Journal of New Theory
ISSN: 2149-1402

39 (2022) 8-18
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

Smarandache-Based Ruled Surfaces with the Darboux Vector According to Frenet Frame in E^3

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Article Info
Received: 20 Apr 2022
Accepted: 27 Jun 2022
Published: 30 Jun 2022
doi:10.53570/jnt.1106331
Research Article

Abstract — The paper introduces a new kind of special ruled surface. The base of each ruled surface is taken to be one of the Smarandache curves of a given curve according to Frenet frame, and the generator (ruling) is chosen to be the corresponding unit Darboux vector. The characteristics of these newly defined ruled surfaces are investigated by means of first and second fundamental forms and their corresponding curvatures. An example is provided by considering both the helix curve and the Viviani's curve.

Keywords — *Ruled surfaces, Smarandache curve, fundamental forms, curvatures, developable and minimal surfaces*

Mathematics Subject Classification (2020) — 53A04, 53A05

1. Introduction

A surface is defined as an image of a function with two real valued variables (domain) by a mapping to two or three dimensional space. The surfaces are characterized by means of their curvatures and accordingly they are used in many areas such as engineering, architectural designs, computer graphics, automobile industry, etc. Researches on surface curvature went through various stages starting from Ancient Greece, and gained momentum with the calculations developed by Newton and Leibniz in the 17th century after the studies of Descartes, Kepler, Fermat and Huygens. The curvature theory for curves and surfaces is an important subject of differential geometry. The theory was first introduced by Gauss in 19th century and it was named by his name as the Gaussian curvature. It is related to the dimensions of the surface. The developability of a surface can be determined by its Gaussian curvature. A surface with zero Gaussian curvature at every point is known as a developable surface. Another kind of curvature for a surface is named as mean curvature. Since a mean curvature corresponds to a ratio, it is independent of the size of the surfaces. Surfaces with a mean curvature of zero at every point are known as minimal surfaces. In the theory of surfaces, there is a special kind that is known as ruled surfaces. A ruled surface is constructed by infinitely many straight lines moving along a given curve. Among other types, the ruled surfaces are mostly referred in computer based geometric designs to deal with real world problems on modeling the real objects. For this reason, introducing new ruled surfaces may lead new potentials to the related fields. Providing their characteristics by means of curvatures may also enable easy adaptations for interested researchers.

The basic theory related to ruled surfaces can be found in many differential geometry textbooks such as [1–4]. However, the generalization of those was first studied by Juza in the 1960s [5]. The ruled surfaces with rulings of Frenet vectors are already covered in textbooks. Apart from Frenet frame,

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Şentürk and Yüce (2015) examined the characterizations of ruled surfaces with Darboux frame in [6]. Tunçer (2015) in [7] and Masal and Azak (2019) in [8], separately, studied some characteristics of the ruled surfaces according to Bishop frame (introduced by Bishop, (1975) in [9]), whereas Ouarab et al. (2018) provided the main properties of ruled surfaces according to alternative frame in [10]. Moreover, some characteristics of the ruled surface with Frenet frame of a non-cylindrical one were investigated in 2020 by Ouarab and Chahdi in [11].

Recently, Ouarab, (2021a) put forth a method to generate new ruled surfaces by taking the advantage of the idea of Smarandache geometry introduced in [12, 13]. By assigning the base curve as one of the Smarandache curves and taking the generator as the another vector element of Frenet frame, she introduced these ruled surfaces as Smarandache ruled surfaces according to Frenet frame in [14]. The same method of generating such ruled surfaces is applied to the Darboux frame by Ouarab, (2021b) in [15] and according to the alternative frame by Ouarab, (2021c) in [16].

Motivated by these studies and acknowledging the great potential use of ruled surfaces, it is of interest for us to define and introduce new kinds of ruled surfaces incorporated with the Darboux vector and Smarandache curves. The geometric properties of these have been examined by means of fundamental forms and the corresponding curvatures.

2. Preliminaries

In this section, we recall some basic notions of which we refer through out the paper. Let $\alpha : I \to E^3$ be a regular unit speed curve. We define the quantities of the Frenet apparatus and Frenet formulae as in the following way:

$$
\begin{aligned}
&T = \alpha', \;\; N = \frac{\alpha''}{\|\alpha''\|}, \;\; B = T \wedge N, \;\; \kappa = \left\|\alpha''\right\|, \;\; \tau = \left\langle N', B \right\rangle \\
&T' = \kappa N, \;\; N' = -\kappa T + \tau B, \;\; B' = -\tau N
\end{aligned}
$$

The Darboux vector W of Frenet frame is defined as $W = \tau T + \kappa B$. Corresponding to this, the unit Darboux vector is

$$
\begin{aligned}
&C = \sin\omega\, T + \cos\omega B \\
&\cos\omega = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \quad \sin\omega = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}, \quad \omega' = \left(\frac{\tau}{\kappa}\right)'\left(1+\frac{\tau^2}{\kappa^2}\right)
\end{aligned} \tag{1}
$$

where $\omega = \angle(B, W)$. Moreover, a unit vector with a linear combination of Frenet vectors can be defined as

$$
\gamma = \frac{fT + gN + hB}{\sqrt{f^2+g^2+h^2}} \tag{2}
$$

where $f, \; g, \; h : \Re \to \Re$ are some arbitrary functions. For $\forall s \in I \subset \Re$, the image of the vector γ is a special differentiable curve. If, specifically, each function $f, \; g$ and h is considered to be a constant valued function and the vector γ is taken to be the position vector, then the generated curves are known to be the special Smarandache curves [12]. These curves were outlined in many studies by using different frames and considering different spaces, as well [4, 13, 17–23].

On the other hand, a surface is said to be ruled if it is formed with a straight line $r(s)$ sharing the same parameter of the given curve α. The parametric form of a ruled surface is as following:

$$
X(s, v) = \alpha(s) + v r(s) \tag{3}
$$

The unit normal vector field of a ruled surface $X(s, v)$ is computed as

$$
\mathrm{N}_X = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|} \tag{4}
$$

and are the curvatures as:

$$
K = -\frac{f^2}{EG-F^2} \quad \text{and} \quad H = \frac{eG - 2fF}{2\left(EG-F^2\right)} \tag{5}
$$

where the corresponding coefficients are defined by

$$\begin{aligned} E &= \langle X_s, X_s \rangle, \quad F = \langle X_s, X_v \rangle, \quad \text{and} \quad G = \langle X_v, X_v \rangle \\ e &= \langle X_{ss}, N_X \rangle, \quad f = \langle X_{sv}, N_X \rangle, \quad \text{and} \quad g = \langle X_{vv}, N_X \rangle \end{aligned} \tag{6}$$

3. Smarandache ruled surfaces by Darboux vector according to Frenet frame in $\mathbf{E}^3$

In this section, we define and examine some special ruled surfaces where the base curve is considered to be one of the Smarandache curves of $\alpha = \alpha(s)$ which we define them by referring the equation (2) as:

if $h = 0$ and $f = g = 1$ $\Rightarrow$ the vector $\gamma = \dfrac{T+N}{\sqrt{2}}$ draws the TN-Smarandache curve that is $\gamma_1 = \dfrac{T+N}{\sqrt{2}}$

if $g = 0$ and $f = h = 1$ $\Rightarrow$ the vector $\gamma = \dfrac{T+B}{\sqrt{2}}$ draws the TB-Smarandache curve that is $\gamma_2 = \dfrac{T+B}{\sqrt{2}}$

if $f = 0$ and $g = h = 1$ $\Rightarrow$ the vector $\gamma = \dfrac{N+B}{\sqrt{2}}$ draws the NB-Smarandache curve that is $\gamma_3 = \dfrac{N+B}{\sqrt{2}}$

if $f = g = h = 1$ $\Rightarrow$ the vector $\gamma = \dfrac{T+N+B}{\sqrt{3}}$ draws the TNB-Smarandache curve that is $\gamma_4 = \dfrac{T+N+B}{\sqrt{3}}$

and the generator for each ruled surface is taken to be the unit Darboux vector C given by the relation (1).

Definition 3.1. The ruled surface generated by unit Darboux vector C along the TN- Smarandache curve γ_1 is given as

$$\Phi(s,v) = \frac{1}{\sqrt{2}}(T+N) + vC$$

The first and second partial derivatives of the surface Φ are given in respective order as follows:

$$\begin{cases} \Phi_s = \dfrac{1}{\sqrt{2}}\left(\left(-\kappa + \sqrt{2}v\omega' \cos\omega\right)T + \kappa N + \left(\tau - \sqrt{2}v\omega' \sin\omega\right)B\right) \\ \Phi_{ss} = \dfrac{1}{\sqrt{2}}\left\{ \begin{array}{l} \left(\left(-\kappa + \sqrt{2}v\omega' \cos\omega\right)' - \kappa^2\right)T + \left(\left(\tau - \sqrt{2}v\omega' \sin\omega\right)' + \kappa\tau\right)B \\ + \left(-\left(\kappa^2 + \tau^2\right) + v\omega'\sqrt{2\kappa^2 + 2\tau^2}\right)N \end{array} \right\} \\ \Phi_{sv} = \omega'\left(\cos\omega T - \sin\omega B\right), \quad \Phi_v = \sin\omega T + \cos\omega B, \quad \Phi_{vv} = 0 \end{cases}$$

By considering (4), we first compute

$$\Phi_s \wedge \Phi_v = \frac{1}{\sqrt{2}}\left(\kappa\cos\omega T + \left(\sqrt{\kappa^2+\tau^2} - \sqrt{2}v\omega'\right)N - \kappa\sin\omega B\right)$$

$$\|\Phi_s \wedge \Phi_v\| = \frac{1}{\sqrt{2}}\sqrt{2\kappa^2 + \tau^2 - v\omega'\sqrt{2\kappa^2+2\tau^2} + 2(v\omega')^2}$$

to provide the unit normal vector as

$$N_\Phi = \frac{\kappa\cos\omega T + \left(\sqrt{\kappa^2+\tau^2} - \sqrt{2}v\omega'\right)N - \kappa\sin\omega B}{\sqrt{2\kappa^2 + \tau^2 - v\omega'\sqrt{2\kappa^2+2\tau^2} + 2(v\omega')^2}}$$

From (6), the coefficients of the first and second fundamental forms are

$$E_\Phi = \frac{1}{2}\left(2\kappa^2+\tau^2-2\sqrt{2}v\omega'\sqrt{\kappa^2+\tau^2}+2\big(v\omega'\big)^2\right), \quad F_\Phi=0, \quad G_\Phi=1$$

$$e_\Phi = \frac{\begin{array}{l}-\kappa\Big(\sqrt{\kappa^2+\tau^2}\Big)'-\kappa\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}+\sqrt{2}v\kappa\left(\omega''+2\omega'\left(\kappa^2+\tau^2\right)\right)\\ -2v^2\omega'^2\kappa\sqrt{\kappa^2+\tau^2}\end{array}}{\sqrt{2}\sqrt{2\kappa^2+\tau^2-v\omega'\sqrt{2\kappa^2+2\tau^2}+2(v\omega')^2}}$$

$$f_\Phi = \frac{\omega'\kappa}{\sqrt{2\kappa^2+\tau^2-v\omega'\sqrt{2\kappa^2+2\tau^2}+2(v\omega')^2}}, \quad g_\Phi=0$$

Thus, by using (5) we compute the mean and Gaussian curvatures as following:

$$K_\Phi = -2\left(\frac{\omega'\kappa}{2\kappa^2+\tau^2-v\omega'2\sqrt{2}\sqrt{\kappa^2+\tau^2}+2(v\omega')^2}\right)^2$$

$$H_\Phi = \frac{\begin{array}{l}-\kappa\Big(\sqrt{\kappa^2+\tau^2}\Big)'-\kappa\big(\kappa^2+\tau^2\big)^{\frac{3}{2}}+\sqrt{2}v\kappa\left(\omega''+2\omega'\left(\kappa^2+\tau^2\right)\right)\\ -2v^2\omega'^2\kappa\sqrt{\kappa^2+\tau^2}\end{array}}{\sqrt{2}\Big(2\kappa^2+\tau^2-v\omega'2\sqrt{2}\sqrt{\kappa^2+\tau^2}+2(v\omega')^2\Big)^{\frac{3}{2}}}$$

Corollary 3.2. If the curve $\alpha=\alpha(s)$ is planar or general helix, then the surface Φ is developable.

PROOF. From (1), $\omega' = \left(\frac{\tau}{\kappa}\right)'\left(1+\frac{\tau^2}{\kappa^2}\right)$. If α is either planar or general helix, then $\omega'=0$ which corresponds to that $K_\Phi=0$. □

Definition 3.3. Another ruled surface generated by unit Darboux vector C along the TB- Smarandache curve γ_2 is given as

$$\Psi(s,v) = \frac{1}{\sqrt{2}}(T+B)+vC$$

The first and second partial derivatives of $\Psi(s,v)$ are given in respective order as follows:

$$\begin{cases} \Psi_s = v\omega'\cos\omega T+\dfrac{\kappa-\tau}{\sqrt{2}}N-v\omega'\sin\omega B \\ \Psi_{ss} = \left(\big(v\omega'\cos\omega\big)'-\dfrac{\kappa^2-\kappa\tau}{\sqrt{2}}\right)T+\left(v\omega'\sqrt{\kappa^2+\tau^2}+\dfrac{\kappa'-\tau'}{\sqrt{2}}\right)N \\ \qquad +\left(\big(-v\omega'\sin\omega\big)'+\dfrac{\kappa\tau-\tau^2}{\sqrt{2}}\right)B \\ \Psi_{sv} = \omega'\cos\omega T-\omega'\sin\omega B, \quad \Psi_v=\sin\omega T+\cos\omega B, \quad \Psi_{vv}=0 \end{cases}$$

By considering (4), we first compute

$$\Psi_s\wedge\Psi_v = \frac{(\kappa-\tau)\cos\omega}{\sqrt{2}}T-v\omega'N-\frac{(\kappa-\tau)\sin\omega}{\sqrt{2}}B$$

$$\|\Psi_s\wedge\Psi_v\| = \frac{1}{\sqrt{2}}\sqrt{(\kappa-\tau)^2+2(v\omega')^2}$$

to get the unit normal vector denoted by N_Φ as

$$N_\Psi = \frac{(\kappa-\tau)\cos\omega T-\sqrt{2}v\omega'N-(\kappa-\tau)\sin\omega B}{\sqrt{(\kappa-\tau)^2+2(v\omega')^2}}$$

Next, by using (6), the coefficients of the first and second fundamental forms are

$$E_\Psi = \frac{1}{2}\left((\kappa-\tau)^2 + 2\left(v\omega'\right)^2\right), \quad F_\Psi = 0, \quad G_\Psi = 1$$

$$e_\Psi = \frac{\sqrt{2}(v\omega')'\,(\kappa-\tau) - \sqrt{\kappa^2+\tau^2}\left(2(v\omega')^2 + (\kappa-\tau)^2\right) - \sqrt{2}v\omega'\,(\kappa'-\tau')}{\sqrt{2}\sqrt{(\kappa-\tau)^2 + 2(v\omega')^2}}$$

$$f_\Psi = \frac{\omega'\,(\kappa-\tau)}{\sqrt{(\kappa-\tau)^2 + 2(v\omega')^2}}, \quad g_\Psi = 0.$$

Thus, from (5) the mean and Gaussian curvatures can be written as in the following way:

$$K_\Psi = -2\left(\frac{\omega'\,(\kappa-\tau)}{(\kappa-\tau)^2 + 2v^2\omega'^2}\right)^2, \qquad \kappa-\tau \neq 0,$$

$$H_\Psi = \frac{\sqrt{2}\,(\kappa-\tau)\,(v\omega')' - \sqrt{\kappa^2+\tau^2}\left(2(v\omega')^2 + (\kappa-\tau)^2\right) - \sqrt{2}v\omega'\,(\kappa'-\tau')}{\sqrt{2}\left((\kappa-\tau)^2 + 2(v\omega')^2\right)^{\frac{3}{2}}}$$

Corollary 3.4. If the curve $\alpha = \alpha(s)$ is planar or general helix, then the surface Ψ is developable.

Proof. The proof is as same as of the proof of Corollary 3.2 □

Definition 3.5. The ruled surface generated by unit Darboux vector C along the NB- Smarandache curve γ_3 is given as

$$\Omega\,(s,v) = \frac{1}{\sqrt{2}}\,(N+B) + vC$$

The first and second partial derivatives of $\Psi(s,v)$ are given in respective order as follows:

$$\begin{cases} \Omega_s = \left(-\dfrac{\kappa}{\sqrt{2}} + v\omega'\cos\omega\right)T - \dfrac{\tau}{\sqrt{2}}N + \left(\dfrac{\tau}{\sqrt{2}} - v\omega'\sin\omega\right)B \\ \Omega_{ss} = \dfrac{1}{\sqrt{2}}\left(\begin{array}{l} \left(\kappa\tau - \kappa' + \sqrt{2}\left(v\omega'\cos\omega\right)'\right)T + \left(\sqrt{2}v\omega'\sqrt{\kappa^2+\tau^2} - \tau' - \kappa^2 - \tau^2\right)N \\ + \left(\tau' - \tau^2 - \sqrt{2}\left(v\omega'\sin\omega\right)'\right)B \end{array}\right) \\ \Omega_{sv} = \omega'\cos\omega T - \omega'\sin\omega B, \quad \Omega_v = \sin\omega T + \cos\omega B, \quad \Omega_{vv} = 0 \end{cases}$$

By considering (4), we first compute

$$\Omega_s \wedge \Omega_v = \frac{1}{\sqrt{2}}\left(-\tau\cos\omega T + \left(\sqrt{\kappa^2+\tau^2} - \sqrt{2}v\omega'\right)N + \tau\sin\omega B\right)$$

$$\|\Omega_s \wedge \Omega_v\| = \frac{1}{\sqrt{2}}\sqrt{\kappa^2 + 2\tau^2 - v\omega'\sqrt{2\kappa^2+2\tau^2} + 2(v\omega')^2}$$

to get the unit normal vector denoted by N_Φ as

$$N_\Omega = \frac{-\tau\cos\omega T + \left(\sqrt{\kappa^2+\tau^2} - \sqrt{2}v\omega'\right)N + \tau\sin\omega B}{\sqrt{\kappa^2 + 2\tau^2 - v\omega'\sqrt{2\kappa^2+2\tau^2} + 2(v\omega')^2}}$$

Next, by using (6), the coefficients of the first and second fundamental forms are

$$E_\Omega = \frac{\kappa^2+2\tau^2}{2} + \left(v\omega'\right)^2 - \sqrt{2\kappa^2+2\tau^2}v\omega', \quad F_\Omega = 0, \quad G_\Omega = 1$$

$$e_\Omega = \frac{\begin{array}{c}\tau\left(\sqrt{\kappa^2+\tau^2}-\sqrt{2}v\omega'\right)' - \tau^2\sqrt{\kappa^2+\tau^2} \\ +\left(\sqrt{\kappa^2+\tau^2}-\sqrt{2}v\omega'\right)\left(v\omega'\sqrt{2}\sqrt{\kappa^2+\tau^2}-\tau'-\kappa^2-\tau^2\right)\end{array}}{\sqrt{2}\sqrt{\kappa^2+2\tau^2-v\omega'\sqrt{2\kappa^2+2\tau^2}+2(v\omega')^2}}$$

$$f_\Omega = -\frac{\omega'\tau}{\sqrt{\kappa^2+2\tau^2-v\omega'\sqrt{2\kappa^2+2\tau^2}+2(v\omega')^2}}$$

Thus, from (5) the mean and Gaussian curvatures can be written as in the following way:

$$K_\Omega = -2\left(\frac{\omega'\tau}{\kappa^2+2\tau^2+2(v\omega')^2 - v\omega'2\sqrt{2\kappa^2+2\tau^2}}\right)^2$$

$$H_\Omega = \frac{\begin{array}{c}\tau\left(\sqrt{\kappa^2+\tau^2}-\sqrt{2}v\omega'\right)' - \tau^2\sqrt{\kappa^2+\tau^2} \\ +\left(\sqrt{\kappa^2+\tau^2}-\sqrt{2}v\omega'\right)\left(v\omega'\sqrt{2}\sqrt{\kappa^2+\tau^2}-\tau'-\kappa^2-\tau^2\right)\end{array}}{\sqrt{2}\left(\kappa^2+2\tau^2+2(v\omega')^2-\sqrt{2\kappa^2+2\tau^2}v\omega'\right)^{\frac{3}{2}}}$$

Corollary 3.6. If the curve $\alpha = \alpha(s)$ is planar or general helix, then the surface Ω is developable.

PROOF. The proof is the same as of the previous proofs. □

Definition 3.7. The ruled surface generated by unit Darboux vector C along the TNB- Smarandache curve γ_4 is given as:

$$\xi(s,v) = \frac{1}{\sqrt{3}}(T+N+B) + vC$$

The first and second partial derivatives of $\Psi(s,v)$ are given in respective order as follows:

$$\begin{cases} \xi_s = \left(-\frac{\kappa}{\sqrt{3}} + v\omega'\cos\omega\right)T + \frac{\kappa-\tau}{\sqrt{3}}N + \left(\frac{\tau}{\sqrt{3}} - v\omega'\sin\omega\right)B \\ \xi_{ss} = \begin{array}{l}\left(\left(-\frac{\kappa}{\sqrt{3}} + v\omega'\cos\omega\right)' - \frac{\kappa^2-\kappa\tau}{\sqrt{3}}\right)T + \left(-\frac{\kappa^2+\tau^2-\kappa'+\tau'}{\sqrt{3}} + v\omega'\sqrt{\kappa^2+\tau^2}\right)N \\ +\left(\left(\frac{\tau}{\sqrt{3}} - v\omega'\sin\omega\right)' + \frac{\kappa\tau-\tau^2}{\sqrt{3}}\right)B\end{array} \\ \xi_{sv} = \omega'\cos\omega T - \omega'\sin\omega B, \quad \xi_v = \sin\omega T + \cos\omega B, \quad \xi_{vv} = 0 \end{cases}$$

By considering (4), we first compute

$$\xi_s \wedge \xi_v = \frac{1}{\sqrt{3}}\left((\kappa-\tau)\cos\omega T + \left(\sqrt{\kappa^2+\tau^2}-\sqrt{3}v\omega'\right)N - (\kappa-\tau)\sin\omega B\right)$$

$$\|\xi_s \wedge \xi_v\| = \frac{1}{\sqrt{3}}\sqrt{(\kappa-\tau)^2 + \left(\sqrt{\kappa^2+\tau^2}-\sqrt{3}v\omega'\right)^2}$$

to get the unit normal vector denoted by N_Φ as

$$N_\xi = \frac{(\kappa-\tau)\cos\omega T + \left(\sqrt{\kappa^2+\tau^2}-\sqrt{3}v\omega'\right)N - (\kappa-\tau)\sin\omega B}{\sqrt{(\kappa-\tau)^2 + \left(\sqrt{\kappa^2+\tau^2}-\sqrt{3}v\omega'\right)^2}}$$

Next, by using (6), the coefficients of the first and second fundamental forms are

$$E_\xi = \frac{1}{3}\left((\kappa-\tau)^2 + \left(\sqrt{\kappa^2+\tau^2} - \sqrt{3}v\omega'\right)^2\right), \quad F_\xi = 0, \quad G_\xi = 1$$

$$e_\xi = \frac{\begin{array}{c}\left(\kappa'-\tau'-\kappa^2-\tau^2+v\omega'\sqrt{3}\sqrt{\kappa^2+\tau^2}\right)\left(\kappa^2+\tau^2-v\omega'\sqrt{3}\sqrt{\kappa^2+\tau^2}\right)\\ +(\kappa-\tau)\left(\sqrt{3}v\omega'-\sqrt{\kappa^2+\tau^2}\right)'-(\kappa-\tau)^2\sqrt{\kappa^2+\tau^2}\end{array}}{\sqrt{3}\sqrt{(\kappa-\tau)^2+\left(\sqrt{\kappa^2+\tau^2}-\sqrt{3}v\omega'\right)^2}}$$

$$f_\xi = \frac{\omega'(\kappa-\tau)}{\sqrt{(\kappa-\tau)^2+\left(\kappa^2+\tau^2-\sqrt{3}v\omega'\right)^2}}, \quad g_\xi = 0$$

Thus, from (5) the mean and Gaussian curvatures can be written as in the following way:

$$K_\xi = -3\left(\frac{\omega'(\kappa-\tau)}{(\kappa-\tau)^2+\left(\sqrt{\kappa^2+\tau^2}-\sqrt{3}v\omega'\right)^2}\right)^2$$

$$H_\xi = \frac{\begin{array}{c}\left((\kappa-\tau)'-\left(\kappa^2+\tau^2\right)+v\omega'\sqrt{3\kappa^2+3\tau^2}\right)\left(\kappa^2+\tau^2-v\omega'\sqrt{3\kappa^2+3\tau^2}\right)\\ +(\kappa-\tau)\left(\sqrt{3}v\omega'-\sqrt{\kappa^2+\tau^2}\right)'-(\kappa-\tau)^2\sqrt{\kappa^2+\tau^2}\end{array}}{2\left(\sqrt{3}\right)^{-1}\left((\kappa-\tau)^2+\left(\sqrt{\kappa^2+\tau^2}-\sqrt{3}v\omega'\right)^2\right)^{\frac{3}{2}}}$$

Corollary 3.8. If the curve $\alpha = \alpha(s)$ is planar or general helix, then the surface ξ is developable.

Proof. The proof is the same as of the previous proofs. □

Example 3.9. Let us consider the well known Viviani's curve parameterized as

$$\gamma(t) = \left(a(1+cost), asint, 2asin\frac{1}{2}t\right), \quad t \in [-2\pi, 2\pi], \qquad [2]$$

For $a = 0.5$ and by changing the parameter as $t = 2s$, we easily represent the given Viviani's curve as in the following way

$$\alpha(s) = \left(\cos^2(s), \sin(s)\cos(s), \sin(s)\right) \quad s \in [-\pi, \pi]$$

Then, the Frenet apparatus of $\alpha = \alpha(s)$ are given as

$$T(s) = \frac{2}{\sqrt{2\cos(2s)+6}}\left(-\sin(2s), \cos(2s), \cos(s)\right)$$

$$N(s) = \frac{-1}{\sqrt{2\cos(2s)+6}\sqrt{6\cos(2s)+26}}\begin{pmatrix}\cos(4s)+12\cos(2s)+3\\ \sin(4s)+12\sin(2s)\\ 4\sin(s)\end{pmatrix}$$

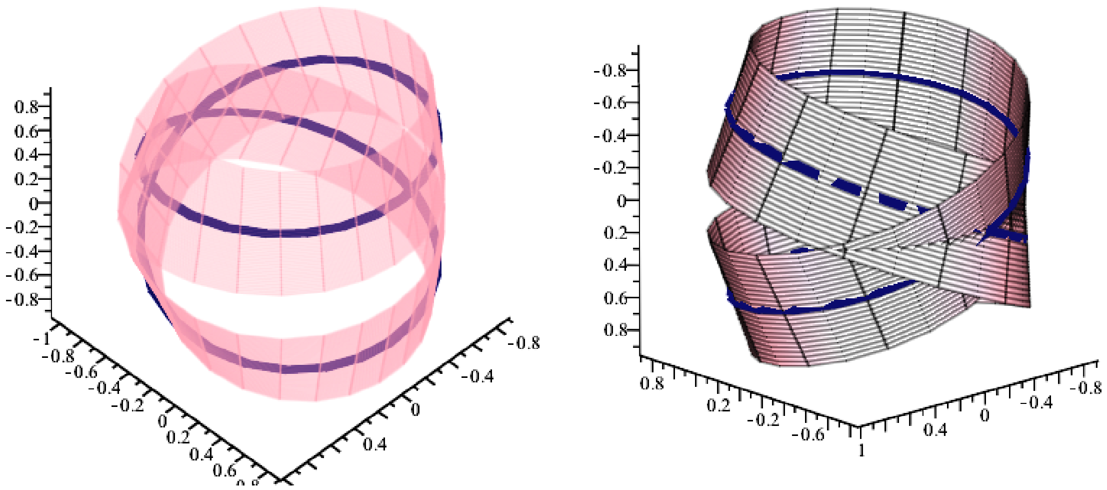
$$B(s) = \frac{1}{\sqrt{6\cos(2s)+26}}\left(\sin(3s)+3\sin(s), -\cos(3s)-3\cos(s), 4\right)$$

$$\kappa = \frac{2\sqrt{3cos(2s)+13}}{(3+cos(2s))^{\frac{3}{2}}} \quad \text{and} \quad \tau = \frac{12cos(s)}{3cos(2s)+13}$$

and the unit Darboux vector is found by

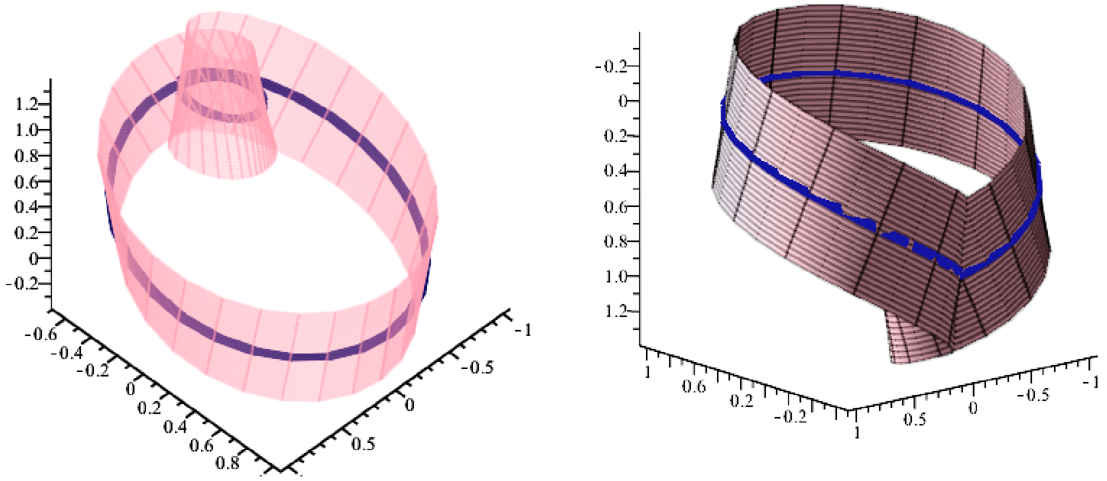
$$C(s) = \begin{pmatrix} \dfrac{2\sqrt{2}\sin(s)^3\left(6\cos(s)^2+5\right)}{\sqrt{18\cos(2s)^4+207\cos(2s)^3+999\cos(2s)^2+2493\cos(2s)+2683}} \\ \dfrac{4\sqrt{2}\cos(s)\left(3\cos(s)^4-2\cos(s)^2-3\right)}{\sqrt{18\cos(2s)^4+207\cos(2s)^3+999\cos(2s)^2+2493\cos(2s)+2683}} \\ \dfrac{\sqrt{2}\left(3\cos(2s)^2+18\cos(2s)+35\right)}{\sqrt{18\cos(2s)^4+207\cos(2s)^3+999\cos(2s)^2+2493\cos(2s)+2683}} \end{pmatrix}$$

The figures of each ruled surface for $-0.5 \leq v \leq 0.5$ and $-\pi \leq s \leq \pi$ is presented in the following:



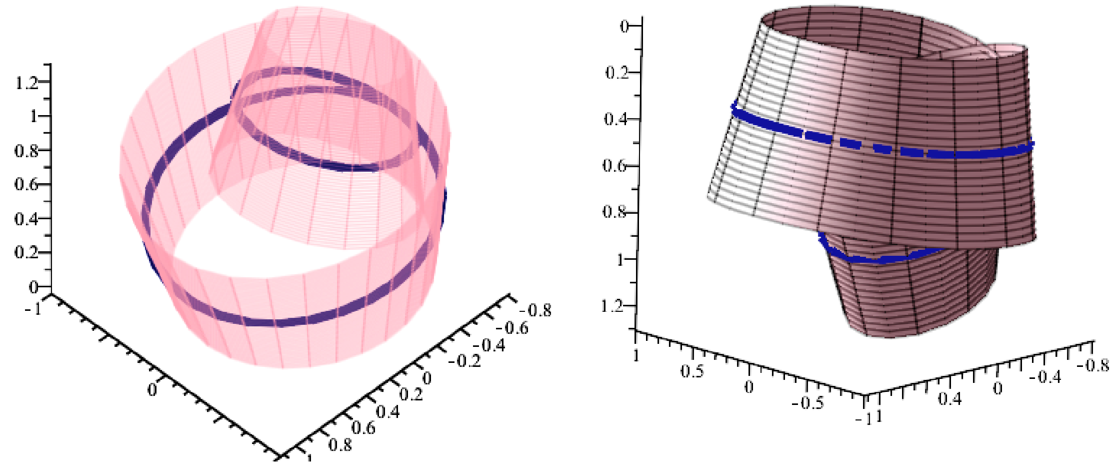
(a) Transparent (default view) **(b)** Solid (oriented)

Fig. 1. The ruled surface $\Phi(s, v)$ from different orientations



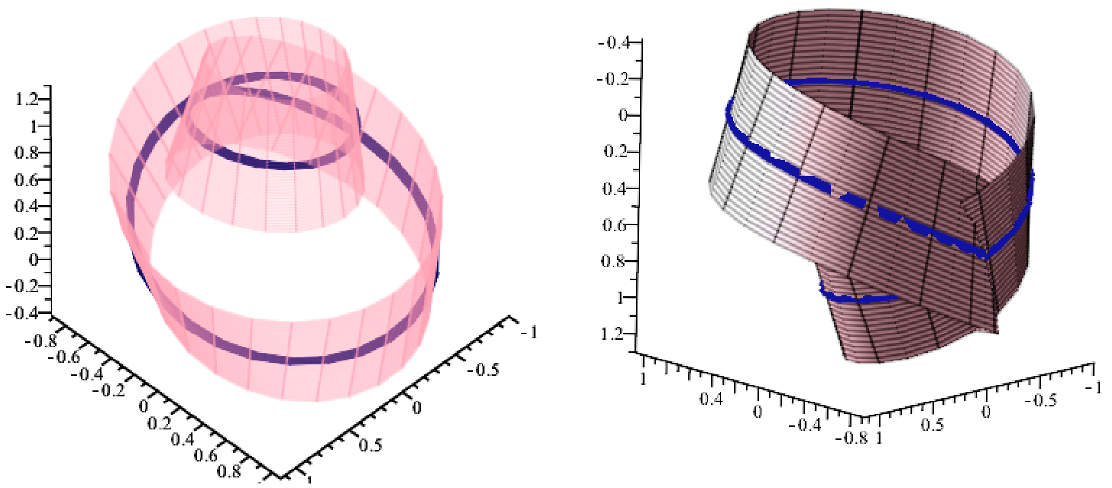
(a) Transparent (default view) **(b)** Solid (oriented)

Fig. 2. The ruled surface $\Psi(s, v)$ from different orientations



(a) Transparent (default view) **(b)** Solid (oriented)

Fig. 3. The ruled surface $\Omega(s, v)$ from different orientations



(a) Transparent (default view) **(b)** Solid (oriented)

Fig. 4. The ruled surface $\xi(s, v)$ from different orientations

Example 3.10. To generate a developable surface, we consider this time the regular unit speed circular helix parameterized as $\beta = \frac{1}{\sqrt{2}}\Big(\cos(s), \sin(s), s\Big)$.
The Frenet apparatus and the corresponding unit Darboux vector are given in following:

$$T(s) = \frac{1}{\sqrt{2}}(-\sin(s), \cos(s), 1) \quad \text{and} \quad N(s) = (-\cos(s), -\sin(s), 0)$$
$$B(s) = \frac{1}{\sqrt{2}}(\sin(s), -\cos(s), 1), \quad \kappa(s) = \tau(s) = \frac{1}{\sqrt{2}}, \quad \text{and} \quad C(s) = (0, 0, 1)$$

The figure (5) given below is the image of Φ, Ω, and ξ which simply corresponds to cylinder as a developable ruled surface for $-0.5 \leq v \leq 0.5$ and $-\pi \leq s \leq \pi$. However, the image of Ψ is simply a line the space, because the TB- Smarandache curve corresponds to a single point.

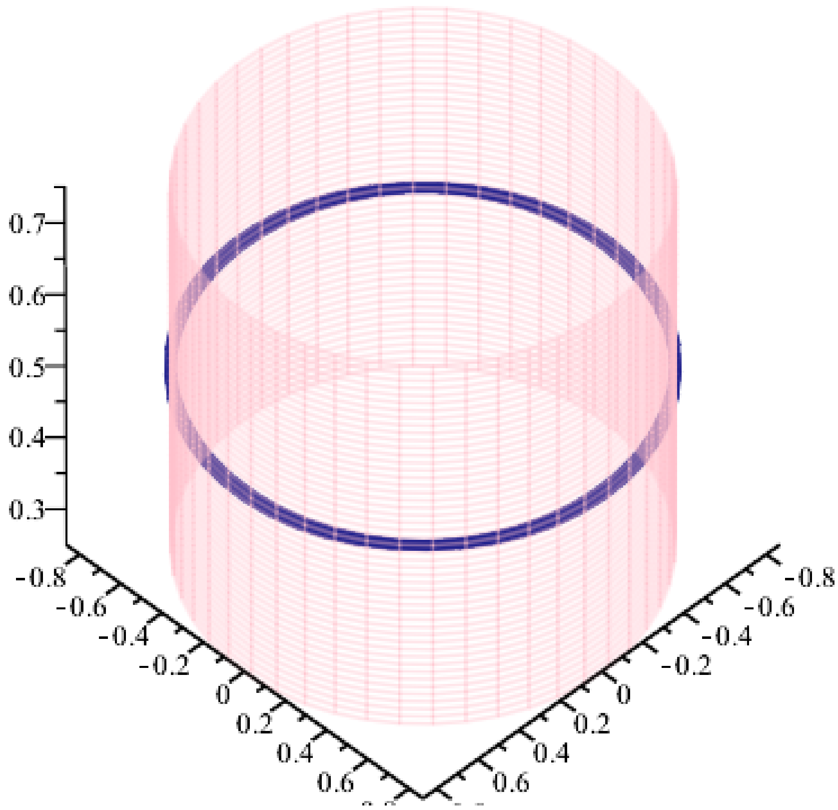


Fig. 5. The image as a developable ruled surface of each Φ, Ω, and ξ is same

4. Conclusion

Overall, in the paper, four new ruled surfaces based on Smarandache curves and ruled by unit Darboux vector have been introduced. The characteristics of each surface have been drawn. It is seen that the characteristics of surfaces are effected if the initial curve α is chosen to be a special curve general or circular helix.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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