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EVALUATION AND COMPARISON OF METAHEURISTIC METHODS TO ESTIMATE THE PARAMETERS OF GAMMA DISTRIBUTION

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ABSTRACT

Parameter estimation of three parameter (3-p) Gamma distribution is very important as it is one of the most popular distributions used to model skewed data. Maximum Likelihood (ML) method based on finding estimators that maximize the likelihood function, is a well-known parameter estimation method. It is rather difficult to maximize the likelihood function formed for the parameter estimation of the 3-p Gamma distribution. In this study, five well known metaheuristic methods, Simulated Annealing (SA), Genetic Algorithm (GA), Particle Swarm Optimization (PSO), Differential Evolution (DE), and Artificial Bee Colony (ABC), are suggested to obtain ML estimates of the parameters for the 3-p Gamma distribution. Monte-Carlo simulations are performed to examine efficiencies of the metaheuristic methods for the parameter estimation problem of the 3-p Gamma distribution. Also, differences between solution qualities and computation time of the algorithms are investigated by statistical tests. Moreover, one of the multi-criteria decision-making methods, Technique for Order Performance by Similarity to Ideal Solution (TOPSIS), is preferred for ranking the metaheuristic algorithms according to their performance in parameter estimation. Results show that Differential Evolution is superior to the others for this problem in consideration of all the

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criteria of solution quality, computation time, simplicity, and robustness of the metaheuristic algorithms. In addition, an analysis of real-life data is presented to demonstrate the implementation of the suggested metaheuristic methods.

Keywords: Gamma Distribution, Maximum Likelihood Estimation, Metaheuristic Methods, Monte-Carlo Simulation, TOPSIS

GAMMA DAĞILIMININ PARAMETRELERİNİN TAHMİNİ İÇİN METASEZGİSEL YÖNTEMLERİN DEĞERLENDİRİLMESİ VE KARŞILAŞTIRILMASI

ÖZ

Üç parametreli (3-p) Gamma dağılımı çarpık verilerin modellenmesinde kullanılan en popular dağılımlardan biri olduğundan bu dağılımın parametrelerinin tahmini çok önemlidir. Olabilirlik fonksiyonunu maksimize eden parametreleri bulan En Çok Olabilirlik (ML) yöntemi yaygın olarak kullanılan bir parametre tahmini yöntemidir.3-p Gamma dağılımının parametrelerinin tahmini için olabilirlik fonksiyonunu maksimize etmek çok zordur. Bu çalışmada, 3-p Gamma dağılımının parametrelerinin ML tahminlerini elde etmek için beş tane iyi bilinen metasezgisel yöntem: Tavlama Benzetimi (SA), Genetik Algoritma (GA), Parçacık Sürüsü Optimizasyonu (PSO), Diferansiyel Gelişim (DE) ve Yapay Arı Kolonisi (ABC) önerilmektedir. 3-p Gamma dağılımının tahmini probleminde metasezgisel yöntemlerin etkinliğinin araştırılması için Monte-Carlo simülasyon çalışmaları yapılmaktadır. Algoritmaların çözüm kalitesi ve hesaplama zamanı arasındaki farklar istatistiksel testler ile araştırılmaktadır. Ayrıca, metasezgisel algoritmaların parametre tahminindeki performanslarına göre sıralanması için çok kriterli karar verme yöntemlerinden biri olan TOPSIS yöntemi önerilmektedir. Sonuçlar, metasezgisel algoritmaların çözüm kalitesi, hesaplama zamanı, basitlik ve sağlamlılık kriterleri göz önüne alındığında DE'nin diğerlerinden daha iyi olduğunu göstermektedir. Ayrıca, önerilen metasezgisel yöntemlerin uygulanabilirliğini göstermek için gerçek bir yaşam verisi analizi sunulmaktadır.

Anahtar Kelimeler: Gamma Dağılımı, En Çok Olabilirlik Tahmini, Metasezgisel Yöntemler, Monte-Carlo Simülasyonu, TOPSIS

1. INTRODUCTION

Gamma distribution is one of the extensively used distributions for modeling skewed data in various fields such as hydrology, finance, especially for reliability or lifetime (Basak and Balakrishnan, 2012; Hirose, 1995; Vaidyanathan and Lakshmi, 2015). Let X be a $gamma(\alpha,\beta,\mu)$ random variable with shape parameter α, scale parameter β and location (or threshold) parameter μ. Probability density function and distribution function of X are expressed as follows, respectively

$$f(x;\alpha,\beta,\mu)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}(x-\mu)^{\alpha-1}\exp\left(-\frac{x-\mu}{\beta}\right),\quad x\geq\mu,\alpha>0,\beta>0. \tag{1}$$

$$F(x;\alpha,\beta,\mu)=\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\frac{x-\mu}{\beta}\right),\quad x\geq\mu,\alpha>0,\beta>0 \tag{2}$$

In Eq. (2), γ indicates incomplete Gamma function (Balakrishnan and Wang, 2000; N. L. Johnson et al., 1994, Vaidyanathan and Lakshmi, 2015).

3-p Gamma distribution contains three special cases according to the values of the shape parameter α. If $\alpha\leq1$ then the distribution is "*J*" shaped, if $\alpha>1$ then the distribution is bell-shaped and if $\alpha=1$ then the distribution becomes an exponential distribution (Basak and Balakrishnan, 2012). It is very important to estimate the parameters of the 3-p gamma distribution since it plays an important role in the applied literature. However, the parameter estimation for the 3-p Gamma distribution has continued as a challenging and interesting problem. The maximum likelihood (ML) method based on the maximization of the likelihood function of any distribution, is the most commonly used parameter estimation method for the distributions because it usually tends to perform better than its competitors (Nagatsuka et al., 2014). Thus, the ML method is used for estimating the parameters of the 3-p Gamma distribution in this study. Since maximizing the likelihood function of the 3-p Gamma distribution is quite difficult, generally iterative techniques such as Newton Raphson and Nelder Mead are used in the literature. However, since iterative techniques have an initial value problem, it may be suggested to use metaheuristic methods.

In the literature, various metaheuristic methods have been used for ML parameter estimation of different distributions. The Simulated Annealing (SA) algorithm (Abbasi et al., 2006), a hybrid metaheuristic method that combines variable neighborhood search and the SA

(Abbasi et al., 2011), Differential Evolution (DE) algorithm (Örkcü et al., 2015), Particle Swarm Optimization (PSO) (Örkcü et al., 2015), PSO with adaptive search space (Acıtaş et al., 2019) and Artificial Bee Colony with Levy flights (LABC) (Yonar and Pehlivan, 2020a) have been proposed to maximize the likelihood function in the ML parameter estimation of a 3-p Weibull distribution. A metaheuristic approach by combining variable neighborhood search (VNS) and iterated local search (ILS) algorithm (Zoraghi et al., 2012) and PSO method (Özsoy et al., 2017) are suggested to maximize likelihood function of the 4-p Burr III distribution. The SA approach (Vera and Díaz-García, 2008) is applied to overcome the problem of ML estimation in any parameterization scheme for the 3-p Lognormal distribution. The Genetic Algorithm (GA) approaches are used for ML parameter estimation of the mixture normal distribution (Shin et al., 2014) and skew normal (SN) distribution (Yalçınkaya et al., 2018). A novel DE algorithm approach is proposed for the ML parameter estimation of the 3-p Gamma distribution (Yonar and Pehlivan, 2020b).

In this study, commonly used metaheuristic methods for parameter estimation, the Simulated Annealing, Genetic Algorithm, Particle Swarm Optimization, Differential Evolution, and Artificial Bee Colony Algorithm, are proposed to estimate the ML estimations for the 3-p Gamma distribution. Differences between these algorithms in terms of solution qualities and computation time have been examined by statistical tests. Also, Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method (Hwang and Yoon, 1981) which is one of the multi-criteria decision-making methods is utilized to determine which algorithm is more reasonable for using the ML parameter estimation problem,. The suggested metaheuristic methods are ranked by TOPSIS method considering all criteria of solution quality, computation time, simplicity, and robustness. In addition, the ML estimates of the parameters for a real-life data are obtained by MLE_P method and MML method as well as suggested metaheuristic algorithms. To compare these algorithms, log-likelihood ($\log L$), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) are used.

The rest of the study is organized as follows. In Section 2, the ML estimation method for parameters of the 3-p Gamma distribution is presented. The SA, GA, PSO, DE and ABC algorithms of metaheuristic methods are briefly introduced in Section 3. In Section 4, TOPSIS method is explained. In Section 5, Monte-Carlo simulations are performed to examine efficiencies of the suggested metaheuristic methods via TOPSIS method for determining the best metaheuristic algorithm. Also, a real-life example is analyzed to show an implementation

of the considered metaheuristic methods. Finally, some concluding remarks are given in the last section.

2. METHODOLOGY

2.1. Maximum likelihood estimation for parameters of the 3-p Gamma distribution

Due to obtaining consistent and asymptotically efficient estimators, the maximum likelihood (ML) method generally outperforms its competitors, e.g.. Least squares method, method of moments and so on for parameter estimation. The ML estimators and their properties for the 3-p Gamma distribution are given in detailed in (Basak and Balakrishnan, 2012; Clifford and Jones, 1982; Cohen and Whitten, 1986; N. L. Johnson et al., 1994).

Let $x_1, x_2, ..., x_n$ be a random variable sample drawn from the $Gamma(\alpha, \beta, \mu)$ distribution. The likelihood function and log-likelihood function of the Gamma distribution are given in Eq. (3) and Eq. (4), respectively.

$$L(\boldsymbol{\theta} / \boldsymbol{x}) = \left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}} \right)^{n} \prod_{i=1}^{n} (x_i - \mu)^{\alpha-1} \exp\left(\sum_{i=1}^{n} \left(-\frac{x_i - \mu}{\beta} \right) \right) \tag{3}$$

$$\log L(\boldsymbol{\theta} / \boldsymbol{x}) = (\alpha - 1) \sum_{i=1}^{n} \log(x_i - \mu) - \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\beta} \right) - n \log \Gamma(\alpha) - n\alpha \log \beta \tag{4}$$

where $\boldsymbol{\theta} = (\alpha, \beta, \mu)$ is a vector of representing unknown parameters (Bowman et al., 1995; N. Johnson et al., 1994). The ML estimates of the parameters α, β and μ are points that maximize the likelihood function or the log-likelihood function. Thus, there is an optimization problem for obtaining the estimates with the ML method. Such problems can be solved by taking the first partial derivatives of $\log L(\boldsymbol{\theta} / \boldsymbol{x})$ according to unknown parameters and equalizing to zero. Then, solutions of these equations known as likelihood equations are called ML estimators of the parameters for the considered distribution. Since the likelihood equations of the 3-p Gamma distribution contain nonlinear functions, they cannot be solved. Therefore, it is recommended to use iterative methods such as Newton-Raphson and Nelder-Mead for maximizing the $\log L(\boldsymbol{\theta} / \boldsymbol{x})$. These iterative methods require an initial value that should be approximated to global optimum value. It is suggested that using some of the metaheuristic methods, SA, GA, PSO, DE, and ABC, to overcome this problem. These metaheuristic methods have some

important advantages such as robustness, simplicity of implementation, high solution quality and reasonable computation time (run time) as well as no initial value problem.

2.2. Metaheuristic Methods

Metaheuristic methods mostly start by generating a random initial solution/population and loop over an iteration process to make the solution/population evolves. For *D*-dimensional optimization problem, $\overset{\mathrm{r}}{x}_i^g = [x_{1,i}^g, x_{2,i}^g, ..., x_{D,i}^g]$ represents the i th vector of the population at iteration g. Each vector is a candidate solution for the optimization problem. The initial population for each element of the vector i is generated by using the prescribed lower limit ($x_{j,\min}$) and upper limit ($x_{j,\max}$) known as search space, as follows (Talbi, 2009; Yang, 2010).

$$x_{j,i}^0 = x_{j,\min} + rand_{ij}[0,1]\left\{x_{j,\max} - x_{j,\min}\right\} \tag{5}$$

where $rand_{ij}[0,1]$ is a uniformly distributed random variable in range $[0,1]$ If the algorithm is the single-solution based metaheuristic, then an initial solution is generated by taking $i = 1$ (Price et al., 2006; Talbi, 2009).

The five metaheuristic algorithms SA, GA, PSO, DE, and ABC used in maximizing the likelihood function of the 3-p Gamma distribution, are briefly explained in the following subsections.

2.2.1. Simulated Annealing

Simulated Annealing (SA) algorithm is based on the principles of statistical mechanics whereby the annealing process requires heating and then slowly freezing a substance to obtain crystalline structure. Application of the SA is firstly proposed by Kirkpatrick et al. (1983) to use optimization problems by taking inspiration the research of Metropolis et al. (1953) in the field of statistical mechanics (Abbasi et al., 2006; Talbi, 2009; Yang, 2010; Yonar, 2020).

The SA is a stochastic algorithm which enables accepting a worse solution. This is the most important feature of the SA algorithm. The aim is to delay convergence by escaping from local optimum. The SA is single solution-based metaheuristic and proceeds in several iterations. At each iteration, a random neighbor solution is generated. The neighbor solution is accepted if

it improves the objective function, otherwise, it is selected with a given probability. Depending on current temperature and amount of degradation of the objective function, the probability is calculated by following

$$P(\Delta E, T) = \exp\left(\frac{f(x') - f(x)}{k_B T}\right) \tag{6}$$

where x is the current solution, x' is the neighbor solution, $f(x)$is the objective function value at the solution x, k_B is the Boltzmann's constant and it can be set $k_B = 1$ for simplicity, and T denotes current temperature for controlling the annealing process (Abbasi et al., 2006; Talbi, 2009; Yang, 2010).

2.2.2. Genetic Algorithm

Genetic Algorithm (GA) is one of the most prominent and widely used search techniques to find approximate solutions for optimization and search problems. It was firstly developed by Holland (1975) for understanding Darwin's evolution process of natural systems and then, it was applied to optimization and machine learning by Goldberg and Holland (1988).

The GA is a population-based method, which is a very popular class of evolutionary algorithms and has no initial value problem. Generally, the GA algorithm starts by generating a random initial population and then it loops over an iteration process to make the population evolve. Each iteration called a generation consists of selection, reproduction with crossover and mutation operators, evolution, and replacement stages (Talbi, 2009; Yalçınkaya et al., 2018; Yang, 2010; Yonar, 2020).

2.2.3. Particle Swarm Optimization

Particle swarm optimization (PSO) first introduced by Eberhart and Kennedy (1995) is a biologically inspired technique derived from the collective behavior of bird flocking and fish schooling (Acıtaş et al., 2019; Örkcü et al., 2015).

The PSO is a population-based and self-adaptive search optimization method. The population consists of a set of particles. Each particle records its own personal best position

(*pbest*), and knows the best positions found by all particles in the swarm (*gbest*). Then, all particles update the velocity and position in each iteration.

The velocity and the new position of each particle at iteration k+1, can be calculated as follows, respectively:

$$v_i^{g+1} = wv_i^g + c_1 r_1 \left(pbest_i^g - x_i^g \right) + c_2 r_2 \left(gbest^g - x_i^g \right) \tag{7}$$

$$x_i^{g+1} = x_i^g + v_i^g \tag{8}$$

In Eqs. (7) and (8), v_i^g is the velocity of individual i at iteration g, w is the inertia weight, c_1 and c_2 are the acceleration coefficients, r_1 and r_2 are random numbers uniformly distributed between 0 and 1, x_i^g is the position of individual i at iteration g, $pbest_i^g$ is the best position of individual i until iteration g, $gbest^g$ is the best position of the group until iteration g(Örkcü et al., 2015; Talbi, 2009; Yang, 2010; Yonar, 2020).

2.2.4. Differential Evolution

Differential evolution (DE) which is a population-based evolutionary algorithm proposed by Storn (1996) and Storn and Price (1997), is one of the most successful approaches for continuous optimization problems. Similar to the GA, the DE algorithm uses crossover, mutation and selection operators. However, the DE uses the mutation operator, while GA uses crossover operator to obtain better solutions (Gui et al., 2019; Price et al., 2006).

The mutation operator in the DE is based on difference between randomly selected two solutions from the population. The DE algorithm uses the mutation operator as a search mechanism and selection operator to navigate the candidate areas in the search space. In this study, the standard variant of the DE, $DE/rand/1/bin$, is handled (Talbi, 2009; Yonar, 2020).

2.2.5. Artificial Bee Colony

The Artificial Bee Colony (ABC) algorithm was firstly proposed by Karaboga (2005) for numerical optimization problem inspiring the intelligent foraging behavior of honey bees (Karaboğa and Öztürk, 2011). Later, it was adapted to solve optimization problem by Akay and Karaboğa (2012).

In the ABC algorithm, honey bees are classified into three groups: employed bees, onlooker bees, and scout bees. The employed bees that are pioneers of the swarm, discover food sources, gather honey, and share information about the nectar amount the food sources within the hive. Based on this information, onlooker bees choose and exploit better food sources. A bee whose food source is bad, change to be a scout bee for searching randomly new food sources. In this way, three types of artificial bees form effective cooperation with each other (Rajasekhar et al., 2017; Yonar, 2020).

2.3. TOPSIS Method

Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method proposed by Hwang and Yoon (1981) is one of the well-known multi criteria decision making (MCDM) methods. It is based on selection of alternative which has the shortest distance from the positive-ideal solution and the farthest distance from the negative-ideal solution. The steps of the TOPSIS method for a MCDM problem involving *m* alternatives and *n* criteria as follows (Chen, 2000; Opricovic and Tzeng, 2004; Şahin and Pehlivan, 2017):

Step 1: Construct a decision matrix $D=\left[x_{ij}\right]_{m\times n}$ and determine a weight vector $W=\left[w_j\right]_{1\times n}$ Here, x_{ij} is the value of evaluation for alternative i with respect to criterion j and w_j denotes the weight of criterion j.

Step 2: Constitute the normalized decision matrix $D'=\left[x'_{ij}\right]_{m\times n}$ by $x'_{ij}=x_{ij}/\left(\sum_{i=1}^{m}\left(x_{ij}\right)^2\right)$.

Step 3: Construct the weighted decision matrix $D_w=[v_{ij}]_{m\times n}$ by

$v_{ij}=w_j\times x_{ij}$

Step 4: Determine positive ideal solution $A^+=\left(v_1^+,v_2^+,...,v_n^+\right)$and negative ideal solution $A^-=\left(v_1^-,v_2^-,...,v_n^-\right)$ as follows:

$$v_j^+=\left\{\left(\max_{1\le i\le m}\{v_{ij}\}\mid c_j\in J_1\right),\left(\min_{1\le i\le m}\{v_{ij}\}\mid c_j\in J_2\right)\right\} \quad (9)$$

$$v_j^-=\left\{\left(\min_{1\le i\le m}\{v_{ij}\}\mid c_j\in J_1\right),\left(\max_{1\le i\le m}\{v_{ij}\}\mid c_j\in J_2\right)\right\} \quad (10)$$

where J_1 and J_2 express the benefit and cost criteria, respectively.

Step 5: Calculate distance between each alternative from the positive ideal solutions (d_i^+) and negative ideal solutions (d_i^-) as follows:

$$d_i^+ = \sqrt{\sum_{j=1}^{n}(v_{ij} - v_j^+)^2} \tag{11}$$

$$d_i^- = \sqrt{\sum_{j=1}^{n}(v_{ij} - v_j^-)^2} \tag{12}$$

Step 6: Calculate relative degree of closeness CC_i by following;

$$CC_i = d_i^- / \left(d_i^+ + d_i^-\right) \tag{13}$$

Step 7: Rank the values of $CC_i \left(0 \le CC_i \le 1\right)$, $i = 1, 2, ..., m$ in descending order. The alternative with the largest CC_i value is the best alternative.

3. RESULTS AND DISCUSSION

This section is composed of three parts. In the first part, the Monte-Carlo simulation study is carried out to examine efficiencies of the estimators obtained by using suggested metaheuristic methods. In the second part, these metaheuristic methods are ranked by TOPSIS method according to their performances in the ML parameter estimation of the 3-p Gamma distribution to determine the best method. In the last part, an analysis of a real-life example is presented to demonstrate implementation of the metaheuristic methods. All computations in this section are conducted using the "optimization", "GA", "pso", "DEoptim", and "ABCoptim" packages included in the free software RStudio 4.1.3.

3.1. Simulation Study

In this part, results of the Monte-Carlo simulation are presented. At first, Nelder-Mead Algorithm which is one of the classical methods is utilized to estimate the parameters of the 3-p Gamma distribution. However, this algorithm is not provided sufficient parameter estimation

results when the search is started randomly since it has an initial value problem. For this reason, metaheuristic methods which have no initial value problem are proposed to estimate the parameters of the 3-p Gamma distribution. The performances of the suggested metaheuristic methods are compared in terms of solution quality and computation (run) time. The values of the Deficiency (DEF) criteria are used for evaluating the solution quality.

The DEF criterion is a significant tool for comparing the joint efficiencies of the various methods utilized to estimate a set of parameters $\theta=(\alpha,\beta,\mu)$. It is determined as the sum of Mean Square Error (MSE) values for the estimators of the parameters (Yalçınkaya et al., 2018).

For performance evaluation of the methods, real parameter values for the 3-p Gamma distribution are specified as $\theta=(4,0.5,10)$, $\theta=(2.56,0.625,10)$, $\theta=(1.778,0.75,10)$, $\theta=(1,1,10)$, and $\theta=(0.64,1.25,10)$. In addition, samples sizes of 25, 50,100, 250, 500, and 1000 are considered for each value of the parameter to examine effect of the sample size on the performance of the methods in the parameter estimation process.

Parameter values determined by various trials and literature reviews for each metaheuristic algorithm are shown in Table 1. Search space of the algorithms is selected as $[0,1]$ for all parameters and maximum iteration number are taken as 100 due to rapid convergence tendency of the algorithms.

Table 1. Parameters of the algorithms

Algorithms	Parameters
SA	$s=10$, $T_0=0.1$, $\tau=0.99$, nMove = 5
GA	NP=50, $p_m=0.3$, $p_c=0.8$, $m=0.1$
PSO	NP=50, $w=1$, $c_1=c_2=2$
DE	NP=50, $F\in[0.2,0.8]$, $C_r=0.2$
ABC	NP=50, $l=5$

Considered algorithms are run 10.000 times to strengthen the reliability of the estimation strategy and mean simulation results are presented in Tables 2-6. Computation times are also noted in all tables and the best results are highlighted in bold.

Table 2. Simulation results of parameter estimations of the 3-p Gamma distribution for $\theta = (4, 0.5, 10)$

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\mu}$			
		Mean	MSE	Mean	MSE	Mean	MSE	DEF	Comp.Time
25	SA	3.6050	12.5249	1.1801	1.2051	10.1921	0.3717	14.1016	12.6098
	GA	3.3577	**4.6881**	0.8050	**0.4559**	10.1998	**0.2081**	**5.3521**	1.0072
	PSO	4.0844	9.4128	0.8088	0.5196	10.1274	0.3027	10.2351	**0.7844**
	DE	2.7104	7.1858	1.2074	1.1902	10.3162	0.2940	8.6700	1.0093
	ABC	3.1595	10.6467	1.4434	1.9951	10.1706	0.4288	13.0706	1.8507
50	SA	4.2584	6.9467	0.5863	0.0878	10.0514	0.1998	7.2343	12.5680
	GA	3.7019	**2.8637**	0.5902	**0.0552**	10.1008	**0.1186**	**3.0375**	1.1366
	PSO	4.3519	5.8955	0.5619	0.0700	10.0275	0.1739	6.1394	**0.8166**
	DE	3.4678	3.0755	0.6217	0.0826	10.1674	0.1345	3.2926	1.0498
	ABC	4.8904	8.3538	0.6508	0.2356	9.7964	0.3605	8.9499	2.1556
100	SA	4.1258	3.4340	0.5307	0.0197	10.0304	0.0984	3.5521	12.5506
	GA	3.7987	2.1573	0.5616	0.0258	10.0543	0.0811	2.2642	1.1233
	PSO	4.2946	3.0910	0.5138	**0.0181**	9.9965	0.0932	3.2023	**0.8551**
	DE	3.5967	**1.6423**	0.5606	0.0185	10.1129	**0.0677**	**1.7285**	1.1223
	ABC	5.3686	7.7421	0.5182	0.0509	9.6773	0.3427	8.1357	2.2840
250	SA	4.0361	1.1110	0.5117	**0.0069**	10.0137	0.0348	1.1527	14.8486
	GA	3.8364	1.5502	0.5514	0.0188	10.0256	0.0515	1.6205	1.2283
	PSO	4.1988	1.2402	0.4998	**0.0069**	9.9848	0.0386	1.2858	**0.8384**
	DE	3.7187	**0.7556**	0.5352	0.0071	10.0674	**0.0292**	**0.7919**	1.2216
	ABC	5.4813	7.5010	0.4960	0.0306	9.6431	0.3353	7.8668	2.2186
500	SA	3.9864	0.4538	0.5072	**0.0031**	10.0129	**0.0152**	0.4721	17.1158
	GA	3.8688	1.3913	0.5462	0.0158	10.0107	0.0457	1.4528	1.3915
	PSO	4.1895	0.6707	0.4934	0.0036	9.9769	0.0214	0.6956	**1.0899**
	DE	3.7546	**0.4126**	0.5265	0.0037	10.0536	0.0157	**0.4320**	1.3984
	ABC	5.2944	6.9967	0.5016	0.0268	9.6910	0.2909	7.3143	2.3078
1000	SA	3.9795	**0.1962**	0.5043	**0.0015**	10.0087	**0.0067**	**0.2043**	21.0536
	GA	3.9076	1.2942	0.5423	0.0145	9.9966	0.0427	1.3514	1.6647
	PSO	4.1775	0.4177	0.4911	0.0021	9.9742	0.0131	0.4329	**1.3248**
	DE	3.7908	0.2397	0.5209	0.0021	10.0424	0.0088	0.2507	1.6241
	ABC	5.0017	5.9974	0.5110	0.0237	9.7600	0.2375	6.2587	2.5252

Table 3. Simulation results of parameter estimations of the 3-p Gamma distribution for $\theta = (2.56,\ 0.625,\ 10)$

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\mu}$			
		Mean	MSE	Mean	MSE	Mean	MSE	DEF	Comp.Time
25	SA	2.1720	7.5901	1.4700	1.3486	10.1206	0.1731	9.1118	7.5784
	GA	2.5007	**3.3987**	0.9648	**0.5480**	10.0621	**0.1212**	**4.0679**	0.8187
	PSO	2.7555	6.2907	1.0209	0.6702	10.0485	0.1554	7.1163	**0.8009**
	DE	1.7099	3.8043	1.4799	1.3309	10.1812	0.1180	5.2533	1.0541
	ABC	1.9905	6.7478	1.7315	2.1788	10.1039	0.2097	9.1363	2.2390
50	SA	2.5539	2.8254	0.7678	0.1513	10.0529	0.0718	3.0485	8.1300
	GA	2.7438	2.1311	0.6890	**0.0812**	9.9999	0.0778	2.2901	1.0869
	PSO	2.7513	2.7048	0.7152	0.1103	10.0179	0.0725	2.8876	**0.8525**
	DE	2.3160	**1.3711**	0.7810	0.1441	10.0863	**0.0480**	**1.5632**	1.0970
	ABC	3.3310	7.0879	0.8872	0.4186	9.8320	0.2531	7.7596	2.3120
100	SA	2.5238	0.8173	0.6647	0.0239	10.0365	0.0253	0.8665	13.2867
	GA	2.8045	1.6087	0.6422	0.0276	9.9705	0.0607	1.6970	1.1233
	PSO	2.6809	1.0447	0.6447	0.0251	10.0065	0.0320	1.1018	**0.8795**
	DE	2.4301	**0.5346**	0.6749	**0.0231**	10.0510	**0.0200**	**0.5777**	1.0979
	ABC	3.7878	6.8260	0.6663	0.1176	9.7233	0.2700	7.2136	2.4334
250	SA	2.5065	0.1809	0.6423	0.0072	10.0217	0.0061	0.1943	14.7119
	GA	2.8050	1.2905	0.6290	0.0161	9.9552	0.0498	1.3564	1.1600
	PSO	2.6382	0.3937	0.6265	0.0089	9.9981	0.0127	0.4153	**0.1006**
	DE	2.4709	0.1638	0.6483	0.0073	10.0274	0.0059	**0.1770**	1.2283
	ABC	3.6770	5.6510	0.6261	0.0584	9.7406	0.2230	5.9323	2.5921
500	SA	2.5195	**0.0782**	0.6353	**0.0033**	10.0133	**0.0026**	**0.0842**	17.3935
	GA	2.8319	1.2201	0.6219	0.0129	9.9433	0.0500	1.2829	1.1686
	PSO	2.6416	0.2578	0.6205	0.0051	9.9922	0.0081	0.2710	**0.1170**
	DE	2.4979	0.0809	0.6395	0.0035	10.0167	0.0028	0.0872	1.3578
	ABC	3.2384	3.5731	0.6464	0.0433	9.8307	0.1408	3.7573	2.5476
1000	SA	2.5353	**0.0346**	0.6306	**0.0015**	10.0080	**0.0011**	**0.0372**	19.9183
	GA	2.8257	1.1094	0.6197	0.0107	9.9427	0.0443	1.1644	1.6153
	PSO	2.6561	0.2203	0.6162	0.0033	9.9862	0.0071	0.2307	**0.1299**
	DE	2.5143	0.0385	0.6347	0.0017	10.0114	0.0013	0.0414	1.5652
	ABC	2.9440	2.2817	0.6584	0.0333	9.8955	0.0896	2.4046	2.6287

Table 4. Simulation results of parameter estimations of the 3-p Gamma distribution for $\theta = (1.778, 0.75, 10)$

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\mu}$			
		Mean	MSE	Mean	MSE	Mean	MSE	DEF	Comp.Time
25	SA	1.2169	3.3012	1.6660	1.2385	10.0963	0.0633	4.6031	5.5191
	GA	1.8060	2.3949	1.1476	**0.6033**	10.0136	0.0752	**3.0734**	1.0010
	PSO	1.8172	3.5000	1.2374	0.7682	10.0313	0.0722	4.3404	**0.8030**
	DE	1.0518	**1.8715**	1.6726	1.2314	10.1170	**0.0437**	3.1465	1.0367
	ABC	1.1218	2.9718	1.9220	2.0505	10.0919	0.0727	5.0951	2.2336
50	SA	1.5649	0.8411	1.0054	0.2454	10.0502	0.0179	1.1044	6.1675
	GA	2.0361	1.8314	0.8410	**0.1439**	9.9657	0.0670	2.0422	1.0725
	PSO	1.7797	1.1584	0.9128	0.1801	10.0213	0.0261	1.3646	**0.8680**
	DE	1.5239	**0.6160**	1.0063	0.2390	10.0559	**0.0145**	**0.8694**	1.0629
	ABC	1.8356	3.4181	1.2357	0.6197	9.9571	0.0954	4.1332	2.2585
100	SA	1.6850	0.1694	0.8062	**0.0323**	10.0318	0.0048	0.2065	8.3841
	GA	2.0675	1.2917	0.7466	0.0393	9.9549	0.0526	1.3836	1.0836
	PSO	1.7906	0.3808	0.7910	0.0389	10.0129	0.0104	0.4301	**0.9052**
	DE	1.6697	**0.1562**	0.8122	0.0336	10.0333	**0.0046**	**0.1945**	1.1550
	ABC	2.0559	2.7207	0.9588	0.2203	9.9120	0.0883	3.0293	2.3672
250	SA	1.7212	**0.0469**	0.7738	**0.0085**	10.0169	**0.0012**	**0.0566**	10.0441
	GA	2.1008	1.2731	0.7249	0.0224	9.9402	0.0526	1.3480	1.2152
	PSO	1.8048	0.1808	0.7587	0.0118	10.0024	0.0052	0.1978	**1.0134**
	DE	1.7148	0.0491	0.7769	0.0091	10.0174	**0.0012**	0.0595	1.2084
	ABC	1.8660	1.0963	0.8713	0.0888	9.9522	0.0392	1.2243	2.5713
500	SA	1.7417	**0.0205**	0.7641	**0.0037**	10.0099	**0.0005**	**0.0247**	15.6388
	GA	2.1274	1.2778	0.7167	0.0188	9.9313	0.0545	1.3511	1.3907
	PSO	1.8151	0.1286	0.7493	0.0067	9.9974	0.0038	0.1391	**1.1789**
	DE	1.7352	0.0222	0.7671	0.0041	10.0104	**0.0005**	0.0268	1.4208
	ABC	1.7985	0.6199	0.8480	0.0602	9.9722	0.0229	0.7031	2.9035
1000	SA	1.7571	**0.0093**	0.7576	**0.0017**	10.0057	**0.0002**	**0.0112**	19.7226
	GA	2.1161	1.1832	0.7153	0.0165	9.9323	0.0504	1.2501	1.6143
	PSO	1.8251	0.1204	0.7448	0.0046	9.9939	0.0039	0.1289	**1.4486**
	DE	1.7509	0.0103	0.7602	0.0019	10.0063	**0.0002**	0.0124	1.6428
	ABC	1.8010	0.4230	0.8238	0.0458	9.9777	0.0160	0.4848	3.3278

Table 5. Simulation results of parameter estimations of the 3-p Gamma distribution for $\theta = (1, 1, 10)$

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\mu}$			
		Mean	MSE	Mean	MSE	Mean	MSE	DEF	Comp.Time
25	SA	0.5988	0.3782	1.7379	0.7463	10.0364	0.0049	1.1295	6.1487
	GA	1.1018	1.4889	1.3692	**0.5094**	9.9655	0.0516	2.0499	0.9317
	PSO	0.8677	0.8354	1.5609	0.7004	10.0108	0.0133	1.5491	**0.7997**
	DE	0.5867	**0.2739**	1.7375	0.7439	10.0376	**0.0039**	**1.0217**	1.0754
	ABC	0.5812	0.3559	2.0319	1.6269	10.0362	0.0053	1.9881	1.0214
50	SA	0.7434	0.1127	1.3716	0.2226	10.0186	0.0009	0.3363	6.1915
	GA	1.1618	1.2099	1.1587	**0.1957**	9.9534	0.0478	1.4534	1.0698
	PSO	0.8636	0.2795	1.3009	0.2259	10.0061	0.0048	0.5102	**0.8669**
	DE	0.7423	**0.1001**	1.3709	0.2222	10.0187	**0.0008**	**0.3230**	1.0831
	ABC	0.7305	0.1967	1.5969	0.6567	10.0162	0.0021	0.8555	1.2136
100	SA	0.8521	0.0394	1.1850	0.0712	10.0095	**0.0002**	0.1107	6.5034
	GA	1.2070	1.0464	1.0468	0.0913	9.9495	0.0443	1.1820	1.0772
	PSO	0.9160	0.1437	1.1572	0.0794	10.0012	0.0030	0.2262	**0.9207**
	DE	0.8507	**0.0392**	1.1861	**0.0710**	10.0096	**0.0002**	**0.1104**	1.1183
	ABC	0.8262	0.0806	1.3774	0.3327	10.0080	0.0006	0.4139	1.5680
250	SA	0.9347	0.0115	1.0746	0.0185	10.0039	**0.0000**	0.0300	8.0523
	GA	1.2807	1.1432	0.9730	0.0546	9.9398	0.0505	1.2484	1.1483
	PSO	0.9824	0.0948	1.0547	0.0256	9.9970	0.0024	0.1228	**0.9780**
	DE	0.9328	**0.0112**	1.0760	**0.0182**	10.0039	**0.0000**	**0.0294**	1.1146
	ABC	0.9162	0.0542	1.2356	0.1936	10.0021	0.0003	0.2481	1.5694
500	SA	0.9671	0.0046	1.0363	0.0072	10.0020	**0.0000**	0.0118	11.9227
	GA	1.2971	1.0705	0.9463	0.0456	9.9391	0.0485	1.1646	1.2685
	PSO	1.0071	0.0799	1.0226	0.0140	9.9955	0.0023	0.0962	1.1618
	DE	0.9658	**0.0044**	1.0372	**0.0071**	10.0020	**0.0000**	**0.0116**	**1.1578**
	ABC	0.9534	0.0470	1.1833	0.1521	9.9999	0.0004	0.1995	1.8515
1000	SA	0.9821	**0.0019**	1.0184	0.0031	10.0010	**0.0000**	0.0050	16.6259
	GA	1.3004	1.0254	0.9345	0.0417	9.9401	0.0450	1.1121	1.3761
	PSO	1.0189	0.0643	1.0055	0.0093	9.9952	0.0019	0.0755	**1.3333**
	DE	0.9814	**0.0019**	1.0189	**0.0030**	10.0010	**0.0000**	**0.0049**	1.3658
	ABC	0.9666	0.0439	1.1610	0.1370	9.9993	0.0002	0.1811	2.6882

Table 6. Simulation results of parameter estimations of the 3-p gamma distribution for $\theta = (0.64, 1.25, 10)$

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\mu}$			
		Mean	MSE	Mean	MSE	Mean	MSE	DEF	Comp.Time
25	SA	0.4725	0.0404	1.7167	0.4728	10.0088	**0.0003**	0.5135	7.6273
	GA	0.8282	1.1742	1.4611	0.4342	9.9544	0.0416	1.6500	0.9449
	PSO	0.6050	0.3324	1.6355	0.5118	9.9959	0.0046	0.8488	**0.6951**
	DE	0.4715	**0.0344**	1.7164	**0.4723**	10.0089	**0.0003**	**0.5070**	0.7928
	ABC	0.4607	0.0519	2.1172	1.4948	10.0088	0.0004	1.5471	1.1183
50	SA	0.5496	**0.0130**	1.4743	**0.1545**	10.0031	**0.0000**	**0.1675**	13.2066
	GA	0.8580	0.9997	1.3106	0.2169	9.9500	0.0412	1.2578	1.0308
	PSO	0.6221	0.1488	1.4221	0.1775	9.9952	0.0028	0.3291	0.8009
	DE	0.5496	**0.0130**	1.4742	**0.1545**	10.0031	**0.0000**	**0.1675**	**0.7323**
	ABC	0.5353	0.0277	1.8069	0.7885	10.0031	**0.0000**	0.8163	1.7725
100	SA	0.5977	**0.0057**	1.3489	**0.0585**	10.0011	**0.0000**	**0.0641**	14.4397
	GA	0.8855	0.9812	1.2201	0.1272	9.9489	0.0420	1.1504	1.1519
	PSO	0.6450	0.1007	1.3248	0.0760	9.9949	0.0023	0.1790	**0.8404**
	DE	0.5977	**0.0057**	1.3490	**0.0585**	10.0011	**0.0000**	0.0642	0.8528
	ABC	0.5817	0.0213	1.6523	0.5234	10.0011	**0.0000**	0.5447	2.0264
250	SA	0.6256	**0.0022**	1.2840	**0.0202**	10.0003	**0.0000**	**0.0224**	9.4667
	GA	0.8908	0.9395	1.1810	0.0886	9.9494	0.0427	1.0708	1.1487
	PSO	0.6645	0.0754	1.2611	0.0351	9.9951	0.0021	0.1126	**0.8456**
	DE	0.6256	**0.0022**	1.2839	**0.0202**	10.0003	**0.0000**	**0.0224**	1.0555
	ABC	0.6077	0.0184	1.5722	0.3947	10.0002	**0.0000**	0.4130	2.2037
500	SA	0.6336	**0.0011**	1.2661	**0.0095**	10.0001	**0.0000**	**0.0105**	15.9482
	GA	0.8895	0.8861	1.1705	0.0772	9.9502	0.0404	1.0037	1.3762
	PSO	0.6650	0.0603	1.2465	0.0226	9.9959	0.0017	0.0846	**0.8639**
	DE	0.6337	**0.0011**	1.2660	**0.0095**	10.0001	**0.0000**	0.0106	1.3105
	ABC	0.6132	0.0174	1.5558	0.3750	10.0001	**0.0000**	0.3924	2.7671
1000	SA	0.6376	**0.0006**	1.2563	**0.0047**	10.0000	**0.0000**	**0.0053**	17.3379
	GA	0.8969	0.9054	1.1621	0.0739	9.9495	0.0408	1.0201	1.7012
	PSO	0.6650	0.0603	1.2465	0.0226	9.9959	0.0017	0.0846	**1.2426**
	DE	0.6377	**0.0006**	1.2563	**0.0047**	10.0000	**0.0000**	**0.0053**	1.3528
	ABC	0.6189	0.0167	1.5316	0.3436	10.0000	**0.0000**	0.3604	3.3038

From Tables 2-6, DEF values decrease for all cases as sample size increases. In other words, more efficient parameter estimates are procured as sample size increases with respect to MSE criteria. However, it is explicitly noted that the run time of the metaheuristic methods will be longer as the increase in sample size complicates the maximizing the likelihood function more. Thus, selection of the sample size is very important for the ML estimators. In view of solution quality, the DE exhibits good performance with the lowest DEF values in most of the cases. Furthermore, the GA for small sample size ($n = 25, n = 50$) and the SA for the large sample size ($n = 500, n = 1000$) shows good performance as well as the DE. The SA produces very close results to the DE especially for sample sizes.

In terms of computation time, the PSO gives the best results with the lowest run time for almost all cases except for a few cases where the DE performed well. When compared to other methods, the SA shows very poor performance about computation time because it is based on single solution

Kruskal-Wallis (K-W) tests are also carried out to show whether there is a statistical difference between the algorithms in terms of solution quality and computation time. According to the K-W test results for solution quality of the algorithms, p-values are computed as 0.893, 0.548 ,0.131, 0.008, 0.003, and 0.002 for the sample sizes (n) of 25, 50,100, 250, 500, and 1000, respectively. There is no statistically significant difference between solution qualities of the estimated parameters of the algorithms for $n=25, n=50$ and $n=100$, whereas a statistically significant difference is found for $n=250, n=500$, and $n=1000$ at 95% confidence level. As a result of post-hoc comparison tests, significant differences exist between the DE and GA for $n=250, n=500$, and $n=1000$ and also between the SA and GA for $n=500$ and $n=1000$. According to the K-W test results for computation time of the algorithms, p-values are computed as 0.000 for all sample sizes. There are statistically significant differences between computation time of the algorithms at 95% confidence level. As a result of post-hoc comparison tests, there are significant differences between the PSO and ABC, and also between the PSO and SA for all sample sizes. Moreover, there are significant differences between the GA and SA for $n=25$, and also between the DE and SA for $n=50, n=100$, and $n=1000$ with respect to computation time.

3.2. Comparison of the Metaheuristic Methods via TOPSIS Method

TOPSIS method is performed for ranking the suggested metaheuristic methods according to their performance in the ML parameter estimation of the 3-p Gamma distribution for determining the best method among them. For this aim, suggested metaheuristic methods of SA, GA, PSO, DE, and ABC are taken as alternatives. In addition, solution quality (C_1), computation time (C_2), simplicity (C_3), and robustness (C_4) of these methods are considered as criteria.

Solution quality, simplicity and robustness criteria are evaluated as benefit criteria, while computation time is assessed as cost criterion. Solution quality is obtained as to be inversely proportional to means of the DEF values. Computation time is also calculated by

taking means of run time in all simulations conducted in this study. Simplicity of the method is considered as inversely proportional to number of its parameters. For example, simplicity of the SA is $1/4$ since it has four parameters (s, T_0, nMove, and τ). Finally, robustness of the method is determined as to be inversely proportional to variation (variance) of the DEF values obtained for different samples.

Since the criteria are considered as equally important, weights of the criteria are taken as 0.25. According to the solution quality, simplicity, robustness, and computation time of the suggested metaheuristic methods, decision matrix (D) is formed as follows.

$$D = \begin{array}{c} \\ SA \\ GA \\ PSO \\ DE \\ ABC \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.6189 & 12.3571 & 0.2500 & 34.5414 \\ 0.5687 & 1.2042 & 0.2500 & 26.5129 \\ 0.6691 & 0.8719 & 0.2500 & 13.6154 \\ 1.0186 & 1.1788 & 0.3333 & 27.4304 \\ 0.2717 & 2.2287 & 0.5000 & 0.1091 \end{bmatrix} \end{array}$$

Positive ideal solution nd negative ideal solution are found as $A^+ = (0.1692, 0.1720, 1.6880, 1.6222)$ and $A^- = (0.0451, 0.2433, 0.0844, 0.0005)$, respectively. Also, calculated d_i^+, $d_i^-$and CC_ivalues are presented in Table 7.

Table 7. Calculated d_i^+, $d_i^-$and CC_ivalues for the alternatives

Alternatives	d_i^+	d_i^-	CC_i
SA	0.0627	0.0295	0.3199
GA	0.0142	0.0660	0.8233
PSO	0.0201	0.0595	0.7471
DE	0.0043	0.0811	0.9495
ABC	0.0423	0.0469	0.5260

According to Table 7, ranking of the metaheuristic methods is obtained as $DE > GA > PSO > ABC > SA$ due to descending order of CC_i values. The TOPSIS method allows to rank five suggested metaheuristic methods considering all criteria, and as a result, the best method among the alternatives is determined as the DE. When evaluations are performed separately for each criterion in the decision matrix D, ranking of the alternatives are obtained as $DE > PSO > SA > GA > ABC$ for solution quality, $PSO > DE > GA > ABC > SA$ for computation time, $ABC > DE > SA = GA = PSO$ for simplicity, and $SA > DE > GA > PSO > ABC$ for robustness.

3.3. Real Life Example

In this part, a real-life data given by Dumonceaux and Antle (1973) is considered. The data known as flood data represent the maximum flood levels of Susquehanna River at Harrisburg, Pennsylvania over four periods years between 1969 and 1980 in millions of cubic feet per second and given as: 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.3235, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265 (Balakrishnan and Wang, 2000; Lakshmi and Vaidyanathan, 2016; Vaidyanathan and Lakshmi, 2015).

As well as the value of log-likelihood function ($\log L$), Akaike Information Criteria (AIC) (Akaike et al., 1973) and Bayesian Information Criteria (BIC) (Stone, 1979) are used for comparing the methods.

The ML estimates of the parameters, log-likelihood values, AIC and BIC values corresponding to MLE_P method suggested by Lakshmi and Vaidyanathan (2016) and MML method proposed by Cohen and Whitten (1986) in addition to the suggested metaheuristic methods in this study, are given in Table 8.

Table 8. Parameter estimates, $\log L$, AIC and BIC for flood data

Methods	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$	$\log L$	AIC	BIC
SA	1.3865	0.1180	0.2595	16.8010	-27.6021	-15.6276
GA	1.3638	0.1195	0.2600	16.8009	-27.6018	-15.6274
PSO	1.7143	0.1020	0.2519	16.7684	-27.5368	-15.5624
DE	0.9681	0.1658	0.2646	**16.8371**	**-27.6742**	**-15.6998**
ABC	1.3972	0.1299	0.2417	15.8753	-25.7506	-13.7762
MLE_P	1.3700	0.1300	0.2700	16.8190	-27.6380	-15.6636
MML	1.1940	0.1343	0.2628	16.7971	-27.5942	-15.6198

As seen from Table 8, all the suggested metaheuristic methods have very close AIC and BIC values except for the ABC algorithm. However, the DE algorithm has the biggest $\log L$ and the smallest AIC and BIC values among not only metaheuristic methods but also MLE_P and MML methods. Therefore, it can be concluded that the DE algorithm shows better performance than the others for ML estimates of the parameters for flood data.

4. CONCLUSION

In this study, some metaheuristic methods, SA, GA, PSO, DE and ABC, are proposed to find ML estimates of the parameters for 3-p Gamma distribution. For examining the efficiencies of the estimates obtained by using the metaheuristic methods, Monte-Carlo

simulation studies are conducted, and the algorithms are compared with respect to the DEF criteria for solution quality and computation time. Simulation results show that the DE is more efficient method in terms of solution quality and the PSO is more efficient method regarding computation time in almost all cases when compared to other metaheuristic methods.

TOPSIS method is also suggested for ranking the metaheuristic methods regarding all the criteria of solution quality, computation time, simplicity, and robustness. According to the results obtained from the TOPSIS method, the DE is determined as the best method among the suggested metaheuristic methods considering all criteria.

A real-life data is analyzed to show applicability of the metaheuristic methods. The superiority of the DE method is also seen in this real-life example. Finally, it could be said that the DE gives the most effective results for ML estimating parameters of the 3-p Gamma distribution. Therefore, it may be preferred more than other metaheuristic methods.

In future studies, it can be suggested to obtain optimal solutions using various metaheuristic optimization algorithms for parameter estimation of 3-p Gamma distribution via the Maximum Product Range method proposed by Ranneby (1984), which generalizes the ML Estimation.

ETHICAL DECLARATION

In the writing process of the study titled "Evaluation and Comparison of Metaheuristic Methods to Estimate the Parameters of Gamma Distribution", there were followed the scientific, ethical and the citation rules; was not made any falsification on the collected data and this study was not sent to any other academic media for evaluation.

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