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Hypersphere and the Third Laplace-Beltrami Operator

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Abstract
In this work, we examine the differential geometric objects of the hypersphere $\mathbf{h}$ in four dimensional Euclidean geometry $\mathbb{E}^4$. Giving some notions of four dimension, we consider the ith curvature formulas of the hypersurfaces of $\mathbb{E}^4$. In addition, we reveal the hypersphere satisfying $\Delta^{\mathrm{III}}\mathbf{h} = \mathcal{A}\mathbf{h}$ for some 4×4 matrix $\mathcal{A}$.

1. Introduction

Surfaces, hypersurfaces (hypfaces), and also sphere and hypersphere have been studied by mathematicians for centuries.

Almost sixty years ago, Obata [1] worked the conditions for a Riemannian manifold to be isometric with a sphere; Takahashi [2] gave the related Euclidean submanifold (subfold) is 1-type (1-t), if and only if it is minimal or minimal in a hypersphere in $\mathbb{E}^m$; Chern et al. [3] focused the minimal subfolds of a sphere; Cheng and Yau [4] introduced the hypfaces having constant curvature; Chen et al. [5-11] researched the subfolds of finite type (f-t) whose immersion into $\mathbb{E}^m$ (or $\mathbb{E}^m_v$) taking the finite number eigenfunctions of its Laplacian. Garay studied [12] expanded the Takahashi theorem in m-space. Chen and Piccinni [11] focused the subfolds with f-t the Gauss map ($\mathbf{G}$) in $\mathbb{E}^m$. Dursun [13] considered the hypfaces having pointwise 1-t $\mathbf{G}$ in $\mathbb{E}^{n+1}$.

In $\mathbb{E}^3$; Takahashi [2] proved the spheres, minimal surfaces are the unique supplying $\Delta r = \lambda_{\in\mathbb{R}} r$; Ferrandez et al. [14] found the surfaces holding $\Delta H = A_{\in Mat(3,3)} H$, are the right circular cylinder, or open sphere, or minimal; Choi and Kim [15] classifed the minimal helicoid having pointwise 1-t (p1-t) $\mathbf{G}$ of the first type; Garay [16] studied f-t rotational surface; Dillen et al. [17] obtained that the unique surfaces supplying $\Delta r = A_{\in Mat(3,3)} r + B_{\in Mat(3,1)}$ are the circular cylinders, minimal surfaces, spheres; Stamatakis and Zoubi [18] focused the rotational surfaces holding $\Delta^{III} x = Ax$; Senoussi and Bekkar [19] gave the helical surfaces M^2 of f-t depends on I, II and III; Kim et al. [20] introduced the Cheng-Yau operator with its $\mathbf{G}$ of the rotational surfaces.

In $\mathbb{E}^4$; Moore [21,22] considered the general rotational surfaces; Hasanis and Vlachos [23] obtained the hypfaces having harmonic mean curvature; Cheng and Wan [24] gave the complete hypfaces having CMC; Kim and Turgay [25] worked the surfaces having L_1-p1-t $\mathbf{G}$; Arslan et al. [26] studied the Vranceanu surface having p1-t $\mathbf{G}$; Arslan et al. [27] worked the generalized rotational surfaces; Güler et al. [28] introduced the helicoidal hypfaces; Güler et al. [29] worked the $\mathbf{G}$ and the third Laplace-Beltrami operator (LBo) of the rotational hypfaces.

In Minkowski geometry $\mathbb{E}^4_1$; Ganchev and Milousheva [30] studied the analogue surfaces of [21,12]; Arvanitoyeorgos et al. [31] indicated if M^3_1 has $\Delta H = \alpha_{\in\mathbb{R}} H$, then M^3_1 covers CMC; Arslan and Milousheva [32] introduced the meridian surfaces having p1-t $\mathbf{G}$; Turgay [33] considered some

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classifications of Lorentzian surfaces f-t **G**; Dursun and Turgay [34] worked spacelike surfaces having p1-t **G**.

Do Carmo and Dajczer [35] considered the rotational hypersurfaces in spaces of constant curvature; Alias and Gürbüz [36] worked an extension theorem of Takahashi.

We introduce the hypersphere in $\mathbb{E}^4$. In Section 2, we recall the notions of $\mathbb{E}^4$. We consider the curvature formulas of a hypface of $\mathbb{E}^4$. We define the hypersphere in Section 3. Finally, we give the hypersphere satisfying $\Delta^{\mathbf{III}}\mathbf{h} = \mathcal{A}_{\in Mat(4,4)}\mathbf{h}$ in Section 4. In Section 5, we serve the results and discussion. We present the conclusion and suggestions in the last section.

2. Preliminaries

We give basic elements, definitions, etc. considered in this paper. Let $\mathbb{E}^{n+1}$ describe a Euclidean $(n+1)$-space with a Euclidean inner product defined by $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n+1} x_i y_i$, where $\vec{\boldsymbol{x}} = (x_1, x_2, \dots, x_{n+1})$, $\vec{y} = (y_1, y_2, \dots, y_{n+1})$ are the vectors in $\mathbb{E}^{n+1}$.

Let **h** be an hypface in $\mathbb{E}^{n+1}$, **S** be its shape operator. The characteristic polynomial of **S** is given by

$$P_{\mathbf{S}}(\lambda) = det(S - \lambda \mathfrak{I}_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = 0.$$

Here, $i = 0,1, \dots, n$, $\mathfrak{I}_n$ is the identity n-matrix. See [36] for details. Then, the curvature formulas of **h** are $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1}\mathfrak{C}_1 = s_1, \dots$, $\binom{n}{n}\mathfrak{C}_n = s_n = K$. The k-th fundamental form of the hypface **h** is given by $\mathbf{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y \rangle$. Then, we obtain

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \mathbf{I}(\mathbf{S}^{n-i}(X), Y) = 0.$$

Any vector will be identified with its transpose in the paper. Considering the curve $\mathcal{C}$ as follows

$$\gamma(w) = \big(f(w), 0, 0, \varphi(w)\big),$$

where f, φ are the differentiable functions, and taking ℓ as the axis x_4, the orthogonal transformation of $\mathbb{E}^4$ has the following

$$Z(u,v) = \begin{pmatrix} \cos u \cos v & -\sin u & -\cos u \sin v & 0 \\ \sin u \cos v & \cos u & -\sin u \sin v & 0 \\ \sin v & 0 & \cos v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1)$$

and $u, v \in \mathbb{R}$.

Then, the rotational hypface is stated by $\mathbf{h}(u,v,w) = Z(u,v).\gamma(w)$. Supposing **h** be the immersion $M^3 \subset \mathbb{E}^3 \to \mathbb{E}^4$, the multiple vector product is given by

$$\vec{x} \times \vec{y} \times \vec{z} = det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix},$$

where e_i are the standart base elements, x_i, y_i, z_i are the elements of the vectors $\vec{x}, \vec{y}, \vec{z}$ respectively, of $\mathbb{E}^4$. We have

$$\mathbf{I} = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix},$$

$$\mathbf{II} = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix},$$

$$\mathbf{III} = \begin{pmatrix} X & Y & O \\ Y & Z & J \\ O & J & U \end{pmatrix},$$

where $\mathbf{I}, \mathbf{II}, \mathbf{III}$ are the fundamental form matrices with the following coefficients

$E = \langle \mathbf{h}_u, \mathbf{h}_u \rangle$, $F = \langle \mathbf{h}_u, \mathbf{h}_v \rangle$, $G = \langle \mathbf{h}_v, \mathbf{h}_v \rangle$,
$A = \langle \mathbf{h}_u, \mathrm{h}_w \rangle$, $B = \langle \mathbf{h}_v, \mathbf{h}_w \rangle$, $C = \langle \mathbf{h}_w, \mathbf{h}_w \rangle$,
$L = \langle \mathbf{h}_{uu}, \mathbf{G} \rangle$, $M = \langle \mathbf{h}_{uv}, \mathbf{G} \rangle$, $N = \langle \mathbf{h}_{vv}, \mathbf{G} \rangle$,

$P=\langle \mathbf{h}_{uw},\mathbf{G}\rangle, T=\langle \mathbf{h}_{vw},\mathbf{G}\rangle, V=\langle \mathbf{h}_{ww},\mathbf{G}\rangle,$
$X=\langle \mathbf{G}_{u},\mathbf{G}_{u}\rangle, Y=\langle \mathbf{G}_{u},\mathbf{G}_{v}\rangle, Z=\langle \mathbf{G}_{v},\mathbf{G}_{v}\rangle,$
$O=\langle \mathbf{G}_{u},\mathbf{G}_{w}\rangle, J=\langle \mathbf{G}_{v},\mathbf{G}_{w}\rangle, U=\langle \mathbf{G}_{w},\mathbf{G}_{w}\rangle$

of the hypface **h**. Here,

$$\mathbf{G}=\frac{\mathbf{h}_u\times\mathbf{h}_v\times\mathbf{h}_w}{\|\mathbf{h}_u\times\mathbf{h}_v\times\mathbf{h}_w\|} \tag{2}$$

is the Gauss map of the **h**. Hence, $\mathbf{I}^{-1}\cdot\mathbf{II}$ holds the shape operator matrix **S**. See [28,29] for details. Any hypface **h** in $\mathbb{E}^4$ has the following: $\mathfrak{C}_0=1$, and

$$\mathfrak{C}_1=\frac{\left\{\begin{array}{c}(EN+GL-2FM)C\\ +(EG-F^2)V-LB^2-NA^2\\ -2(APG-BPF-ATF+BTE-ABM)\end{array}\right\}}{3[(EG-F^2)C-EB^2+2FAB-GA^2]}, \tag{3}$$

$$\mathfrak{C}_2=\frac{\left\{\begin{array}{c}(EN+GL-2FM)V\\ +(LN-M^2)C-ET^2-GP^2\\ -2(APN-BPM-ATM+BTL-PTF)\end{array}\right\}}{3[(EG-F^2)C-EB^2+2FAB-GA^2]}, \tag{4}$$

$$\mathfrak{C}_3=\frac{(LN-M^2)V-LT^2+2MPT-NP^2}{3[(EG-F^2)C-EB^2+2FAB-GA^2]}. \tag{5}$$

See [37] for details. The hypface **h** is i-minimal, when $\mathfrak{C}_i=0$.

3. Hypersphere in 4-Space

We reveal the hypersphere, then obtain its geometric objects in $\mathbb{E}^4$. Assume $\gamma: I\subset\mathbb{R}\to\Pi$ be a curve in a plane Π, ℓ be a line on Π in $\mathbb{E}^4$.

Definition 1. A rotational hypface in $\mathbb{E}^4$ is called hypersphere, when the profile curve

$$\gamma(w)=(r\,cosw,0,0,r\,sinw)$$

rotates by (1) around the *axis* $\ell=(0,0,0,1)$ for $r>0$. So, the hypersphere spanned by the vector ℓ, is defined by $\mathbf{h}(u,v,w)=Z(u,v)\cdot\gamma(w)$. Therefore, more clear form of **h** is written by

$$\mathbf{h}(u,v,w)=\begin{pmatrix} r\cos u\cos v\cos w\\ r\sin u\cos v\cos w\\ r\sin v\cos w\\ r\sin w\end{pmatrix}. \tag{6}$$

Here, $r>0$, $0\le u,v,w\le 2\pi$. When $w=0$, we have the sphere in $\mathbb{E}^4$. See [38] for details.

Next, we will obtain the **G** and the $\mathfrak{C}_i$ of the hypersphere (6). The first quantities of (6) are given by

$$\mathbf{I}=diag(r^2\cos^2 v\cos^2 w, r^2\cos^2 w,\ r^2). \tag{7}$$

By (2), we obtain the **G** of the hypersphere (6) as follows

$$\mathbf{G}=\begin{pmatrix}\cos u\cos v\cos w\\ \sin u\cos v\cos w\\ \sin v\cos w\\ \sin w\end{pmatrix}. \tag{8}$$

By taking the second derivatives of (6) with respect to $u,v,w,$ and by the **G** (8) of the hypersphere (6), we have

$$\mathbf{II}=diag(-r\cos^2 v\cos^2 w,-r\cos^2 w,-r). \tag{9}$$

Computing the shape operator matrix of the hypersphere (6): $\mathbf{S}=-\frac{1}{r}\mathfrak{I}_3$, we find the following third quantities

$$\mathbf{III}=diag(r^2\cos^2 v\cos^2 w, r^2\cos^2 w, 1). \tag{10}$$

Finally, by using (3), (4), (5), with (7), (9), respectively, we obtain the following.

Theorem 1. Suppose $\mathbf{h}: M^3\subset\mathbb{E}^3\to\mathbb{E}^4$ be the hypface given by (6). Then, the hypersphere **h** has the following curvatures

$$\mathfrak{C}_1=-\frac{1}{r},\quad \mathfrak{C}_2=\frac{1}{r^2},\quad \mathfrak{C}_3=-\frac{1}{r^3}.$$

4. Hypersphere Satisfying $\Delta^{\mathbf{III}}\mathbf{h}=\mathcal{A}\mathbf{h}$

In this section, we give the third LBo of a function, and then calculate it by using the hypersphere (6).

Definition 2. The third LBo of $\phi = \phi(x^1, x^2, x^3)|_{D\subset\mathbb{R}^3}$ of C^3 depends on the third fundamental form is defined by

$$\Delta^{\mathbf{III}}\phi = \frac{1}{\sqrt{|t|}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(\sqrt{|t|}t^{ij}\frac{\partial\phi}{\partial x^j}\right),$$

where $\mathbf{III} = (t_{ij})_{3\times 3}, (t^{ij}) = (t_{kl})^{-1}$ and $t = det(t_{ij})$. See [29] for details.

Therefore, the third LBo of the hypersphere (6) transforms to

$$\Delta^{\mathbf{III}}\mathbf{h} = \frac{1}{\sqrt{|det\,\mathbf{III}|}}\left(\frac{\partial}{\partial u}\Phi - \frac{\partial}{\partial v}\Omega + \frac{\partial}{\partial w}\Psi\right), \qquad (11)$$

where

$$\Phi = \frac{(OZ - J^2)\frac{\partial\mathbf{h}}{\partial u} - (JU - OY)\frac{\partial\mathbf{h}}{\partial v} + (JY - UZ)\frac{\partial\mathbf{h}}{\partial w}}{\sqrt{|det\,\mathbf{III}|}},$$

$$\Omega = \frac{(JU - OY)\frac{\partial\mathbf{h}}{\partial u} - (OX - U^2)\frac{\partial\mathbf{h}}{\partial v} + (UY - JX)\frac{\partial\mathbf{h}}{\partial w}}{\sqrt{|det\,\mathbf{III}|}},$$

$$\Psi = \frac{(JY - UZ)\frac{\partial\mathbf{h}}{\partial u} - (UY - JX)\frac{\partial\mathbf{h}}{\partial v} + (XZ - Y^2)\frac{\partial\mathbf{h}}{\partial w}}{\sqrt{|det\,\mathbf{III}|}}.$$

By using the derivatives $\frac{\partial\Phi}{\partial u}, \frac{\partial\Omega}{\partial v}, \frac{\partial\Psi}{\partial w}$, and substituting them into (11), respectively, we obtain the following.

Theorem 2. Let $\mathbf{h} : M^3 \subset \mathbb{E}^3 \to \mathbb{E}^4$ be an hypersphere (6). Then, $\mathbf{h}$ has the following

$$\Delta^{\mathbf{III}}\mathbf{h} = -\,3\,r\,\mathbf{G},$$

where $r > 0$.

Proof. By direct computation, it is clear.

5. Results and Discussion

Considering all findings in the previous section, we give the following results.

Corollary 1. Assume that $\mathbf{h} : M^3 \subset \mathbb{E}^3 \to \mathbb{E}^4$ be an hypersphere (6). Therefore, the hypersphere $\mathbf{h}$ has $\Delta^{\mathbf{III}}\mathbf{h} = \mathcal{A}\mathbf{h}$, where

$$\mathcal{A}_{\in Mat(4,4)} = (-1)^{i+1}3r^i\mathfrak{C}_i\mathfrak{I}_4, \qquad i = 0,1,2,3.$$

and $\mathfrak{I}_4$ is the 4×4 identity matrix.

6. Conclusion and Suggestions

In this paper, we introduce the the hypersphere $\mathbf{h}$ in four dimensional Euclidean geometry $\mathbb{E}^4$. Recalling some notions of 4-dimension, we give the ith curvature formulas of the hypersurfaces of $\mathbb{E}^4$. Moreover, we present the hypersphere supplying $\Delta^{\mathbf{III}}\mathbf{h} = \mathcal{A}\mathbf{h}$ for some 4×4 matrix $\mathcal{A}$. It can be studied in other space forms.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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