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Obtaining the Parametric Equation of the Curve of the Sun's Apparent Movement by Using Quaternions

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Abstract

This article aims to express the daily and yearly apparent movement of the Sun in the same curve by using quaternions as a rotation operator. To achieve this, the daily and yearly apparent movement of the Sun, the algebraical structure of quaternions, and how quaternions work as rotation operators have been examined. For each of the apparent movements of the Sun, a quaternion that will work as a rotation operator has been determined. Afterward, these two rotation operators have been applied to the vector that is found between point $(0,0,0)$ and the accepted starting point of the apparent movement of the Sun. As a result, a curve on a sphere is obtained. The importance of this study is to emphasize the use of quaternions in other areas of study and to provide the science of astronomy with a new outlook with regards to expressing the apparent movement of the Sun.

1. Introduction

Astronomy is considered the oldest science in the world. Humankind has always observed the stars in the sky and especially the Sun. At the end of these observations, it was noticed that the daily and yearly movement of the Sun followed a certain cycle. By observing the Sun's movement in the sky the formation of the night-day and the seasons was noted.

For thousands of years, mankind accepted that Earth was the center of the universe and believed that the Sun, like all other celestial bodies rotated around the Earth. However, Copernicus proved that this belief was not accurate because it was the Earth that rotated around the Sun [1]. After this discovery, the expression of "the Sun's movement" was replaced with the expression of "the Sun's apparent movement". Even though the daily and yearly apparent movement of the Sun occurs at the same time, in calculation these movements are considered separable. The two main reasons for why these movements are considered separable are: firstly, the dyad Earth-Sun is not alone in the solar system which means that the problem does not remain limited to the two-body problem. Secondly, the difference between the periods of the daily and yearly movement is too big.

Showing the daily and yearly apparent movement of the Sun in the same curve is important in helping understand these movements, especially for young astronomers. At the same time, there exist situations in which great precision is not required but where nonetheless finding these two movements in the same curve would be useful. In many areas, such as using solar panels, planning agricultural activities, and in determining prayer time, doing the calculation of this curve would bring many benefits.

In our time astronomy problems that have in their base periodical repetition of the movement find a solution by using spherical trigonometry and Kepler's Laws [2]. Solving this problem by using the rotation matrix is theoretically possible from the mathematical perspective, however, using this method is considerably difficult. Therefore, the question arises, is it possible to obtain a faster mathematical approach to calculate the apparent movement of the Sun that would take the place of the rotation matrices or the long calculations of Kepler's equations? There are some studies done in this direction in the relevant literature. In 1996, M. Kummer proved that one can obtain the orbit's parameters by solving Kepler's equations with the Hamilton systems [3]. This study, on the other hand, has researched whether there can be easier and faster solutions done by using quaternions and the conclusion has been that quaternions can indeed be used in analyzing the apparent movement of the Sun.

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To understand and present the problem the author has benefited from the references [1]- [3] and [5] - [15]. The details about the quaternions can be viewed from the references [4] and [16] - [22]. The information needed for the other calculations is found in the references [23] - [24].

2. Notations and Preliminaries

2.1. Quaternion algebra

The quaternion, a hyper-complex number of rank 4, was invented by Hamilton. The most important rule of this invention is:

$$i^2 = j^2 = k^2 = ijk = -1$$

i, j and k are the components of the vector part of the quaternion.
Henceforth the quaternions will be denoted with the letters q, p or r. i, j and k will be used to represent the standard ortogonal base of $\mathbb{R}^3$. Accordingly:

$$i = (1,0,0)\,,\ j = (0,1,0)\,,\ k = (0,0,1)$$

The quaternion, from the Latin kuattur meaning four, can be thought of as a quadruplet of the real numbers. This makes it an element of $\mathbb{R}^4$. Accordingly, quaternion q can be expressed as below where q_0, q_1, q_2, q_3 are each a real number

$$q = (q_0, q_1, q_2, q_3)$$

or the quaternion q is accordingly:

$$\alpha = iq_1 + jq_2 + kq_3$$
$$q = q_0 + \alpha = q_0 + iq_1 + jq_2 + kq_3$$

where q_0 is the scalar part and α is the vector part. Throughout the article, q will be displayed with $q = q_0 + \alpha$.
Some algebraic properties of the quaternions are given as follows:

$$q + p = (q_0 + p_0) + i(q_1 + p_1) + j(q_2 + p_2) + k(q_3 + p_3)$$
$$aq = aq_0 + iaq_1 + jaq_2 + kaq_3 \quad , \quad a \in \mathbb{R}$$

Multiplication of quaternions is done according to the following rule

$$i^2 = j^2 = k^2 = ijk = -1 \text{ and } ij = k = -ij, jk = i = -kj, ki = j = -ij$$

for $p = p_0 + \alpha_p = p_0 + ip_1 + jp_2 + kp_3$ and $q = q_0 + \alpha_q = q_0 + iq_1 + jq_2 + kq_3$

$$\begin{aligned} p \times q &= (p_0 + ip_1 + jp_2 + kp_3) \times (q_0 + iq_1 + jq_2 + kp_3) \\ &= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0(iq_1 + jq_2 + kq_3) + q_0(p_0 + ip_1 + jp_2 + kp_3) \\ &+ i(p_2q_3 - p_3q_2) + j(p_3q_1 - p_1q_3) + k(p_1q_2 - p_2q_1) \\ &= p_0q_0 - \langle \alpha_p, \alpha_q \rangle + p_0\alpha_q + q_0\alpha_p + \alpha_p \wedge \alpha_q \end{aligned}$$

"$\langle\ ,\ \rangle$" represents the scalar product of vectors and "$\wedge$" represents the cross-produc of vectors.
Let q be a quaternion $q = q_0 + iq_1 + jq_2 + kq_3$ then q's complex conjugent is:

$$q^* = q_0 - iq_1 - jq_2 - kq_3$$

Finally, we can state that the set of quaternions together with the addition and multiplication operation satisfies the properties of a field except that multiplication is not commutative. Before quaternions are expressed as a rotation operator the definition of pure quaternions will be given.

Definition 2.1. *The quaternion whose scalar part is zero is called a pure quaternion.*

According to the definition above, the set of pure quaternions is one-to-one correspondent with the $v \in \mathbb{R}^3$ vector set. It can be shown that for any $v \in \mathbb{R}^3$ and for whichever $q \in \mathbb{R}^4$, there can be found $w_1 = q \times v \times q^*$ vector $w_1 \in \mathbb{R}^3$ and $w_2 = q^* \times v \times q$ vector $w_2 \in \mathbb{R}^3$.
The unit quaternion $q = q_0 + \alpha$ satisfies the following equality $q_0^2 + |\alpha|^2 = 1$. It is known that for whichever φ angle $\cos^2\varphi + \sin^2\varphi = 1$. In this case, a φ angle which would make possible the equations below can be found:

$$\cos^2\varphi = q_0^2 \text{ and } \sin^2\varphi = |\alpha|^2.$$

If we select the φ angle in $-\pi < \varphi \leq \pi$, this angle will simultaneously have a singular value. In light of this data, the quaternion that will be used as a rotation operator is:

$$q = q_0 + \alpha = \cos\varphi + u\sin\varphi \text{ and } q^* = q_0 - \alpha = \cos\varphi - u\sin\varphi$$

where

$$u = \frac{\alpha}{|\alpha|} = \frac{\alpha}{\sin\varphi}.$$

Theorem 2.2. *For any $Q = Q_0 + \mathbf{Q} = \cos\varphi + u\sin\varphi$ unit quaternion (where Q_0 is the scalar part and $\mathbf{Q}$ is the vector part of the quaternion) and for any vector $v \in \mathbb{R}^3$ the action of the operator*

$$L_Q(v) = Q \times v \times Q^*$$

on v may be interpreted geometrically as a rotation of the vector v through an angle 2φ about $\mathbf{Q}$ as the axis of the rotation, [4].

In addition: the action of the operator $L_Q(v) = Q^* \times v \times Q$ on v may be interpreted geometrically as a rotation of the vector v through an angle 2φ in a negative direction about $\mathbf{Q}$ as the axis of the rotation.

Theorem 2.3. *Suppose that k and r are unit quaternions that define the quaternion rotation operators:*

$$L_k(u) = k \times u \times k^* \text{ and } L_r(v) = r \times v \times r^*.$$

Then the quaternion product $r \times k$ defines a quaternion operator L_{rk} which represents a sequence of operators, L_k followed by L_r. The axis and the angels of rotation are those represented by the quaternion product, $q = r \times k$ [4].

In this study, two methods will be used to solve the problem. The first method will benefit from the characteristic of quaternions used as rotation operators. The second method will use the rotation matrix, which is a product of the unit quaternion. This matrix is as below:
for $Q = q_0 + iq_1 + jq_2 + kq_3$ unit quaternion, the rotation matrix D_Q is shown below [4].

$$D_Q = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix} \tag{2.1}$$

and let $\beta = (\beta_1, \beta_2, \beta_3)$ be the vector that is obtained by the rotation of vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ then:

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}. \tag{2.2}$$

2.2. The Sun's daily and yearly apparent movement

2.2.1. The Sun's daily apparent movement

The Earth rotates around its axis in a positive direction every day, so from the west to the east. Because the movement of the Earth cannot be felt, it is perceived instead that it's the other celestial bodies that rotate from the east to the west around the axis of the celestial sphere which in itself is the lengthening of the axis of the Earth. Among these celestial bodies, there is the Sun. So it can be said that the Sun in appearance moves every day in the negative direction in the celestial sphere. This movement occurs with a particular velocity in an orbit parallel to the celestial equator plane. The celestial equator plane is the lengthening of the Earth's equator plane [5].

2.2.2. The Sun's yearly apparent movement

The Earth orbits around the Sun in an elliptical orbit and a positive direction, in the elliptical plane throughout the year. However, in appearance, it is the Sun that orbits around the Earth in the same plane and a positive direction. The angle between the elliptical plane and the equatorial plane is $23^0 27'$. This plane forms a $23^0 27'$ angle with the plane of the celestial equator. If in the center of the celestial system instead of the Sun we placed the Earth and then drew the apparent elliptical orbit of the Sun,the orbit in Figure 2.1 would be obtained. To obtain this orbit the Earth will be imagined as fixed and the Sun as the body that rotates around it. Because the Earth's orbit is well-known the Earth will be fixed in what will be called point A henceforth which is found in its orbit. When the Earth is on day 21 March at the Y_1 point the Sun appears in the direction of Aries. If we transfer point Y_1 to point A and find point G_1 for which $AG_1 = Y_1G$ and AG_1 is parallel to Y_1G, it would mean that the Sun would appear at point G_1 at this date. In the same manner, if P_1G to AP_2, Y_2G to AG_2, Y_3G to AG_3, and Y_4G to AG_4 are transferred a new ellipse is formed which has at its center point A. This is the Sun's yearly apparent elliptical orbit. Every year the Sun moves in this elliptical orbit. Below are five important points that concern this orbit [5].

1. Both orbits are found in the same plane and this plane is the elliptical plane.
2. The Earth is found in one of the focal points of the apparent elliptic orbit.
3. These two ellipses are equal in shape and size.
4. The rotation period is the same in both and it is a one-star year long.
5. Both rotations are in the positive direction.

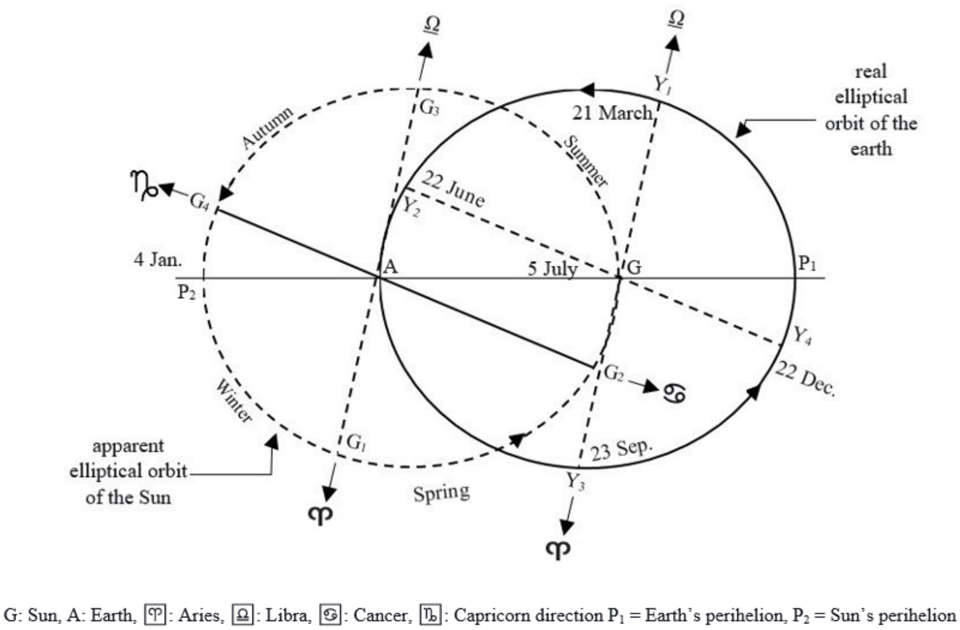

Figure 2.1: The Earth's orbit and the Sun's apparent orbit

3. Obtaining the Parametric Equation of the Curve of Both the Daily and Yearly Apparent Movement the Sun Makes in the Celestial Sphere by Using Quaternions

In this paper, it is assumed the apparent movement of the Sun occurs in ideal conditions. This means that the Earth will rotate around the Sun with a constant angular velocity (this velocity will be accepted as equal to the yearly average angular velocity of the Earth around the Sun) and it will be accepted that the orbit of rotation will be circular instead of elliptic. So, it will be accepted that the apparent movement of the Sun in the ecliptic plane will occur in a circular orbit with a constant angular velocity.
Firstly, it is necessary to define the problem in physical terms.
Let us accept that a celestial body completes a circular motion in plane E that intersects with plane XY in axis x and forms with it an ε angle. Let us also accept that this movement starts from point $P=(1,0,0)$ in a positive direction, and under force, F_1 completes a circular movement with a constant angular velocity w_1. Lastly, let us also accept that a force $F_2 = c\,F_1$, $c>2$ (there is a linear relationship between the scalar magnitude of the forces), forces the same celestial body to move parallel to plane XY in a positive direction with a constant angular velocity w_2. In this case, the celestial body whose vectors are linear independent is under the effect of two forces and is bound to both velocities. This body, however, will not move parallel to either plane XY or plane E instead it will move with the unified velocity in a different direction. How can we express the celestial body's interaction with the velocities w_1 and w_2?
Between the scalar magnitudes of w_1 and w_2 velocities, a linear relation is found. This linear relation will be the same as the linear relation between the scalar magnitudes of F_1 and F_2. In the same manner, the θ and φ angles these angular velocities trace in the same unit of time will also have the same linear relationship between their magnitudes. So $\varphi = c\theta$ because the forces are directly proportional to the angular velocities and the angular velocities are directly proportional to the angles they trace. To conclude, the curve that this celestial body traces on the sphere is a product of two rotations. One of the rotations will be in a positive direction around the axis of the plane E (let this axis be called N) and the other will be in a positive direction around axis Z.
Let plane E represent the elliptic plane while plane XY represents the plane of the celestial equator and angle ε represents the angle $\varepsilon = 23^0 27'$ which is the angle that is formed from the intersection of the celestial equatorial plane and the ecliptic plane (Figure 3.1). In this case, point $(0,0,0)$ represents the Earth. In addition, the positive direction of axis X will represent the Aries constellation. The direction of the vector $(0,-\cos\varepsilon,-\sin\varepsilon)$ will represent the Capricorn constellation. The direction of the vector $(0,\cos\varepsilon,\sin\varepsilon)$ will represent the Cancer constellation. The negative direction of axis X will represent the Libra constellation.
Now let us show the daily apparent movement of the Sun. This movement occurs in a negative direction parallel to the celestial equatorial plane. In this case, the second rotation movement in the negative direction of the celestial body that was presented in the problem above represents the movement of the daily apparent movement of the Sun.
Finally, above, it was stated that between the scalar magnitudes of w_1 (if we adapt w_1 to the velocity of the Sun this corresponds with the velocity of the Sun's movement in the elliptical plane) and w_2 (if we adapt w_2 to the velocity of the Sun this corresponds with the velocity of the movement the Sun makes parallel to the celestial equatorial plane) exists a linear relation. The same linear relation exists between the angles these velocities trace. In this case; because $w_2 = 365,25\,w_1$ (when the Sun rotates once around the ecliptic axis it rotates $365,25$ times parallel to the celestial equatorial plane) $\varphi = 365,25\theta$. So, $c = 365,25$.

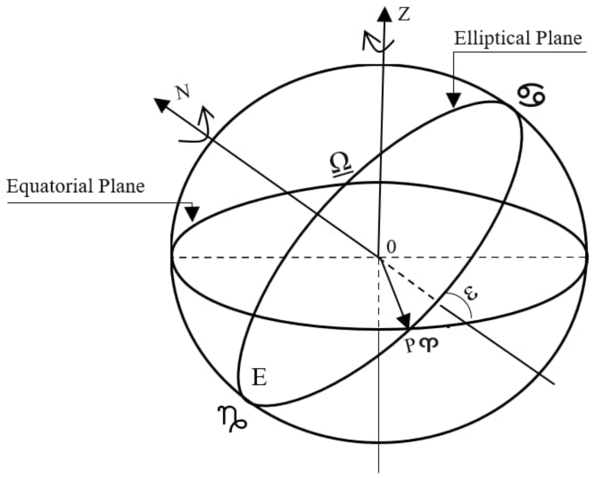

Figure 3.1: The system in which the apparent movement of the Sun occurs

Let Q_1 be the quaternion that will realize the movement in the positive direction around axis N. Let Q_2 be the quaternion that will realize the movement in the positive direction around axis Z. With the help of these two quaternions, the parametric equation of the curve of the daily and yearly apparent movement the Sun makes in the celestial sphere will be obtained. The starting point of the movement is $P = (1,0,0)$ which coincides with the Aries constellation. The vector OP that is found in the direction of the Earth-Aries constellation is $v = (1,0,0)$. First, let this vector be transferred to the quaternion space so:
$v_1 = (1,0,0)$ vector $\rightarrow w_1 = 0+i+0j+0k = i$ corresponds to a pure quaternion. The first rotation movement will be realized around axis $u = -j\sin\varepsilon + k\cos\varepsilon$ with θ angle. The second rotation movement will be realized around axis k with a φ angle in a negative direction. In this case, the Q_1 and Q_2 quaternions that will operate as rotation operators are: For $a = \sin\varepsilon$ and $b = \cos\varepsilon$,

$$Q_1 = \cos\left(\frac{\theta}{2}\right) - ja\sin\left(\frac{\theta}{2}\right) + kb\sin\left(\frac{\theta}{2}\right)$$

and

$$Q_2 = \cos\left(\frac{\varphi}{2}\right) + k\sin\left(\frac{\varphi}{2}\right).$$

It is stated that the second rotation movement (daily movement) occurs around axis k in the negative direction. If the necessary adjustments are made, instead of $Q_2 = \cos\left(\frac{\varphi}{2}\right) + k\sin\left(\frac{\varphi}{2}\right)$ for the second rotation, the complex conjugate of Q_2 will be used.

$$Q_2^* = \cos\left(\frac{\varphi}{2}\right) - k\sin\left(\frac{\varphi}{2}\right).$$

According to Theorem 2.3, for $L_{Q_1}(w_1) = Q_1 \times w_1 \times Q_1^*$, $L_{Q_2^*}(w_2) = Q_2^* \times w_2 \times Q_2$, and $w_2 = Q_1 \times w_1 \times Q_1^*$

$$L_{Q_2^* Q_1}(w_1) = (Q_2^* \times Q_1) \times w_1 \times (Q_2^* \times Q_1)^*.$$

If $Q_2^* \times Q_1 = Q$ and $w_1 = i$ then

$$L_{Q_2^* Q_1}(w_1) = Q \times i \times Q^*.$$

So the calculations are as such:

$$Q = Q_2{}^* \times Q_1 = \left(\cos\left(\frac{\varphi}{2}\right) - k\sin\left(\frac{\varphi}{2}\right)\right) \times \left(\cos\left(\frac{\theta}{2}\right) - ja\sin\left(\frac{\theta}{2}\right) + kb\sin\left(\frac{\theta}{2}\right)\right)$$

$$Q = \left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right) + b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right) - ia\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) - ja\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)$$
$$+k\left(b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)$$

$$L = Q \times i \times Q^* = L_0 + iL_1 + jL_2 + kL_3$$

$$L_0 = 0$$

$$
\begin{aligned}
L_1 &= \left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2+a^2\sin^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)\\
&-a^2\cos^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)-\left(b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^2\\
&=\cos^2\left(\frac{\varphi}{2}\right)\cos^2\left(\frac{\theta}{2}\right)+2b\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)+b^2\sin^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)\\
&-a^2\sin^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)-b^2\cos^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)\\
&+2b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\\
&=\cos^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)-b^2\sin^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)\\
&-a^2\sin^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)+\left(2b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\right)\left(2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\\
&=\cos^2\left(\frac{\theta}{2}\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)-\sin^2\left(\frac{\theta}{2}\right)\left(a^2+b^2\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)\\
&+\left(2b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\right)\left(2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\\
&=\left(\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)\right)\left(\cos^2\left(\frac{\varphi}{2}\right)-\sin^2\left(\frac{\varphi}{2}\right)\right)+\left(2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\left(2b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right)\right)
\end{aligned}
$$

$$L_1=\cos\varphi\cos\theta+b\sin\varphi\sin\theta.$$

Likewise:

$$L_2=b\cos\varphi\sin\theta-\sin\varphi\cos\theta$$

$$L_3=a\sin\theta$$

then

$$
\begin{aligned}
L_{Q_2{}^*Q_1}(w_1)=Q\times i\times Q^*&=i(\cos\varphi\cos\theta+b\sin\varphi\sin\theta)+j(b\cos\varphi\sin\theta-\sin\varphi\cos\theta)\\
&+ka\sin\theta=w.
\end{aligned}
$$

When vector w that was obtained in the quaternion space is transferred to vector v in the real space:

$$v=(x,y,z)=(\cos\varphi\cos\theta+b\sin\varphi\sin\theta\,,\ b\cos\varphi\sin\theta-\sin\varphi\cos\theta\,,\ a\sin\theta).$$

If $c>2$, $0\leq\theta\leq 2\pi$, $0\leq\varphi\leq n\pi$, n and c are constants and $\varphi=c\theta$ are kept in mind then:

$$
\begin{aligned}
X&=\cos\theta\cos(c\theta)+b\sin\theta\sin(c\theta)\\
Y&=b\sin\theta\cos(c\theta)-\cos\theta\sin(c\theta)\\
Z&=a\sin\theta
\end{aligned}
$$

$c=365,25$ and $0\leq\theta\leq 2\pi$, $a=\sin 23^0 27'$ and $b=\cos 23^0 27'$.

The quaternion that will be used for the first rotation movement, was defined before as:
$Q_1=\cos\left(\frac{\theta}{2}\right)-ja\sin\left(\frac{\theta}{2}\right)+kb\sin\left(\frac{\theta}{2}\right)$. From here, we have:

$$
\begin{aligned}
q_{10}&=\cos\left(\frac{\theta}{2}\right)\\
q_{11}&=0\\
q_{12}&=-a\sin\left(\frac{\theta}{2}\right)\\
q_{13}&=b\sin\left(\frac{\theta}{2}\right)
\end{aligned}
$$

According to (2.1) rotation matrix A which is produced by the unit quaternion above is:

$$A=\begin{bmatrix} 2\cos^2\left(\frac{\theta}{2}\right)-1 & -2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & -2a\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ 2b\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & 2\cos^2\left(\frac{\theta}{2}\right)-1+2\left(-a\sin\left(\frac{\theta}{2}\right)\right)^2 & -2ab\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ 2\cos\left(\frac{\theta}{2}\right)a\sin\left(\frac{\theta}{2}\right) & -2ab\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & 2\cos^2\left(\frac{\theta}{2}\right)-1+2\left(b\sin\left(\frac{\theta}{2}\right)\right)^2 \end{bmatrix}.$$

The quaternion that will be used for the second rotation movement, was defined before as:
$Q_2{}^*=\cos\left(\frac{\varphi}{2}\right)-k\sin\left(\frac{\varphi}{2}\right)$ From here:

$$q_{20}{}^*=\cos\left(\frac{\varphi}{2}\right)$$
$$q_{21}{}^*=0$$
$$q_{22}{}^*=0$$
$$q_{23}{}^*=-\sin\left(\frac{\varphi}{2}\right)$$

According to (2.1) rotation matrix B which is produced by the unit quaternion above is:

$$B=\begin{bmatrix} 2\cos^2\left(\frac{\varphi}{2}\right)-1 & 2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right) & 0 \\ -2\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\varphi}{2}\right) & 2\cos^2\left(\frac{\varphi}{2}\right)-1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let matrix be the resultant matrix of matrixes and then:

$$C=BA$$

When the necessary calculations are done:

$$C=\begin{bmatrix} 2q_0^2-1+2q_1^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & 2q_0^2-1+2q_2^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & 2q_0^2-1+2q_3^2 \end{bmatrix}$$

where

$$\begin{aligned} q_0&=\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ q_1&=-a\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ q_2&=-a\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ q_3&=b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right) \end{aligned} \tag{3.1}$$

As expected, the values in equation (3.1) are the same as the values of $Q=Q_2{}^*\times Q_1$.
According to (2.2), the vector $w=(w_1,w_2,w_3)$ obtained when rotation matrix C is applied in vector $\vec{v}=(1,0,0)$ is:

$$w=C\vec{v}$$

$$w(w_1,w_2,w_3)=\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}=\begin{bmatrix} 2q_0^2-1+2q_1^2 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & 2q_0^2-1+2q_2^2 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & 2q_0^2-1+2q_3^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$w_1=2q_0^2-1+2q_1^2=2\left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2-1+2\left(-a\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)^2$$

$$w_1=\cos\varphi\cos\theta+b\sin\varphi\sin\theta$$

$$
\begin{aligned}
w_2 &= (2q_1q_2 + 2q_0q_3) \\
&= 2\left(-a\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\left(-a\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right) \\
&+2\left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+b\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\left(b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)
\end{aligned}
$$

$$w_2 = b\cos\varphi\,\sin\theta - \sin\varphi\,\cos\theta$$

$$
\begin{aligned}
w_3 &= 2\left(-a\sin\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right)\left(b\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)\right) \\
&-2\left(\cos\left(\frac{\varphi}{2}\right)\cos\left(\frac{\theta}{2}\right)+bsin\left(\frac{\varphi}{2}\right)\ sin\left(\frac{\theta}{2}\right)\right)\left(-a\cos\left(\frac{\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right)\right) = 2a\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)
\end{aligned}
$$

$$w_3 = a\sin\theta$$

$$w = (w_1, w_2, w_3) = (\cos\varphi\cos\theta + b\sin\varphi\sin\theta\,,\ b\cos\varphi\sin\theta - \sin\varphi\cos\theta\,,\ a\sin\theta)\,.$$

If $c > 2$, $0 \le \theta \le 2\pi$, $0 \le \varphi \le n\pi$, n and c constants and $\varphi = c\theta$, are kept in mind then:

$$
\begin{aligned}
w_1 &= X = \cos\theta\cos(c\theta) + b\sin\theta\sin(c\theta) \\
w_2 &= Y = b\sin\theta\cos(c\theta) - \cos\theta\sin(c\theta) \\
w_3 &= Z = a\sin\theta
\end{aligned} \tag{3.2}
$$

$c = 365,25$ and $0 \le \theta \le 2\pi$, $a = \sin 23^0 27'$ and $b = \cos 23^0 27'$

If the graphic of the equation (3.2) we obtained above was drawn, the three-dimensional graphic shown in Figure 3.2 will be acquired. This curve covers the entirety of the sphere found between the planes $z = -\sin 23^0 27'$ and $z = \sin 23^0 27'$ because the constant c is $c = 365,25$. For this reason, to be able to comprehend the shape of the curve, $c = 12$ is chosen instead of $c = 365,25$ and this way the graphic shown in Figure 3.3 is obtained. As shown in Figure 3.3, the curve is a spherical spiral limited between the planes $z = -\sin 23^0 27'$ and $z = \sin 23^0 27'$. If in equation (3.2) $\varepsilon = 90^0$ then the parametric equation of the spherical spiral is procured.

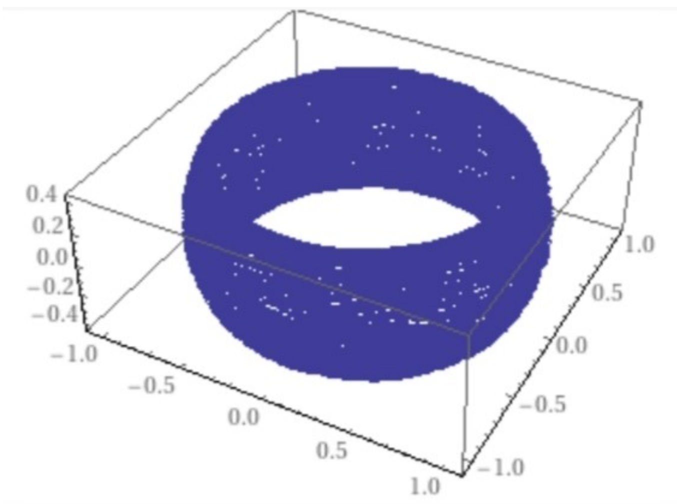

Figure 3.2: The curve of the apparent movement of the Sun for $c = 365,25$

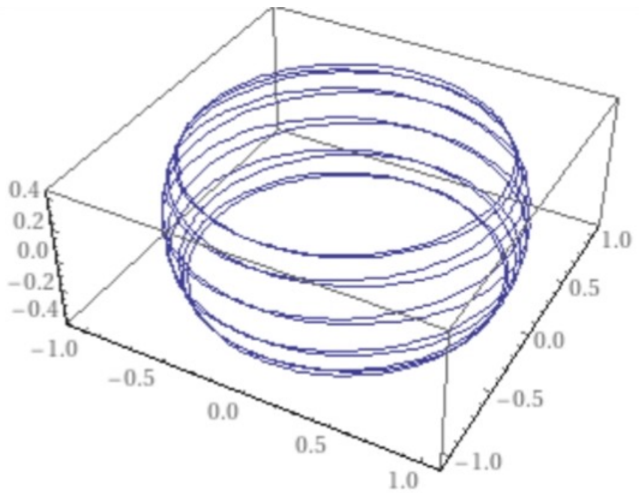

Figure 3.3: The curve of the apparent movement of the Sun for $c = 12$

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Author's contributions

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