N),$$

$$\left(z_1^*,z_2^*\right)\to z_1^*\otimes_1 z_2^*=\left(\dot{a}_1,\ddot{b}_1\right)\otimes_1\left(\dot{a}_2,\ddot{b}_2\right)=\left(\alpha\left(a_1a_2-b_1b_2\right),\beta\left(a_1b_2+b_1a_2\right)\right).$$

On the other hand, a bicomplex number is defined as $z=z_1+jz_2$ where $j^2=-1$, $ij=ji$, z_1 and z_2 are complex numbers, and i and j are independent imaginary units. Also, the set of bicomplex numbers is denoted by $\mathbb{BC}$ and the set forms a Banach space with the following operations $+$, $\cdot$ and the norm $|.|$

$$z+w=\left(z_1+jz_2\right)+\left(w_1+jw_2\right)=\left(z_1+w_1\right)+j\left(z_2+w_2\right),$$

$$\lambda.z=\lambda.\left(z_1+jz_2\right)=\lambda z_1+j\lambda z_2,$$

$$|.|:\mathbb{BC}\to\mathbb{R},\ \ z\to|z|=\sqrt{|z_1|^2+|z_2|^2}$$

for all $z=z_1+jz_2, w=w_1+jw_2\in\mathbb{BC}$ and for all $\lambda\in\mathbb{R}$ [2].

In [22], we defined the concept of a non-Newtonian bicomplex number which forms the basis of this study, as follows:

Let $\dot{a},\dot{c}\in\left(A,\dot{+},\dot{-},\dot{\times},\dot{/},\dot{\leq}\right)$ and $\ddot{b},\ddot{d}\in\left(B,\ddot{+},\ddot{-},\ddot{\times},\ddot{/},\ddot{\leq}\right)$. Then, $\left(\dot{a},\ddot{b},\dot{c},\ddot{d}\right)$ is called as a $*-$bicomplex number (non-Newtonian bicomplex number). The set of these numbers is denoted by $\mathbb{BC}^*$ or $\mathbb{BC}(N)$; that is,

$$\mathbb{BC}(N)=\left\{\left(\dot{a},\ddot{b},\dot{c},\ddot{d}\right)\ :\ \dot{a},\dot{c}\in A\subseteq\mathbb{R},\ddot{b},\ddot{d}\in B\subseteq\mathbb{R}\right\}$$

$$=\left\{\left(z^*,w^*\right)\ :\ z^*=\left(\dot{a},\ddot{b}\right),w^*=\left(\dot{c},\ddot{d}\right),\dot{a},\dot{c}\in A\subseteq\mathbb{R},\ddot{b},\ddot{d}\in B\subseteq\mathbb{R}\right\}.$$

Also, $\mathbb{BC}(N)$ forms a vector space over the field $\mathbb{C}(N)$ and a ring w.r.t. the algebraic operations addition $\oplus_2$, multiplication $\otimes_2$ and scalar multiplication $\odot_2$ defined on $\mathbb{BC}(N)$ as follows:

$$\oplus_2:\mathbb{BC}(N)\times\mathbb{BC}(N)\to\mathbb{BC}(N),$$

$$\left(\zeta_1^*,\zeta_2^*\right)\to\zeta_1^*\oplus_2\zeta_2^*=\left(z_1^*,w_1^*\right)\oplus_2\left(z_2^*,w_2^*\right)=\left(z_1^*\oplus_1 z_2^*,w_1^*\oplus_1 w_2^*\right),$$

$$\otimes_2:\mathbb{BC}(N)\times\mathbb{BC}(N)\to\mathbb{BC}(N),$$

$$\left(\zeta_1^*,\zeta_2^*\right)\to\zeta_1^*\otimes_2\zeta_2^*=\left(z_1^*,w_1^*\right)\otimes_2\left(z_2^*,w_2^*\right)=\left(\left(z_1^*\otimes_1 z_2^*\right)\Theta_1\left(w_1^*\otimes_1 w_2^*\right),\left(z_1^*\otimes_1 w_2^*\right)\oplus_1\left(z_2^*\otimes_1 w_1^*\right)\right),$$

$$\odot_2:\mathbb{C}(N)\times\mathbb{BC}(N)\to\mathbb{BC}(N),$$

$$\left(z^*,\zeta_1^*\right)\to z^*\odot_2\zeta_1^*=z^*\odot_2\left(z_1^*,w_1^*\right)=\left(z^*\otimes_1 z_1^*,z^*\otimes_1 w_1^*\right)$$

where $\zeta_1^*=\left(z_1^*,w_1^*\right)$, $\zeta_2^*=\left(z_2^*,w_2^*\right)\in\mathbb{BC}(N)$ and $z^*\in\mathbb{C}(N)$.

3. Results

In this part, we obtain new concepts and results by rewriting some definitions and concepts given in the second part w.r.t.g.c.

3.1. Complex numbers with respect the geometric calculus

In this section, we examine the concept of a non-Newtonian complex number and some related properties given by Tekin and Başar w.r.t.g.c.

Definition 3.1.1. The ordered pair $\left(a, e^{b}\right) \in \mathbb{R} \times \mathbb{R}_{\exp}$ is called as a complex number w.r.t.g.c. The set of all complex numbers w.r.t.g.c. is denoted by $\mathbb{C}(GC)$ that is,

$$\mathbb{C}(GC)=\left\{\left(a, e^{b}\right): a, b \in \mathbb{R}\right\}.$$

Theorem 3.1.2. The set of complex numbers w.r.t.g.c. $\mathbb{C}(GC)$ forms a field w.r.t. addition $\oplus_{1,GC}$ and multiplication $\otimes_{1,GC}$ for all $z_1^{GC}=\left(a_1, e^{b_1}\right),\ z_2^{GC}=\left(a_2, e^{b_2}\right) \in \mathbb{C}(GC)$ defined as

$$\begin{aligned}
&\oplus_{1,GC}: \mathbb{C}(GC) \times \mathbb{C}(GC) \rightarrow \mathbb{C}(GC), \\
&\qquad \left(z_1^{GC}, z_2^{GC}\right) \rightarrow z_1^{GC} \oplus_{1,GC} z_2^{GC}=\left(a_1, e^{b_1}\right) \oplus_{1,GC}\left(a_2, e^{b_2}\right)=\left(a_1+a_2, e^{b_1+b_2}\right)
\end{aligned}$$

$$\begin{aligned}
&\otimes_{1,GC}: \mathbb{C}(GC) \times \mathbb{C}(GC) \rightarrow \mathbb{C}(GC), \\
&\qquad \left(z_1^{GC}, z_2^{GC}\right) \rightarrow z_1^{GC} \otimes_{1,GC} z_2^{GC}=\left(a_1, e^{b_1}\right) \otimes_{1,GC}\left(a_2, e^{b_2}\right)=\left(a_1 a_2-b_1 b_2, e^{a_1 b_2+b_1 a_2}\right).
\end{aligned}$$

Proof. The proof depends on definitions of of algebraic operations $\oplus_{1,GC}$ and $\otimes_{1,GC}$.

Definition 3.1.3. The distance d_1^{GC} between two elements $z_1^{GC}=\left(a_1, e^{b_1}\right),\ z_2^{GC}=\left(a_2, e^{b_2}\right) \in \mathbb{C}(GC)$ of the set $\mathbb{C}(GC)$ is defined by

$$\begin{aligned}
&d_1^{GC}: \mathbb{C}(GC) \times \mathbb{C}(GC) \rightarrow \mathbb{R}_{\exp}, \\
&\qquad \left(z_1^{GC}, z_2^{GC}\right) \rightarrow d_1^{GC}\left(z_1^{GC}, z_2^{GC}\right)=d_1^{GC}\left(\left(a_1, e^{b_1}\right),\left(a_2, e^{b_2}\right)\right)=e^{\sqrt{\left(a_1-a_2\right)^2+\left(b_1-b_2\right)^2}}.
\end{aligned}$$

Definition 3.1.4. The number $d_1^{GC}\left(z^{GC}, 0^{GC}\right)$ is called norm of $z^{GC}=\left(a, e^{b}\right)$, denoted by $\|.\|_1^{GC}$, that is,

$$\left\|z^{GC}\right\|_1^{GC}=d_1^{GC}\left(z^{GC}, 0^{GC}\right)=d_1^{GC}\left(\left(a, e^{b}\right),\left(0, e^{0}\right)\right)=e^{\sqrt{a^2+b^2}}.$$

Definition 3.1.5. A sequence $\left(s_n^{GC}\right)$ in $\mathbb{C}(GC)$ is a function defined by $s: \mathbb{N} \rightarrow \mathbb{C}(GC)$. This sequence is called a complex sequence w.r.t.g.c. It converges to a limit $s^{GC} \in \mathbb{C}(GC)$ w.r.t. the metric d_1^{GC} if and only if for every $\varepsilon >_{GC} e^{0}$ there is a $n_0 \in \mathbb{N}$ such that $d_1^{GC}\left(s_n^{GC}, s^{GC}\right) <_{GC} \varepsilon$ for all $n \geq n_0$. It is denoted by $\lim\limits_{n \rightarrow \infty}^{1,GC} s_n^{GC}=s^{GC}$. The sequence $\left(s_n^{GC}\right)$ is Cauchy w.r.t. d_1^{GC} if and only if for every $\varepsilon >_{GC} e^{0}$ there is a $n_0 \in \mathbb{N}$ such that $d_1^{GC}\left(s_n^{GC}, s_m^{GC}\right) <_{GC} \varepsilon$ for all $n, m \geq n_0$.

Theorem 3.1.6. Let $\left(s_n^{GC}\right)=\left(\left(a_n, e^{b_n}\right)\right)$ be a complex sequence w.r.t.g.c. and $s^{GC}=\left(a, e^{b}\right)$. Then, $\lim\limits_{n \rightarrow \infty}^{1,GC} s_n^{GC}=s^{GC}$ if and only if $\lim\limits_{n \rightarrow \infty} a_n=a$ and $\lim\limits_{n \rightarrow \infty} b_n=b$.

Proof. The proof follows from definitions of complex sequences w.r.t.g.c.

Definition 3.1.7. Let $\left(s_n^{GC}\right)$ be a complex sequence w.r.t.g.c. Then, the infinite sum

$$\underset{\oplus_{1,GC}}{\sum_{k=1}^{\infty}} s_k^{GC} = s_1^{GC} \oplus_{1,GC} s_2^{GC} \oplus_{1,GC} \ldots \oplus_{1,GC} s_n^{GC} \oplus_{1,GC} \ldots \tag{1}$$

is called a complex series w.r.t.g.c. Define the complex sequence w.r.t.g.c. $S:\mathbb{N} \to \mathbb{C}(GC),\ n \to S_n^{GC} = \underset{\oplus_{1,GC}}{\sum_{k=1}^{n}} s_k^{GC}$. The infinite series (1) converges to a limit $S^{GC} \in \mathbb{C}(GC)$ w.r.t. the metric d_1^{GC} if and only if $\left(S_n^{GC}\right)$ converges to $S^{GC} \in \mathbb{C}(GC)$ w.r.t. the metric d_1^{GC}. Then, S^{GC} is called the sum of bicomplex series w.r.t.g.c., and $\underset{\oplus_{1,GC}}{\sum_{k=1}^{\infty}} s_k^{GC} = S^{GC}$.

Theorem 3.1.8. Let $\left(s_n^{GC}\right) = \left(\left(a_n, e^{b_n}\right)\right)$ be a complex sequence w.r.t.g.c., $S^{GC} = \left(a, e^b\right) \in \mathbb{C}(GC)$. Then, $\underset{\oplus_{1,GC}}{\sum_{k=1}^{\infty}} s_k^{GC} = S^{GC}$ if and only if $\sum_{k=1}^{\infty} a_k = a$ and $\sum_{k=1}^{\infty} b_k = b$.

Proof. The proof of this theorem is directly seen from the definitions of convergence for complex series w.r.t.g.c.

Theorem 3.1.9. $\mathbb{C}(GC)$ is complete w.r.t. the metric d_1^{GC}.

Proof. Let $\left(s_n^{GC}\right) = \left(\left(a_n, e^{b_n}\right)\right)$ be a bicomplex Cauchy sequence w.r.t.g.c. Then, to each $\varepsilon >_{\exp} e^0$ there corresponds a natural number $n_0 \in \mathbb{N}$ such that $d_1^{GC}\left(s_n^{GC}, s_m^{GC}\right) <_{\exp} \varepsilon$ for all $n,m \geq n_0$. So, $d_1^{GC}\left(s_n^{GC}, s_m^{GC}\right) = e^{\sqrt{(a_n-a_m)^2+(b_n-b_m)^2}} <_{\exp} \varepsilon$. This implies that $\sqrt{(a_n-a_m)^2+(b_n-b_m)^2} < \ln\varepsilon$ and so $|a_n - a_m| < \ln\varepsilon$, $|b_n - b_m| < \ln\varepsilon$. Then, since (a_n) and (b_n) are Cauchy sequences in $\mathbb{R}$ and $\mathbb{R}$ is Banach, there exist $a,b \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. So, we obtain

$$d_1^{GC}\left(s_n^{GC}, s^{GC}\right) = e^{\sqrt{(a_n-a)^2+(b_n-b)^2}} \leq_{\exp} e^{\sqrt{(a_n-a)^2}+\sqrt{(b_n-b)^2}} = e^{|a_n-a_m|+|b_n-b_m|} < e^{\ln\varepsilon} = \varepsilon,$$

This means that $\lim_{n\to\infty} s_n^{GC} = s^{GC}$. Then, $\mathbb{C}(GC)$ is complete w.r.t. the metric d_1^{GC}.

Corollary 3.1.10. $\mathbb{C}(GC)$ is Banach w.r.t. the norm $\|.\|_1^{GC}$.

Proof. The proof is based on Theorem 3.1.9.

3.2. Bicomplex numbers with respect to geometric calculus and some inequalities

In this section, we consider the concept of a non-Newtonian bicomplex number w.r.t.g.c. and give some definitions about it. Also, we obtain that $\mathbb{BC}(GC)$ is a Banach space w.r.t. the norm $\|.\|_2^{GC}$ by proving some inequalities w.r.t. the norm $\|.\|_2^{GC}$.

Definition 3.2.1. The point $\left(a,e^{b},c,e^{d}\right)$ is called as a bicomplex number w.r.t.g.c. The set of all bicomplex numbers w.r.t.g.c. is denoted by $\mathbb{BC}(GC)$ that is,

$$\mathbb{BC}(GC)=\left\{\left(a,e^{b},c,e^{d}\right):a,b,c,d\in\mathbb{R}\right\}=\left\{\left(z^{GC},w^{GC}\right):z^{GC}=\left(a,e^{b}\right),w^{GC}=\left(c,e^{d}\right)\in\mathbb{C}(GC)\right\}.$$

Theorem 3.2.2. The set $\mathbb{BC}(GC)$ forms a vector space over the field $\mathbb{C}(GC)$ and a ring w.r.t. addition $\oplus_{2,GC}$, multiplication $\otimes_{2,GC}$ and scalar multiplication $\odot_{2,GC}$ for all $\zeta_1^{GC}=\left(z_1^{GC},w_1^{GC}\right)$, $\zeta_2^{GC}=\left(z_2^{GC},w_2^{GC}\right)\in\mathbb{BC}(GC)$ and $z^{GC}\in\mathbb{C}(GC)$ defined as

$$\oplus_{2,GC}:\mathbb{BC}(GC)\times\mathbb{BC}(GC)\to\mathbb{BC}(GC),$$

$$\left(\zeta_1^{GC},\zeta_2^{GC}\right)\to\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}=\left(z_1^{GC},w_1^{GC}\right)\oplus_{2,GC}\left(z_2^{GC},w_2^{GC}\right)=\left(z_1^{GC}\oplus_{1,GC}z_2^{GC},w_1^{GC}\oplus_{1,GC}w_2^{GC}\right)$$

$$\otimes_{2,GC}:\mathbb{BC}(GC)\times\mathbb{BC}(GC)\to\mathbb{BC}(GC),$$

$$\begin{aligned}\left(\zeta_1^{GC},\zeta_2^{GC}\right)\to\zeta_1^{GC}\otimes_{2,GC}\zeta_2^{GC}&=\left(z_1^{GC},w_1^{GC}\right)\otimes_{2,GC}\left(z_2^{GC},w_2^{GC}\right)\\&=\left(\left(z_1^{GC}\otimes_{1,GC}z_2^{GC}\right)\Theta_{1,GC}\left(w_1^{GC}\otimes_{1,GC}w_2^{GC}\right),\left(z_1^{GC}\otimes_{1,GC}w_2^{GC}\right)\oplus_{1,GC}\left(z_2^{GC}\otimes_{1,GC}w_1^{GC}\right)\right)\end{aligned}$$

$$\odot_{2,GC}:\mathbb{C}(GC)\times\mathbb{BC}(GC)\to\mathbb{BC}(GC),$$

$$\left(z^{GC},\zeta_1^{GC}\right)\to z^{GC}\odot_{2,GC}\zeta_1^{GC}=z^{GC}\odot_{2,GC}\left(z_1^{GC},w_1^{GC}\right)=\left(z^{GC}\otimes_{1,GC}z_1^{GC},z^{GC}\otimes_{1,GC}w_1^{GC}\right).$$

Proof. The proof of Theorem 3.2.2 follows from definitions of algebraic operations $\oplus_{2,GC}$, $\otimes_{2,GC}$ and $\odot_{2,GC}$.

Definition 3.2.3. The distance d_2^{GC} between two elements $\zeta_1^{GC}=\left(z_1^{GC},w_1^{GC}\right)$, $\zeta_2^{GC}=\left(z_2^{GC},w_2^{GC}\right)\in\mathbb{BC}(GC)$ is defined by

$$d_2^{GC}:\mathbb{BC}(GC)\times\mathbb{BC}(GC)\to\mathbb{R}_{\exp},$$

$$\left(\zeta_1^{GC},\zeta_2^{GC}\right)\to d_2^{GC}\left(\zeta_1^{GC},\zeta_2^{GC}\right)=e^{\sqrt{\ln\left[\left(\left\|z_1^{GC}\Theta_1 z_2^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_1^{GC}\Theta_1 w_2^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}}.$$

Theorem 3.2.4. The function d_2^{GC} is a metric on $\mathbb{BC}(GC)$.

Proof. It is clear that $d_2^{GC}\left(\zeta_1^{GC},\zeta_2^{GC}\right)\geq_{\exp}e^{0}$, $d_2^{GC}\left(\zeta_1^{GC},\zeta_2^{GC}\right)=d_2^{GC}\left(\zeta_2^{GC},\zeta_1^{GC}\right)$ for all ζ_1^{GC}, $\zeta_2^{GC}\in\mathbb{BC}(GC)$ and $d_2^{GC}\left(\zeta_1^{GC},\zeta_2^{GC}\right)=e^{0}\Leftrightarrow\zeta_1^{GC}=\zeta_2^{GC}$. On the other hand,

$$\begin{aligned}d_2^{GC}\left(\zeta_1^{GC},\zeta_2^{GC}\right)&=e^{\sqrt{\ln\left[\left(\left\|z_1^{GC}\Theta_{1,GC}z_2^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_1^{GC}\Theta_{1,GC}w_2^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}}\\
&=e^{\sqrt{\ln\left[\left(\left\|\left(z_1^{GC}\Theta_{1,GC}z_3^{GC}\right)\oplus_{1,GC}\left(z_3^{GC}\Theta_{1,GC}z_2^{GC}\right)\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|\left(w_1^{GC}\Theta_{1,GC}w_3^{GC}\right)\oplus_{1,GC}\left(w_3^{GC}\Theta_{1,GC}w_2^{GC}\right)\right\|_1^{GC}\right)^{2\exp}\right]}}\\
&\leq_{\exp}e^{\sqrt{\ln\left[\left(\left\|z_1^{GC}\Theta_{1,GC}z_3^{GC}\right\|_1^{GC}+_{\exp}\left\|z_3^{GC}\Theta_{1,GC}z_2^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_1^{GC}\Theta_{1,GC}w_3^{GC}\right\|_1^{GC}+_{\exp}\left\|w_3^{GC}\Theta_{1,GC}w_2^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}}\\
&\leq_{\exp}e^{\sqrt{\ln\left[\left(\left\|z_1^{GC}\Theta_{1,GC}z_3^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_1^{GC}\Theta_{1,GC}w_3^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}+\sqrt{\ln\left[\left(\left\|z_3^{GC}\Theta_{1,GC}z_2^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_3^{GC}\Theta_{1,GC}w_2^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}}\\
&=d_2^{GC}\left(\zeta_1^{GC},\zeta_3^{GC}\right)+_{\exp}d_2^{GC}\left(\zeta_3^{GC},\zeta_2^{GC}\right)\end{aligned}$$

for all $\zeta_1^{GC} = \left(z_1^{GC}, w_1^{GC}\right)$, $\zeta_2^{GC} = \left(z_2^{GC}, w_2^{GC}\right) \in \mathbb{BC}(GC)$. Then, d_2^{GC} is a metric on $\mathbb{BC}(GC)$.

Definition 3.2.5. A sequence $\left(s_n^{GC}\right)$ in $\mathbb{BC}(GC)$ is a function defined by $s : \mathbb{N} \to \mathbb{BC}(GC)$. This sequence is called a bicomplex sequence w.r.t.g.c. It is convergent to a limit $s^{GC} \in \mathbb{BC}(GC)$ w.r.t. the metric d_2^{GC} if and only if for every $\varepsilon >_{\exp} e^0$ there is a $n_0 \in \mathbb{N}$ such that $d_2^{GC}\left(s_n^{GC}, s^{GC}\right) <_{\exp} \varepsilon$ for all $n \geq n_0$. It is denoted by $\lim\limits_{n\to\infty}^{2,GC} s_n^{GC} = s^{GC}$. The sequence $\left(s_n^{GC}\right)$ is Cauchy w.r.t. d_2^{GC} if and only if for every $\varepsilon >_{\exp} e^0$ there is a $n_0 \in \mathbb{N}$ such that $d_2^{GC}\left(s_n^{GC}, s_m^{GC}\right) <_{\exp} \varepsilon$ for all $n, m \geq n_0$.

Theorem 3.2.6. Let $\left(s_n^{GC}\right) = \left(\left(z_n^{GC}, w_n^{GC}\right)\right)$ be a bicomplex sequence w.r.t.g.c. and $s^{GC} = \left(z^{GC}, w^{GC}\right)$. Then, $\lim\limits_{n\to\infty}^{2,GC} s_n^{GC} = s^{GC}$ if and only if $\lim\limits_{n\to\infty}^{1,GC} z_n^{GC} = z^{GC}$ and $\lim\limits_{n\to\infty}^{1,GC} w_n^{GC} = w^{GC}$.

Proof. The proof follows directly from the definitions of convergence for bicomplex sequences w.r.t.g.c.

Definition 3.2.7. Let $\left(s_n^{GC}\right)$ be a bicomplex sequence w.r.t.g.c. Then, the infinite sum

$$\underset{\oplus_{2,GC}}{\sum_{k=1}^{\infty}} s_k^{GC} = s_1^{GC} \oplus_{2,GC} s_2^{GC} \oplus_{2,GC} \ldots \oplus_{2,GC} s_n^{GC} \oplus_{2,GC} \cdots \tag{2}$$

is called a bicomplex series w.r.t.g.c. Define the bicomplex sequence w.r.t.g.c. $S : \mathbb{N} \to \mathbb{BC}(GC),\ n \to S_n^{GC} = \underset{\oplus_{2,GC}}{\sum_{k=1}^{n}} s_k^{GC}$. (2) converges to a limit $S^{GC} \in \mathbb{BC}(GC)$ w.r.t. the metric d_2^{GC} if and only if $\left(S_n^{GC}\right)$ converges to a limit $S^{GC} \in \mathbb{BC}(GC)$ w.r.t. the metric d_2^{GC}. Then, S^{GC} is called the sum of bicomplex series w.r.t.g.c., and we $\underset{\oplus_{2,GC}}{\sum_{k=1}^{\infty}} s_k^{GC} = S^{GC}$.

Theorem 3.2.8. Let $\left(s_n^{GC}\right) = \left(\left(z_n^{GC}, w_n^{GC}\right)\right)$ be a bicomplex sequence w.r.t.g.c., $S^{GC} = \left(z^{GC}, w^{GC}\right) \in \mathbb{BC}(GC)$. Then, $\underset{\oplus_{2,GC}}{\sum_{k=1}^{\infty}} s_k^{GC} = S^{GC}$ if and only if $\underset{\oplus_{1,GC}}{\sum_{k=1}^{\infty}} z_k^{GC} = z^{GC}$ and $\underset{\oplus_{1,GC}}{\sum_{k=1}^{\infty}} w_k^{GC} = w^{GC}$.

Proof. The proof depends on definitions of convergence of bicomplex series w.r.t.g.c.

Theorem 3.2.9. $\mathbb{BC}(GC)$ is complete w.r.t. the metric d_2^{GC}.

Proof. Let $\left(s_n^{GC}\right) = \left(\left(z_n^{GC}, w_n^{GC}\right)\right)$ be a bicomplex Cauchy sequence w.r.t.g.c. Then, for every $\varepsilon >_{\exp} e^0$ there is a $n_0 \in \mathbb{N}$ such that $d_2^{GC}\left(s_n^{GC}, s_m^{GC}\right) <_{\exp} \varepsilon$ for all $n, m \geq n_0$. So, $d_2^{GC}\left(s_n^{GC}, s_m^{GC}\right) = e^{\sqrt{\ln\left[\left(\left\|z_n^{GC}\Theta_{1,GC} z_m^{GC}\right\|_1^{GC}\right)^{2_{\exp}} +_{\exp}\left(\left\|w_n^{GC}\Theta_{1,GC} w_m^{GC}\right\|_1^{GC}\right)^{2_{\exp}}\right]}} <_{\exp} \varepsilon$. This implies that $\sqrt{\ln\left[\left(\left\|z_n^{GC}\Theta_{1,GC} z_m^{GC}\right\|_1^{GC}\right)^{2_{\exp}} +_{\exp}\left(\left\|w_n^{GC}\Theta_{1,GC} w_m^{GC}\right\|_1^{GC}\right)^{2_{\exp}}\right]} < \ln\varepsilon$ and so $\left\|z_n^{GC}\Theta_{1,GC} z_m^{GC}\right\|_1^{GC} < e^{\frac{\ln\varepsilon}{\sqrt{2}}}$, $\left\|w_n^{GC}\Theta_{1,GC} w_m^{GC}\right\|_1^{GC} < e^{\frac{\ln\varepsilon}{\sqrt{2}}}$. Then, $\left(z_n^{GC}\right)$ and $\left(w_n^{GC}\right)$ are Cauchy sequences w.r.t. the norm $\|.\|_1^{GC}$. Since $\mathbb{C}(GC)$

is Banach, there exist $z^{GC}, w^{GC} \in \mathbb{C}(GC)$ such that $\lim_{n\to\infty}^{1,GC} z_n^{GC} = z^{GC}$ and $\lim_{n\to\infty}^{1,GC} w_n^{GC} = w^{GC}$. This implies that to each $\varepsilon >_{GC} e^0$ there corresponds a natural number $n_1 \in \mathbb{N}$ such that $d_1^{GC}\left(z_n^{GC}, z^{GC}\right) <_{GC} e^{\frac{\ln\varepsilon}{\sqrt{2}}}$ for all $n \geq n_1$ and there corresponds a natural number $n_2 \in \mathbb{N}$ such that $d_1^{GC}\left(w_n^{GC}, w^{GC}\right) <_{GC} e^{\frac{\ln\varepsilon}{\sqrt{2}}}$ for all $n \geq n_2$.

Then, we obtain

$$d_2^{GC}\left(s_n^{GC}, s^{GC}\right) = e^{\sqrt{\ln\left[\left(\left\|z_n^{GC}\Theta_{1,GC}z^{GC}\right\|_1^{GC}\right)^{2\exp} +_{\exp}\left(\left\|w_n^{GC}\Theta_{1,GC}w^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}} <_{\exp} e^{\ln\varepsilon} = \varepsilon$$

for all $n \geq n_0 = \max\{n_1, n_2\}$. This means that $\lim_{n\to\infty}^{GC} s_n^{GC} = s^{GC}$. Then, $\mathbb{BC}(GC)$ is complete w.r.t. the metric d_2^{GC}.

Definition 3.2.10. The number $d_2^{GC}\left(\zeta^{GC}, 0^{GC}\right)$ is called norm of $\zeta^{GC} = \left(z^{GC}, w^{GC}\right)$, denoted by $\|.\|_2^{GC}$, that is,

$$\left\|\zeta^{GC}\right\|_2^{GC} = d_2^{GC}\left(\zeta^{GC}, 0^{GC}\right) = d_2^{GC}\left(\left(z^{GC}, w^{GC}\right), \left(0, e^0, 0, e^0\right)\right) = e^{\sqrt{\ln\left[\left(\left\|z^{GC}\right\|_1^{GC}\right)^{2\exp} +_{\exp}\left(\left\|w^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}}.$$

Corollary 3.2.11. $\mathbb{BC}(GC)$ is Banach w.r.t. the norm $\|.\|_2^{GC}$.

Proof. The proof is easily derived from Theorem 3.2.9.

Lemma 3.2.12. The following inequalities are satisfied:

i. $\left\|\zeta_1^{GC}\Theta_{2,GC}\zeta_2^{GC}\right\|_2^{GC} \leq_{\exp} \left\|\zeta_1^{GC}\right\|_2^{GC} +_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}$ for all $\zeta_1^{GC}, \zeta_2^{GC} \in \mathbb{BC}(GC)$.

ii. $\left|\left\|\zeta_1^{GC}\right\|_2^{GC} -_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}\right|_{\exp} \leq_{\exp} \left\|\zeta_1^{GC}\Theta_{2,GC}\zeta_2^{GC}\right\|_2^{GC}$ for all $\zeta_1^{GC}, \zeta_2^{GC} \in \mathbb{BC}(GC)$.

iii. $\left|\left\|\zeta_1^{GC}\right\|_2^{GC} -_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}\right|_{\exp} \leq_{\exp} \left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC}$ for all $\zeta_1^{GC}, \zeta_2^{GC} \in \mathbb{BC}(GC)$.

iv. $\dfrac{\left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC}}{1_{\exp} +_{\exp} \left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC}}\exp \leq_{\exp} \dfrac{\left\|\zeta_1^{GC}\right\|_2^{GC}}{1_{\exp} +_{\exp} \left\|\zeta_1^{GC}\right\|_2^{GC}}\exp +_{\exp} \dfrac{\left\|\zeta_2^{GC}\right\|_2^{GC}}{1_{\exp} +_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}}\exp$ for all $\zeta_1^{GC}, \zeta_2^{GC} \in \mathbb{BC}(GC)$.

v. $\left(\sum_{k=1}^{n}{}_{\exp}\left(\left\|s_k^{GC}\oplus_{2,GC}t_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\frac{1_{\exp}}{p}\exp} \leq_{\exp} \left(\sum_{k=1}^{n}{}_{\exp}\left(\left\|s_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\frac{1_{\exp}}{p}\exp} +_{\exp} \left(\sum_{k=1}^{n}{}_{\exp}\left(\left\|t_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\frac{1_{\exp}}{p}\exp}$ for $p \in \mathbb{R}_{\exp}$ with $1_{\exp} <_{\exp} p <_{\exp} \infty_{\exp} = \lim_{x\to\infty} e^x = \infty$ and $s_k^{GC}, t_k^{GC} \in \mathbb{BC}(GC)$ where $k \in \{1, 2, \ldots, n\}$.
(Minkowski's inequality in $\mathbb{BC}(GC)$ with respect to $\|.\|_2^{GC}$)

Proof.

i. Let $\zeta_1^{GC} = \left(z_1^{GC}, w_1^{GC}\right)$, $\zeta_2^{GC} = \left(z_2^{GC}, w_2^{GC}\right) \in \mathbb{BC}(GC)$. Then, we have

$$
\begin{aligned}
\left\|\zeta_1^{GC}\Theta_{2,GC}\zeta_2^{GC}\right\|_2^{GC} &= e^{\sqrt{\ln\left[\left(\left\|z_1^{GC}\Theta_{1,GC}z_2^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_1^{GC}\Theta_{1,GC}w_2^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}} \\
&\leq_{\exp} e^{\sqrt{\ln\left[\left(\left\|z_1^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|z_2^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_1^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_2^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}} \\
&\leq_{\exp} e^{\sqrt{\ln\left[\left(\left\|z_1^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_1^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}} +_{\exp} e^{\sqrt{\ln\left[\left(\left\|z_2^{GC}\right\|_1^{GC}\right)^{2\exp}+_{\exp}\left(\left\|w_2^{GC}\right\|_1^{GC}\right)^{2\exp}\right]}} \\
&\leq_{\exp} \left\|\zeta_1^{GC}\right\|_2^{GC} +_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}.
\end{aligned}
$$

This completes the proof.

ii. Since

$$
\begin{aligned}
\left\|\zeta_1^{GC}\right\|_2^{GC} &= \left\|\zeta_1^{GC}\oplus_{2,GC}\left(\zeta_2^{GC}\Theta_{2,GC}\zeta_2^{GC}\right)\right\|_2^{GC} \\
&= \left\|\left(\zeta_1^{GC}\Theta_{2,GC}\zeta_2^{GC}\right)\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC} \\
&\leq_{\exp} \left\|\zeta_1^{GC}\Theta_{2,GC}\zeta_2^{GC}\right\|_2^{GC} +_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}
\end{aligned}
$$

we have

$$\left\|\zeta_1^{GC}\right\|_2^{GC} -_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC} \leq_{\exp} \left\|\zeta_1^{GC}\Theta_{2,GC}\zeta_2^{GC}\right\|_2^{GC}$$

and similarly

$$-_{\exp}\left\|\zeta_1^{GC}\Theta_{2,GC}\zeta_2^{GC}\right\|_2^{GC} \leq_{\exp} \left\|\zeta_1^{GC}\right\|_2^{GC} -_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}.$$

This implies that $\left|\left\|\zeta_1^{GC}\right\|_2^{GC} -_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}\right|_{\exp} \leq_{\exp} \left\|\zeta_1^{GC}\Theta_{2,GC}\zeta_2^{GC}\right\|_2^{GC}$ for all $\zeta_1^{GC}, \zeta_2^{GC} \in \mathbb{BC}(GC)$.

iii. Since

$$
\begin{aligned}
\left\|\zeta_1^{GC}\right\|_2^{GC} &= \left\|\zeta_1^{GC}\oplus_{2,GC}\left(\zeta_2^{GC}\Theta_{2,GC}\zeta_2^{GC}\right)\right\|_2^{GC} \\
&= \left\|\left(\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right)\Theta_{2,GC}\zeta_2^{GC}\right\|_2^{GC} \\
&\leq_{\exp} \left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC} +_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}
\end{aligned}
$$

from (i), we have

$$\left\|\zeta_1^{GC}\right\|_2^{GC} -_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC} \leq_{\exp} \left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC}$$

and similarly

$$-_{\exp}\left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC} \leq_{\exp} \left\|\zeta_1^{GC}\right\|_2^{GC} -_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}.$$

This implies that $\left|\left\|\zeta_1^{GC}\right\|_2^{GC} -_{\exp} \left\|\zeta_2^{GC}\right\|_2^{GC}\right|_{\exp} \leq_{\exp} \left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC}$ for all $\zeta_1^{GC}, \zeta_2^{GC} \in \mathbb{BC}(GC)$.

iv. We have

$$\frac{\left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC}}{1_{\exp}+_{\exp}\left\|\zeta_1^{GC}\oplus_{2,GC}\zeta_2^{GC}\right\|_2^{GC}}\exp=\frac{\left\|\left(a_1,e^{b_1},c_1,e^{d_1}\right)\oplus_{2,GC}\left(a_2,e^{b_2},c_2,e^{d_2}\right)\right\|_2^{GC}}{1_{\exp}+_{\exp}\left\|\left(a_1,e^{b_1},c_1,e^{d_1}\right)\oplus_{2,GC}\left(a_2,e^{b_2},c_2,e^{d_2}\right)\right\|_2^{GC}}\exp$$

$$=\frac{e^{\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}}}{1_{\exp}+_{\exp}e^{\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}}}\exp$$

$$=e^{\ln e^{\frac{\ln e^{\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}}}{\left[\ln 1_{\exp}+\ln e^{\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}}\right]}}}$$

$$=e^{\frac{\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}}{1+\sqrt{(a_1+a_2)^2+(b_1+b_2)^2+(c_1+c_2)^2+(d_1+d_2)^2}}}$$

$$\leq_{\exp}e^{\left[\frac{\sqrt{a_1^2+b_1^2+c_1^2+d_1^2}}{1+\sqrt{a_1^2+b_1^2+c_1^2+d_1^2}}+\frac{\sqrt{a_2^2+b_2^2+c_2^2+d_2^2}}{1+\sqrt{a_2^2+b_2^2+c_2^2+d_2^2}}\right]}$$

$$=e^{\left[\frac{\ln e^{\sqrt{a_1^2+b_1^2+c_1^2+d_1^2}}}{\ln 1_{\exp}+\ln e^{\sqrt{a_1^2+b_1^2+c_1^2+d_1^2}}}+\frac{\ln e^{\sqrt{a_2^2+b_2^2+c_2^2+d_2^2}}}{\ln 1_{\exp}+\ln e^{\sqrt{a_2^2+b_2^2+c_2^2+d_2^2}}}\right]}$$

$$=e^{\left[\frac{\ln\left\|\zeta_1^{GC}\right\|_2^{GC}}{\ln 1_{\exp}+\ln\left\|\zeta_1^{GC}\right\|_2^{GC}}+\frac{\ln\left\|\zeta_2^{GC}\right\|_2^{GC}}{\ln 1_{\exp}+\ln\left\|\zeta_2^{GC}\right\|_2^{GC}}\right]}$$

$$=e^{\left[\frac{\ln\left\|\zeta_1^{GC}\right\|_2^{GC}}{\ln e^{\left(\ln 1_{\exp}+\ln\left\|\zeta_1^{GC}\right\|_2^{GC}\right)}}+\frac{\ln\left\|\zeta_2^{GC}\right\|_2^{GC}}{\ln e^{\left(\ln 1_{\exp}+\ln\left\|\zeta_2^{GC}\right\|_2^{GC}\right)}}\right]}$$

$$=e^{\ln e^{\frac{\ln\left\|\zeta_1^{GC}\right\|_2^{GC}}{\ln\left(1_{\exp}+_{\exp}\left\|\zeta_1^{GC}\right\|_2^{GC}\right)}}+\ln e^{\frac{\ln\left\|\zeta_2^{GC}\right\|_2^{GC}}{\ln\left(1_{\exp}+_{\exp}\left\|\zeta_2^{GC}\right\|_2^{GC}\right)}}}$$

$$=e^{\left[\ln\frac{\left\|\zeta_1^{GC}\right\|_2^{GC}}{\left(1_{\exp}+_{\exp}\left\|\zeta_1^{GC}\right\|_2^{GC}\right)}\exp+\ln\frac{\left\|\zeta_2^{GC}\right\|_2^{GC}}{\left(1_{\exp}+_{\exp}\left\|\zeta_2^{GC}\right\|_2^{GC}\right)}\exp\right]}$$

$$=\frac{\left\|\zeta_1^{GC}\right\|_2^{GC}}{1_{\exp}+_{\exp}\left\|\zeta_1^{GC}\right\|_2^{GC}}\exp+_{\exp}\frac{\left\|\zeta_2^{GC}\right\|_2^{GC}}{1_{\exp}+_{\exp}\left\|\zeta_2^{GC}\right\|_2^{GC}}\exp.$$

This completes the proof.

v. Since

$$\,_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC}t_k^{GC}\right\|_2^{GC}\right)^p=\,_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC}t_k^{GC}\right\|_2^{GC}\right)^{p-_{\exp}1_{\exp}}\times_{\exp}\left\|s_k^{GC}\oplus_{2,GC}t_k^{GC}\right\|_2^{GC}$$

$$\leq_{\exp}\,_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC}t_k^{GC}\right\|_2^{GC}\right)^{p-_{\exp}1_{\exp}}\times_{\exp}\left(\left\|s_k^{GC}\right\|_2^{GC}+_{\exp}\left\|t_k^{GC}\right\|_2^{GC}\right)$$

$$= {}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{p-_{\exp}1_{\exp}} \times_{\exp} \left\|s_k^{GC}\right\|_2^{GC} +_{\exp} {}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{p-_{\exp}1_{\exp}} \times_{\exp}\left\|t_k^{GC}\right\|_2^{GC},$$

if we choose $q=\left(\frac{p}{p-_{\exp}1_{\exp}}\right)_{\exp}$, we obtain the inequalities

$${}_{\exp}\sum_{k=1}^{n}\left\|s_k^{GC}\right\|_2^{GC}\times_{\exp}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{p-_{\exp}1_{\exp}} \leq_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}\times_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{\left(p-_{\exp}e^{1}\right)\times_{\exp}q}\right)^{\left(\frac{1_{\exp}}{q}\right)_{\exp}}$$

and

$${}_{\exp}\sum_{k=1}^{n}\left\|t_k^{GC}\right\|_2^{GC}\times_{\exp}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{p-_{\exp}1_{\exp}} \leq_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|t_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}\times_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{\left(p-_{\exp}1_{\exp}\right)\times_{\exp}q}\right)^{\left(\frac{1_{\exp}}{q}\right)_{\exp}}.$$

It follows that

$$\begin{aligned}
&{}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{p}\\
&\leq_{\exp}\left[\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}+_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|t_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}\right]\times_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{\left(p-_{\exp}1_{\exp}\right)\times_{\exp}q}\right)^{\left(\frac{1_{\exp}}{q}\right)_{\exp}}\\
&=\left[\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}+_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|t_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}\right]\times_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{q}\right)_{\exp}}
\end{aligned}$$

and so

$$\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\oplus_{2,GC} t_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}\leq_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|s_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}+_{\exp}\left({}_{\exp}\sum_{k=1}^{n}\left(\left\|t_k^{GC}\right\|_2^{GC}\right)^{p}\right)^{\left(\frac{1_{\exp}}{p}\right)_{\exp}}.$$

4. Discussion and Conclusion

In the present study, inspired by the ideas of non-Newtonian bicomplex numbers and geometric calculus, we give bicomplex numbers with respect to the geometric calculus, and we state and prove some inequalities for use in future studies. Also, our findings carry some concepts and results from the recent literature to $\mathbb{BC}(GC)$.

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