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On Lightlike Submanifolds of Poly-Norden Semi-Riemannian Manifolds

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Abstract. In this paper, we introduce and examine invariant and semi-invariant lightlike submanifolds of a poly-Norden semi-Riemannian manifold. Also, we obtain some examples of such types submanifolds and study the conditions for both integrability and totally geodesic foliation description of distributions.

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1. Introduction

In differential geometry, submanifolds equipped with different geometric structure have been studied widely. A submanifold of a semi-Riemannian (briefly s-Riemannian) manifold is called a lightlike submanifold if the induced metric is degenerate. The general view of lightlike submanifold has been introduced in [4] (see also [7]). Many results on lightlike submanifolds have been given in many papers [1, 5, 6, 11, 20].

The golden proportion and the golden rectangle have been found in the harmonious proportion of temples, fractals, paintings etc. Golden structure was charecterized by J. Kepler. The number ϕ, which is the real positive root of

$$x^2 - x - 1 = 0,$$

(hence $\phi = \frac{1+\sqrt{5}}{2}$) is the golden proportion. In [8], inspired by golden ratio, golden Riemannian manifolds were introduced. Then, many authors have studied golden structure on different manifolds [3, 12, 13, 17].

As a generalization of the golden mean family, metallic mean family was introduced in [18]. The positive solution of

$$x^2 - px - q = 0,$$

is called member of the metallic means family, where p and q are fixed two positive integers. These numbers denoted by

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2},$$

are also known $(p, q)-$metallic numbers. Recently many paper about metallic mean have been published [2, 9, 10, 15].

Also in [16], the authors have defined Bronze structure which is different from Bronze mean given in [14]. In [19], B. Şahin introduce as a new type of manifold which is called almost poly-Norden manifolds. An almost poly-Norden

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structure can not be expressed with metallic structure for any positive integer p, q. Recently, Yüksel Perktaş studied submanifolds of almost poly-Norden Riemannian manifolds in [21].

In this article, by inspiring from [19], we study lightlike submanifolds of poly-Norden manifolds. Also, we introduce and examine invariant and semi-invariant lightlike submanifolds of a poly-Norden s-Riemannian manifold. Also, we give some examples of such types submanifolds.

2. Preliminaries

Let $\tilde{M}$ be a diffrentiable manifold and Φ be a $(1,1)-$type tensor field on $\tilde{M}$. If

$$\Phi^2 = m\Phi - I,$$

satisfied, then Φ is called an almost poly-Norden manifold [19].

Example 2.1 ([19]). Consider the 4-tuples real space $\mathbb{R}^4$ and define a map by

$$\begin{aligned} \Phi \quad &: \quad \mathbb{R}^4 \to \mathbb{R}^4 \\ (x_1, x_2, x_3, x_4) \quad &\to \quad (B_m x_1, B_m x_2, \bar{B}_m x_3, \bar{B}_m x_4), \end{aligned}$$

where $B_m = \frac{m+\sqrt{m^2-4}}{2}$ and $\bar{B} = m - B_m$. Thus, $(\mathbb{R}^4, \Phi)$ is an example of almost poly-Norden manifold.

If a s-Riemannian metric $\tilde{g}$ satisfies

$$\tilde{g}(\Phi U, \Phi V) = m\tilde{g}(\Phi U, V) - \tilde{g}(U, V), \tag{2.1}$$

then $\tilde{g}$ is called $\Phi-$compatible. So $(\tilde{M}, \Phi, \tilde{g})$ is called an almost poly-Norden s-Riemannian manifold [19].

By use of (2.1), we get

$$\tilde{g}(\Phi U, V) = \tilde{g}(U, \Phi V).$$

From now on, we will suppose that m is different from zero.

If the induced metric g from $\tilde{g}$ is degenerate on M^m and

$$rank(RadTM) = r, \qquad 1 \leq r \leq m,$$

then (M, g) is called a lightlike submanifold, where $RadTM$ is the radical distribution and $TM^\perp$ the normal bundle of TM with

$$RadTM = TM \cap TM^\perp,$$

and

$$TM^\perp = \bigcup_{p\in\tilde{M}} \Big\{ V_p \in T_p\tilde{M} : g_p(U_p, V_p) = 0, \forall U \in \Gamma(T_pM) \Big\}.$$

Since TM and $TM^\perp$ are non-degenerate vector subbundles, there exists complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $RadTM$ in TM and $TM^\perp$, respectively, which are the screen distribution and screen transversal bundle of M with

$$TM = S(TM) \perp RadTM,$$

$$TM^\perp = S(TM^\perp) \perp RadTM.$$

Also, in view of an orthogonal complement subbundle $S(TM)^\perp$ to $S(TM)$ in $T\tilde{M}$ such that

$$S(TM)^\perp = S(TM^\perp) \perp S(TM^\perp)^\perp, \tag{2.2}$$

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$.

Theorem 2.2 ([4]). *Let $(M, g, S(TM), S(TM^\perp))$ be a $r-$lightlike submanifold of a s-Riemannian manifold $\tilde{M}$. Then, there exists a complementary vector bundle $ltr(TM)$ known a lightlike transversal bundle of $RadTM$ in $S(TM^\perp)^\perp$ and a basis of $\Gamma(ltr(TM))$ consist of sections $\{N_1, ..., N_r\}$ of $S(TM^\perp)^\perp$ such that*

$$\tilde{g}(N_i, N_j) = 0, \qquad \tilde{g}(N_i, E_j) = 1, \ i, j = 1, ..., r,$$

where $\{E_1, ..., E_r\}$ is a basis of $\Gamma(RadTM)$.

This theorem gives that

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.3}$$

$$S(TM^{\perp})^{\perp} = RadTM \oplus ltr(TM). \tag{2.4}$$

So, in view of equations (2.2)–(2.4), we arrive at

$$\begin{aligned} T\tilde{M} &= S(TM) \perp S(TM)^{\perp} \\ &= S(TM) \perp \{RadTM \oplus ltr(TM)\} \perp S(TM)^{\perp} \\ &= TM \oplus tr(TM). \end{aligned} \tag{2.5}$$

The Gauss and Weingarten equations of M are given by

$$\tilde{\nabla}_U V = \nabla_U V + h(U,V), \qquad U, V \in \Gamma(TM), \tag{2.6}$$

$$\tilde{\nabla}_U N = -A_N U + \nabla^t_U N, \qquad N \in \Gamma(ltr(TM)),$$

where $\nabla_U V, A_N U \in \Gamma(TM)$ and $h(U,V), \nabla^t_U N \in \Gamma(tr(TM))$.

By using (2.5) with projection morphisms defined by

$$L : tr(TM) \to ltr(TM), \qquad S : tr(TM) \to S(TM^{\perp}),$$

then every $U, V \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we get

$$\tilde{\nabla}_U V = \nabla_U V + h^l(U,V) + h^s(U,V), \tag{2.7}$$

$$\tilde{\nabla}_U N = -A_N U + \nabla^l_U N + D^s(U,N), \tag{2.8}$$

$$\tilde{\nabla}_U W = -A_W U + D^l(U,W) + \nabla^s_U W, \tag{2.9}$$

where $\nabla^l_U N, D^l(U,W) \in \Gamma(ltr(TM))$ and $D^s(U,N), \nabla^s_U W \in \Gamma(S(TM^{\perp}))$ and

$$h^l(U,V) = Lh(U,V) \in \Gamma(ltr(TM)),$$

$$h^s(U,V) = Sh(U,V) \in \Gamma(S(TM^{\perp})).$$

If we denote the projection of TM on $S(TM)$ with $\tilde{P}$, from (2.6), (2.7), (2.8) and (2.9), we obtain

$$\tilde{g}(h^s(U,V),W) + \tilde{g}(Y, D^l(U,W)) = \tilde{g}(A_W U, Y),$$

$$\tilde{g}(D^s(U,N),W) = \tilde{g}(N, A_W U),$$

$$\nabla_U \tilde{P}V = \nabla^*_U \tilde{P}V + h^*(U, \tilde{P}V), \tag{2.10}$$

$$\nabla_U E = -A^*_E U + \nabla^{*t}_U E, \tag{2.11}$$

for $U, V \in \Gamma(TM)$, $E \in \Gamma(RadTM)$, where ∇^* and ∇^t are induced connections on $S(TM)$ and $RadTM$, respectively. Also, h^* is $\Gamma(RadTM)-$ valued bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ which is called second fundamental form on $S(TM)$ and A^* is $\Gamma(S(TM))-$ valued bilinear forms on $\Gamma(RadTM) \times \Gamma(TM)$ which is called second fundamental form on $RadTM$.

Moreover, induced connection ∇ on$M$ is not a metric connection and satisfies

$$(\nabla_U g)(V,Y) = \tilde{g}(h^l(U,V),Y) + \tilde{g}(h^l(U,Y),V).$$

Also, ∇^* is a metric connection on $S(TM)$.

3. Main Results

In this article, we assume that

$$\tilde{\nabla}\Phi = 0, \tag{3.1}$$

and to avoid repetition in the remain part of this section, $(\tilde{M}, \Phi, \tilde{g})$ will be considered a poly-Norden semi-Riemannian manifold.

Definition 3.1. Let (M, g) be a lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then, M is an invariant lightlike submanifold if the following conditions are satisfied [4]:

$$\Phi(S(TM)) = S(TM),$$
$$\Phi(RadTM) = RadTM.$$

Example 3.2. Let $\tilde{M} = \mathbb{R}^5_1$ be a semi-Euclidean space with coordinate system $(x_1, x_2, ..., x_5)$ and signature $(-, +, +, +, +)$. If we take

$$\Phi(x_1, x_2, ..., x_5) = (B_m x_1, B_m x_2, B_m x_3, B_m x_4, B_m x_5),$$

then Φ is an almost poly-Norden structure on $\tilde{M}$.

Suppose that M is a submanifold given by

$$x_1 = u_3, \quad x_2 = -\sin\alpha u_1 + \cos\alpha u_3$$

$$x_3 = \cos\alpha u_1 + \sin\alpha u_3, \quad x_4 = u_2, \quad x_5 = 0,$$

where (u_1, u_2, u_3) is the local coordinate system of M. Then, TM is given by

$$\Psi_1 = -\sin\alpha\frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial x_3},$$

$$\Psi_2 = \frac{\partial}{\partial x_4},$$

$$\Psi_3 = \frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2} + \sin\alpha\frac{\partial}{\partial x_3}.$$

Hence, $RadTM = Sp\{\Psi_3\}$, $S(TM) = Sp\{\Psi_1, \Psi_2\}$ and $ltr(TM)$ is spanned by

$$N = \frac{1}{2}\left\{-\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2} + \sin\alpha\frac{\partial}{\partial x_3}\right\}.$$

It follows that $\Phi(RadTM) = RadTM$ and $\Phi(S(TM)) = S(TM)$. Hence, M is an invariant lightlike submanifold.

If we show the projection morphisim on $S(TM)$ and $RadTM$ with T and R, respectively. Thus, for $U \in \Gamma(TM)$, we can state

$$U = TU + RU,$$

where $TU \in \Gamma(S(TM))$ and $RU \in \Gamma(RadTM)$.

Applying Φ to above equation, we get

$$\Phi U = \Phi TU + \Phi RU,$$

from which we can write

$$\Phi U = SU + QU, \tag{3.2}$$

where $SU \in \Gamma(S(TM))$ and $QU \in \Gamma(RadTM)$.

If we differentiate (3.2) and by use of (3.1) with (2.7)–(2.9), we arrive at

$$\begin{aligned} S\nabla_U V + Q\nabla_U V + \Phi h^l(U,V) + \Phi h^s(U,V) \quad = \quad & \nabla^*_U SV + h^*(U,SV) + h^l(U,SV) + h^s(U,SV) \\ & - A^*_{QV}U + \nabla^{*t}_U QV + h^l(U,QV) + h^s(U,QV), \end{aligned}$$

which yields

$$\begin{aligned} S\nabla_U V &= \nabla^*_U SV - A^*_{QV}U, \\ Q\nabla_U V &= h^*(U, SV) + \nabla^{*t}_U QV, \\ \Phi h^l(U, V) &= h^l(U, QV), \\ \Phi h^s(U, V) &= h^s(U, QV), \\ \Phi\nabla_U V &= \nabla^*_U SV - A^*_{QV}U + h^*(U, SV) + \nabla^{*t}_U QV. \end{aligned}$$

Theorem 3.3. *Let (M, g) be an invariant lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then, $RadTM$ is integrable if and only if*

$$A^*_{\Phi E_1}E_2 = A^*_{\Phi E_2}E_1 \quad and \quad A^*_{E_1}E_2 = A^*_{E_2}E_1,$$

for any $E_1, E_2 \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.

Proof. For $E_1, E_2 \in \Gamma(RadTM)$, $RadTM$ is integrable if and only if

$$[E_1, E_2] \in \Gamma(RadTM).$$

From above equation, we can say $\tilde{g}([E_1, E_2], Z] = 0$. Therefore, from above equation and using (2.1) with (3.1), we get

$$\begin{aligned} \tilde{g}([E_1, E_2], Z] &= \tilde{g}(\tilde{\nabla}_{E_1}E_2 - \tilde{\nabla}_{E_2}E_1, Z) \\ &= \tilde{g}(\tilde{\nabla}_{E_1}E_2, Z) - \tilde{g}(\tilde{\nabla}_{E_2}E_1, Z) \\ &= -\tilde{g}(\Phi\tilde{\nabla}_{E_1}E_2, \Phi Z) + m\tilde{g}(\tilde{\nabla}_{E_1}E_2, \Phi Z) + \tilde{g}(\Phi\tilde{\nabla}_{E_2}E_1, \Phi Z) - m\tilde{g}(\tilde{\nabla}_{E_2}E_1, \Phi Z) \\ &= -\tilde{g}(\tilde{\nabla}_{E_1}\Phi E_2, \Phi Z) + m\tilde{g}(\tilde{\nabla}_{E_1}E_2, \Phi Z) + \tilde{g}(\tilde{\nabla}_{E_2}\Phi E_1, \Phi Z) - m\tilde{g}(\tilde{\nabla}_{E_2}E_1, \Phi Z). \end{aligned} \tag{3.3}$$

In view of (2.7) with (3.3), we find

$$g(\nabla_{E_2}\Phi E_1, \Phi Z) - g(\nabla_{E_1}\Phi E_2, \Phi Z) + m\Big(g(\nabla_{E_1}E_2, \Phi Z) - g(\nabla_{E_2}E_1, \Phi Z)\Big) = 0.$$

Therefore, using the decomposition (2.5) and (2.11), we arrive at

$$g(A_{\Phi E_2}E_1 - A_{\Phi E_1}E_2, \Phi Z) + m\Big(g(A_{E_1}E_2 - A_{E_2}E_1, \Phi Z)\Big) = 0,$$

which yields the proof. □

Theorem 3.4. *Let (M, g) be an invariant lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then, $S(TM)$ is integrable if and only if*

$$h^*(U, \Phi V) = h^*(V, \Phi U) \quad and \quad h^*(U, V) = h^*(V, U),$$

for any $U, V \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. For $U, V \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$, $S(TM)$ is integrable if and only if

$$\tilde{g}([U, V], N] = 0.$$

Therefore, from above equation and using (2.1) with (3.1), we have

$$\begin{aligned} \tilde{g}([U, V], N] &= \tilde{g}(\tilde{\nabla}_U V - \tilde{\nabla}_V U, N) \\ &= -\tilde{g}(\Phi\tilde{\nabla}_U V, \Phi N) + m\tilde{g}(\tilde{\nabla}_U V, \Phi N) + \tilde{g}(\Phi\tilde{\nabla}_V U, \Phi N) - m\tilde{g}(\tilde{\nabla}_V U, \Phi N) \\ &= -\tilde{g}(\tilde{\nabla}_U \Phi V, \Phi N) + m\tilde{g}(\tilde{\nabla}_U V, \Phi N) + \tilde{g}(\tilde{\nabla}_V \Phi U, \Phi N) - m\tilde{g}(\tilde{\nabla}_V U, \Phi N). \end{aligned} \tag{3.4}$$

By use of (2.10) with (3.4) and using the decomposition (2.5) and (2.11), we can write

$$g(h^*(U, \Phi V) - h^*(V, \Phi U), \Phi N) + mg(h^*(U, V) - h^*(V, U), \Phi N) = 0,$$

which gives the proof. □

Theorem 3.5. *Let (M, g) be an invariant lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then, ∇ on M is a metric connection if and only if*

$$A^*_{\Phi E}U = mA^*_E U, \tag{3.5}$$

for any $U \in \Gamma(TM)$ and $E \in \Gamma(RadTM)$.

Proof. If ∇ is a metric connection, we have

$$\nabla_U E \in \Gamma(RadTM),$$

for any $U \in \Gamma(TM)$ and $E \in \Gamma(RadTM)$. So we can write

$$g(\nabla_U E, Z) = 0,$$

for all $Z \in \Gamma(S(TM))$.

Therefore, using (2.7) with (2.1), we get

$$\tilde{g}(\Phi\tilde{\nabla}_U E, \Phi Z) - m\tilde{g}(\tilde{\nabla}_U E, \Phi Z) = 0,$$

so from (3.1), we have

$$\tilde{g}(\tilde{\nabla}_U \Phi E, \Phi Z) - m\tilde{g}(\tilde{\nabla}_U E, \Phi Z) = 0.$$

Therefore, using (2.11) in above equation, we find

$$g(-A^*_{\Phi E}U + mA^*_E U, \Phi Z) = 0,$$

which gives (3.5). □

Theorem 3.6. *Let (M, g) be an invariant lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then, $RadTM$ defines a totally geodesic foliation on M if and only if*

$$h^l(E, \Phi U) = mh^l(E, U), \tag{3.6}$$

for any $U \in \Gamma(S(TM))$ and $E \in \Gamma(RadTM)$.

Proof. The radical distribution $RadTM$ defines a totally geodesic foliation if and only if

$$\nabla_{E_1} E_2 \in \Gamma(RadTM),$$

for any $E_1, E_2 \in \Gamma(RadTM)$. Because of $\tilde{\nabla}$ is a metric connection we can write for all $U \in \Gamma(S(TM))$

$$g(\nabla_{E_1} E_2, U) = \tilde{g}(\tilde{\nabla}_{E_1} E_2, U) = \tilde{g}(E_2, \tilde{\nabla}_{E_1} U) = 0.$$

Therefore, using (2.7) with (2.1) and (2.10), we get

$$\begin{aligned} 0 &= \tilde{g}(\tilde{\nabla}_{E_1}\Phi U, \Phi E_2) - m\tilde{g}(\tilde{\nabla}_{E_1} U, \Phi E_2) \\ &= g(h^l(E_1, \Phi U), \Phi E_2) - mg(h^l(E_1, U), \Phi E_2). \end{aligned}$$

So, we get (3.6). The converse of proof is clear. □

Now, we introduce semi-invariant lightlike submanifolds of $(\tilde{M}, \Phi, \tilde{g})$. Firstly, we give the following:

Definition 3.7. Let (M, g) be a lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then M is called a semi-invariant lightlike submanifold if the following conditions are satisfied:

$$\Phi(RadTM) = S(TM),$$
$$\Phi(ltr(TM)) = S(TM),$$
$$\Phi(S(TM^\perp)) = S(TM).$$

Taking $\hat{D} = \Phi(RadTM)$, $\tilde{D} = \Phi(ltr(TM))$, $\mathring{D} = \Phi(S(TM^\perp))$ then we can state

$$S(TM) = D_0 \perp \{\hat{D} \oplus \tilde{D}\} \perp \mathring{D}.$$

So, we get

$$TM = D_0 \perp \{\hat{D} \oplus \tilde{D}\} \perp \mathring{D} \perp RadTM,$$
$$T\tilde{M} = D_0 \perp \{\hat{D} \oplus \tilde{D}\} \perp \mathring{D} \perp S(TM^\perp) \perp \{RadTM \oplus ltr(TM)\}.$$

If we take

$$D = D_0 \perp \hat{D} \oplus \perp RadTM \quad \text{and} \quad D^\perp = \tilde{D} \perp \mathring{D},$$

then we can write

$$TM = D \oplus D^\perp.$$

Example 3.8. Let $\tilde{M} = \mathbb{R}^7_2$ be a semi-Euclidean space with coordinate system $(x_1, x_2, ..., x_7)$ and signature $(-, +, -, +, +, +, +)$. If we take

$$\Phi(x_1, x_2, ..., x_7) = (\bar{B}_m x_1, B_m x_2, B_m x_3, \bar{B}_m x_4, B_m x_5, B_m x_6, B_m x_7),$$

where $\bar{B}_m = m - B_m$, then we can say that Φ is an almost poly-Norden structure on $\tilde{M}$.

Suppose that M is a submanifold given by

$$\begin{aligned} x_1 &= B_m u_1 + u_2 + \bar{B}_m u_3, \\ x_2 &= u_1 + B_m u_2 - B_m u_3, \\ x_3 &= u_1 + B_m u_2 + B_m u_3, \\ x_4 &= B_m u_1 + u_2 - \bar{B}_m u_3, \\ x_5 &= x_6 = \bar{B}_m u_4, \quad x_7 = u_5, \end{aligned}$$

where $(u_1, u_2, ..., u_5)$ is the local coordinate system of M.Then, TM is given by

$$\Psi_1 = B_m \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + B_m \frac{\partial}{\partial x_4},$$

$$\Psi_2 = \frac{\partial}{\partial x_1} + B_m \frac{\partial}{\partial x_2} + B_m \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4},$$

$$\Psi_3 = \bar{B}_m \frac{\partial}{\partial x_1} - B_m \frac{\partial}{\partial x_2} + B_m \frac{\partial}{\partial x_3} - \bar{B}_m \frac{\partial}{\partial x_4},$$

$$\Psi_4 = \bar{B}_m \frac{\partial}{\partial x_5} + \bar{B}_m \frac{\partial}{\partial x_6},$$

$$\Psi_5 = \frac{\partial}{\partial x_7}.$$

So, we have $RadTM = Sp\{E = \Psi_1\}$ and $S(TM) = Sp\{\Psi_2, \Psi_3, \Psi_4, \Psi_5\}$ with

$$ltr(TM) = Sp\left\{N = -\frac{1}{2(B_m + 1)}\{\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\}\right\},$$

$$S(TM^{\perp}) = Sp\{L = \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}\}.$$

Moreover, we arrive at $\Phi(E) = \Psi_2$, $\Phi(N) = \Psi_3$ and $\Phi(L) = \Psi_4$. If we consider $D_0 = Sp\{\Psi_5\}$, $\hat{D} = Sp\{\Psi_2\}$, $\tilde{D} = Sp\{\Psi_3\}$ and $\mathring{D} = Sp\{\Psi_4\}$, then M is a semi-invariant lightlike submanifold.

Suppose that M is a semi-invariant lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. If we show the projection morphism on $S(TM)$ and $RadTM$ with T and S, respectively. Therefore, for $U \in \Gamma(TM)$, we can write

$$U = TU + SU,$$

where $TU \in \Gamma(S(TM))$ and $SU \in \Gamma(RadTM)$.

Applying Φ to above equation, we obtain

$$\Phi U = \Phi TU + \Phi SU,$$

from which we can state

$$\Phi U = KU + LU, \tag{3.7}$$

where $KU \in \Gamma(S(TM))$ and $LU \in \Gamma(ltr(TM))$.

Similarly, for any $W \in \Gamma(tr(TM))$, we have

$$\Phi W = BW + CW, \tag{3.8}$$

where BW and CW are the tangential and transversal components of ΦW, respectively.

Lemma 3.9. *Let M be a semi-invariant lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then, we get for any $U, V \in \Gamma(TM)$ and $W \in \Gamma(tr(TM))$,*

$$(\nabla_U K)V = A_{LV}U + Bh(U, V), \tag{3.9}$$

$$L\nabla_U V = \nabla^t_U LV + h(U, KV), \tag{3.10}$$

$$\nabla_U BW = -KA_W U + B\nabla_U W, \tag{3.11}$$

$$h(U, BW) = -LA_W U. \tag{3.12}$$

Proof. From (3.1) and (3.7) with (3.8), one can easily see that (3.9)-(3.12). □

Theorem 3.10. *Let (M, g) be a semi-invariant lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then, D is integrable if and only if*

$$h^l(\Phi U, \Phi V) = mh^l(V, \Phi U) - h^l(V, U), \tag{3.13}$$

$$h^s(\Phi U, \Phi V) = mh^s(V, \Phi U) - h^s(V, U), \tag{3.14}$$

for any $U, V \in \Gamma(D)$.

Proof. From the definition of D, D is integrable if and only if

$$\tilde{g}([\Phi U, V], \Phi E) = 0,$$

and

$$\tilde{g}([\Phi U, V], \Phi L) = 0,$$

for $E \in \Gamma(RadTM)$ and $L \in \Gamma(S(TM^{\perp}))$.

In view of (2.1) and (2.7), we get

$$\begin{aligned}
\tilde{g}([\Phi U, V], \Phi E) &= \tilde{g}(\tilde{\nabla}_{\Phi U} V, \Phi E) - \tilde{g}(\tilde{\nabla}_V \Phi U, \Phi E) \\
&= \tilde{g}(\Phi\tilde{\nabla}_{\Phi U} V, E) - \tilde{g}(\Phi\tilde{\nabla}_V U, \Phi E) \\
&= \tilde{g}(\tilde{\nabla}_{\Phi U} \Phi V, E) - m\tilde{g}(\Phi\tilde{\nabla}_V U, E) + \tilde{g}(\tilde{\nabla}_V U, E) \\
&= \tilde{g}(\tilde{\nabla}_{\Phi U} \Phi V, E) - m\tilde{g}(\tilde{\nabla}_V \Phi U, E) + \tilde{g}(\tilde{\nabla}_V U, E) \\
&= g(h^l(\Phi U, \Phi V), E) - mg(h^l(V, \Phi U), E) + g(h^l(V, U), E),
\end{aligned}$$

which gives (3.13).

Also,

$$\begin{aligned}
\tilde{g}([\Phi U, V], \Phi L) &= \tilde{g}(\tilde{\nabla}_{\Phi U} V, \Phi L) - \tilde{g}(\tilde{\nabla}_V \Phi U, \Phi L) \\
&= \tilde{g}(\Phi\tilde{\nabla}_{\Phi U} V, L) - \tilde{g}(\Phi\tilde{\nabla}_V U, \Phi L) \\
&= \tilde{g}(\tilde{\nabla}_{\Phi U} \Phi V, L) - m\tilde{g}(\Phi\tilde{\nabla}_V U, L) + \tilde{g}(\tilde{\nabla}_V U, L) \\
&= \tilde{g}(\tilde{\nabla}_{\Phi U} \Phi V, L) - m\tilde{g}(\tilde{\nabla}_V \Phi U, L) + \tilde{g}(\tilde{\nabla}_V U, L) \\
&= g(h^s(\Phi U, \Phi V), L) - mg(h^s(V, \Phi U), L) + g(h^s(V, U), L),
\end{aligned}$$

which implies (3.14). □

Theorem 3.11. *Let (M, g) be a semi-invariant lightlike submanifold of $(\tilde{M}, \Phi, \tilde{g})$. Then, $D^{\perp}$ is integrable if and only if*

i) $mg(h^*(U, V) - h^*(V, U), N) = g(A_{W_1}V - A_{W_2}U, N)$,

ii) $g(A_{W_1}V, \Phi N) = g(A_{W_2}U, \Phi N)$,

iii) $g(A_{W_1}V, \Phi Z) = g(A_{W_2}U, \Phi Z)$

for any $U, V \in \Gamma(D^{\perp})$, $N \in \Gamma(ltr(TM))$, $Z \in \Gamma(D_0)$ and $W_1, W_2 \in \Gamma(tr(TM))$.

Proof. From the definition of $D^{\perp}$, $D^{\perp}$ is integrable if and only if

$$\tilde{g}([U, V], \Phi N) = 0,$$

$$\tilde{g}([U, V], N) = 0,$$

and

$$\tilde{g}([U, V], Z) = 0,$$

for any $U, V \in \Gamma(D^{\perp})$, $N \in \Gamma(ltr(TM))$ and $Z \in \Gamma(D_0)$. Taking $U, V \in \Gamma(D^{\perp})$, there are some vector fields $W_1, W_2 \in \Gamma(tr(TM))$ such that

$$U = \Phi W_1 \quad \text{and} \quad V = \Phi W_2.$$

By use of (2.1), (2.7), (2.8) and (2.10), we find

$$\begin{aligned}\tilde{g}([U,V],\Phi N) &= \tilde{g}(\tilde{\nabla}_U V,\Phi N)-\tilde{g}(\tilde{\nabla}_V U,\Phi N)\\ &= \tilde{g}(\tilde{\nabla}_U \Phi W_2,\Phi N)-\tilde{g}(\tilde{\nabla}_V \Phi W_1,\Phi N)\\ &= m\tilde{g}(\tilde{\nabla}_U \Phi W_2,N)-\tilde{g}(\tilde{\nabla}_U W_2,N)-m\tilde{g}(\tilde{\nabla}_V \Phi W_1,\Phi N)+\tilde{g}(\tilde{\nabla}_V W_1,\Phi N)\\ &= mg(\nabla_U \Phi W_2-\nabla_V \Phi W_1,N)+g(A_{W_2}U-A_{W_1}V,N)\\ &= mg(h^*(U,\Phi W_2)-h^*(V,\Phi W_1),N)+g(A_{W_2}U-A_{W_1}V,N),\end{aligned}$$

which gives (i).
Moreover,

$$\begin{aligned}\tilde{g}([U,V],N) &= \tilde{g}(\tilde{\nabla}_U V,N)-\tilde{g}(\tilde{\nabla}_V U,N)\\ &= \tilde{g}(\tilde{\nabla}_U \Phi W_2,N)-\tilde{g}(\tilde{\nabla}_V \Phi W_1,N)\\ &= \tilde{g}(\Phi\tilde{\nabla}_U W_2,N)-\tilde{g}(\Phi\tilde{\nabla}_V W_1,N)\\ &= \tilde{g}(\tilde{\nabla}_U W_2,\Phi N)-\tilde{g}(\tilde{\nabla}_V W_1,\Phi N)\\ &= g(A_{W_1}V-A_{W_2}U,N),\end{aligned}$$

which implies (ii).
Also,

$$\begin{aligned}\tilde{g}([U,V],Z) &= \tilde{g}(\tilde{\nabla}_U V,Z)-\tilde{g}(\tilde{\nabla}_V U,Z)\\ &= \tilde{g}(\tilde{\nabla}_U \Phi W_2,Z)-\tilde{g}(\tilde{\nabla}_V \Phi W_1,Z)\\ &= \tilde{g}(\Phi\tilde{\nabla}_U W_2,Z)-\tilde{g}(\Phi\tilde{\nabla}_V W_1,Z)\\ &= \tilde{g}(\tilde{\nabla}_U W_2,\Phi Z)-\tilde{g}(\tilde{\nabla}_V W_1,\Phi Z)\\ &= g(A_{W_1}V-A_{W_2}U,Z),\end{aligned}$$

from which we obtain (iii). □

Theorem 3.12. *Let (M,g) be a semi-invariant lightlike submanifold of $(\tilde{M},\Phi,\tilde{g})$. Then, $RadTM$ is integrable if and only if*
i) $h^*(E_1,\Phi E_2)=h^*(\Phi E_1,E_2)$,
ii) $h^l(E_1,\Phi E_2)=h^l(\Phi E_1,E_2)$,
iii) $h^s(E_1,\Phi E_2)=h^s(\Phi E_1,E_2)$,
iv) $A^*_{E_1}E_2=A^*_{E_2}E_1$,
for any $E_1,E_2,E_3\in\Gamma(RadTM)$, $N\in\Gamma(ltr(TM))$, $Z\in\Gamma(D_0)$ *and* $L\in\Gamma(S(TM^\perp))$.

Proof. From the definition of $RadTM$, $RadTM$ is integrable if and only if

$$\tilde{g}([E_1,E_2],\Phi N)=0,$$

$$\tilde{g}([E_1,E_2],\Phi E_3)=0,$$

$$\tilde{g}([E_1,E_2],\Phi L)=0,$$

and

$$\tilde{g}([E_1,E_2],Z)=0,$$

for any $E_1,E_2,E_3\in\Gamma(RadTM)$, $N\in\Gamma(ltr(TM))$, $Z\in\Gamma(D_0)$ and $L\in\Gamma(S(TM^\perp))$.
By use of (2.1), (2.7), (2.10) and (2.11), we can write

$$\begin{aligned}\tilde{g}([E_1,E_2],\Phi N) &= \tilde{g}(\tilde{\nabla}_{E_1}E_2,\Phi N)-\tilde{g}(\tilde{\nabla}_{E_2}E_1,\Phi N)\\ &= \tilde{g}(\Phi\tilde{\nabla}_{E_1}E_2,N)-\tilde{g}(\Phi\tilde{\nabla}_{E_2}E_1,N)\\ &= \tilde{g}(\tilde{\nabla}_{E_1}\Phi E_2,N)-\tilde{g}(\tilde{\nabla}_{E_2}\Phi E_1,N)\\ &= g(\nabla_{E_1}\Phi E_2,N)-g(\nabla_{E_2}\Phi E_1,N)\\ &= g(h^*(E_1,\Phi E_2),N)-g(h^*(E_2,\Phi E_1),N),\end{aligned}$$

$$\begin{aligned}
\tilde{g}([E_1,E_2],\Phi E_3) &= \tilde{g}(\tilde{\nabla}_{E_1}E_2,\Phi E_3)-\tilde{g}(\tilde{\nabla}_{E_2}E_1,\Phi E_3)\\
&= \tilde{g}(\Phi\tilde{\nabla}_{E_1}E_2,E_3)-\tilde{g}(\Phi\tilde{\nabla}_{E_2}E_1,E_3)\\
&= \tilde{g}(\tilde{\nabla}_{E_1}\Phi E_2,E_3)-\tilde{g}(\tilde{\nabla}_{E_2}\Phi E_1,E_3)\\
&= g(\nabla_{E_1}\Phi E_2,E_3)-g(\nabla_{E_2}\Phi E_1,E_3)\\
&= g(h^l(E_1,\Phi E_2),E_3)-g(h^l(E_2,\Phi E_1),E_3),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([E_1,E_2],\Phi L) &= \tilde{g}(\tilde{\nabla}_{E_1}E_2,\Phi L)-\tilde{g}(\tilde{\nabla}_{E_2}E_1,\Phi L)\\
&= \tilde{g}(\Phi\tilde{\nabla}_{E_1}E_2,L)-\tilde{g}(\Phi\tilde{\nabla}_{E_2}E_1,L)\\
&= \tilde{g}(\tilde{\nabla}_{E_1}\Phi E_2,L)-\tilde{g}(\tilde{\nabla}_{E_2}\Phi E_1,L)\\
&= g(\nabla_{E_1}\Phi E_2,L)-g(\nabla_{E_2}\Phi E_1,L)\\
&= g(h^s(E_1,\Phi E_2),L)-g(h^s(E_2,\Phi E_1),L),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([E_1,E_2],Z) &= \tilde{g}(\tilde{\nabla}_{E_1}E_2,Z)-\tilde{g}(\tilde{\nabla}_{E_2}E_1,Z)\\
&= g(A^*_{E_1}E_2,Z)-g(A^*_{E_2}E_1,Z),
\end{aligned}$$

which completes the proof. □

Theorem 3.13. *Let (M,g) be a semi-invariant lightlike submanifold of $(\tilde{M},\Phi,\tilde{g})$. Then, $\Phi RadTM$ is integrable if and only if*
i) $g(h^l(\Phi E_1,E_2),E_3)=g(h^l(\Phi E_2,E_1),E_3)$,
ii) $g(h^s(\Phi E_1,E_2),L)=g(h^s(\Phi E_2,E_1),L)$,
iii) $g(\Phi E_2,A_N\Phi E_1)=g(\Phi E_1,A_N\Phi E_2)$,
iv) $g(A_{E_1}\Phi E_2,\Phi Z)=g(A_{E_2}\Phi E_1,\Phi Z)$,
for any $E_1,E_2,E_3\in\Gamma(RadTM)$, $N\in\Gamma(ltr(TM))$, $Z\in\Gamma(D_0)$ and $L\in\Gamma(S(TM^{\perp}))$.

Proof. From the definition of $\Phi RadTM$, $\Phi RadTM$ is integrable if and only if

$$\tilde{g}([\Phi E_1,\Phi E_2],\Phi E_3)=0,$$

$$\tilde{g}([\Phi E_1,\Phi E_2],\Phi L)=0,$$

$$\tilde{g}([\Phi E_1,\Phi E_2],N)=0,$$

and

$$\tilde{g}([\Phi E_1,\Phi E_2],Z)=0,$$

for any $E_1,E_2,E_3\in\Gamma(RadTM)$, $N\in\Gamma(ltr(TM))$, $Z\in\Gamma(D_0)$ and $L\in\Gamma(S(TM^{\perp}))$.

By use of (2.1), (2.7), (2.10) and (2.11), we can write

$$\begin{aligned}
\tilde{g}([\Phi E_1,\Phi E_2],\Phi E_3) &= \tilde{g}(\tilde{\nabla}_{\Phi E_1}\Phi E_2,\Phi E_3)-\tilde{g}(\tilde{\nabla}_{\Phi E_2}\Phi E_1,\Phi E_3)\\
&= \tilde{g}(\Phi\tilde{\nabla}_{\Phi E_1}E_2,\Phi E_3)-\tilde{g}(\Phi\tilde{\nabla}_{\Phi E_2}E_1,\Phi E_3)\\
&= m\tilde{g}(\Phi\tilde{\nabla}_{\Phi E_1}E_2,E_3)-\tilde{g}(\tilde{\nabla}_{\Phi E_1}E_2,E_3)-m\tilde{g}(\Phi\tilde{\nabla}_{\Phi E_2}E_1,E_3)+\tilde{g}(\tilde{\nabla}_{\Phi E_2}E_1,E_3)\\
&= m\tilde{g}(\tilde{\nabla}_{\Phi E_1}\Phi E_2,E_3)-\tilde{g}(\tilde{\nabla}_{\Phi E_1}E_2,E_3)-m\tilde{g}(\tilde{\nabla}_{\Phi E_2}\Phi E_1,E_3)+\tilde{g}(\tilde{\nabla}_{\Phi E_2}E_1,E_3)\\
&= g(h^l(\Phi E_2,E_1),E_3)-g(h^l(\Phi E_1,E_2),E_3),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\Phi E_1,\Phi E_2],\Phi L) &= \tilde{g}(\tilde{\nabla}_{\Phi E_1}\Phi E_2,\Phi L)-\tilde{g}(\tilde{\nabla}_{\Phi E_2}\Phi E_1,\Phi L)\\
&= \tilde{g}(\Phi\tilde{\nabla}_{\Phi E_1}E_2,\Phi L)-\tilde{g}(\Phi\tilde{\nabla}_{\Phi E_2}E_1,\Phi L)\\
&= m\tilde{g}(\Phi\tilde{\nabla}_{\Phi E_1}E_2,L)-\tilde{g}(\tilde{\nabla}_{\Phi E_1}E_2,L)-m\tilde{g}(\Phi\tilde{\nabla}_{\Phi E_2}E_1,L)+\tilde{g}(\tilde{\nabla}_{\Phi E_2}E_1,L)\\
&= m\tilde{g}(\tilde{\nabla}_{\Phi E_1}\Phi E_2,L)-\tilde{g}(\tilde{\nabla}_{\Phi E_1}E_2,L)-m\tilde{g}(\tilde{\nabla}_{\Phi E_2}\Phi E_1,L)+\tilde{g}(\tilde{\nabla}_{\Phi E_2}E_1,L)\\
&= g(h^s(\Phi E_2,E_1),L)-g(h^s(\Phi E_1,E_2),L),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\Phi E_1, \Phi E_2], N) &= \tilde{g}(\tilde{\nabla}_{\Phi E_1}\Phi E_2, N) - \tilde{g}(\tilde{\nabla}_{\Phi E_2}\Phi E_1, N) \\
&= \Phi E_1 g(\Phi E_2, N) - \tilde{g}(\Phi E_2, \tilde{\nabla}_{\Phi E_1} N) - \Phi E_2 g(\Phi E_1, N) + \tilde{g}(\Phi E_1, \tilde{\nabla}_{\Phi E_2} N) \\
&= -\tilde{g}(\Phi E_2, \tilde{\nabla}_{\Phi E_1} N) + \tilde{g}(\Phi E_1, \tilde{\nabla}_{\Phi E_2} N) \\
&= g(\Phi E_2, A_N \Phi E_1) - g(\Phi E_1, A_N \Phi E_2),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\Phi E_1, \Phi E_2], Z) &= \tilde{g}(\tilde{\nabla}_{\Phi E_1}\Phi E_2, Z) - \tilde{g}(\tilde{\nabla}_{\Phi E_2}\Phi E_1, Z) \\
&= \tilde{g}(\Phi\tilde{\nabla}_{\Phi E_1} E_2, Z) - \tilde{g}(\Phi\tilde{\nabla}_{\Phi E_2} E_1, Z) \\
&= \tilde{g}(\tilde{\nabla}_{\Phi E_1} E_2, \Phi Z) - \tilde{g}(\tilde{\nabla}_{\Phi E_2} E_1, \Phi Z) \\
&= g(\nabla_{\Phi E_1} E_2, \Phi Z) - g(\nabla_{\Phi E_2} E_1, \Phi Z) \\
&= -g(A_{E_2}\Phi E_1, \Phi Z) + g(A_{E_1}\Phi E_2, \Phi Z),
\end{aligned}$$

which gives the proof of our assertion. □

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] Acet, B.E., Yüksel Perktaş, S., Kılıç, E., *On lightlike geometry of para-Sasakian manifolds*, Scientific Work J., Article ID 696231, (2014).
[2] Acet, B.E., *Lightlike hypersurfaces of metallic semi-Riemannian manifolds*, Int J Geo Meth in Modern Phys., **15**(2018), 185–201.
[3] Acet, B.E., *Screen pseudo slant lightlike submanifolds of golden semi-Riemannian manifolds*, Hacettepe J Math and Stat., **49**(2020), 2037–2045.
[4] Duggal, K.L., Bejancu, A., Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Mathematics and Its Applications, Kluwer Publisher, 1996.
[5] Duggal, K.L., Şahin, B., *Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds*, Acta Math Hungar., **112**(2006), 107–130.
[6] Duggal, K.L., Şahin, B., *Lightlike submanifolds of indefinite Sasakian manifolds*, Int J Math and Math Sci., Article ID 57585, (2007).
[7] Duggal, K.L., Şahin, B., Differential Geometry of Lightlike Submanifolds, Frontiers in Mathematics, 2010.
[8] Crasmareanu, M.C., Hretcanu, C.E., *Golden differential geometry*, Chaos, Solitons and Fractals, **38**(2008), 1229–1238.
[9] Erdoğan, F.E., *Transversal lightlike submanifolds of metallic semi-Riemannian manifolds*, Turk J Math., **42**(2018), 3133–3148.
[10] Erdoğan, F.E.,Yüksel Perktaş, S., Acet, B.E., Blaga, A.M., *Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds*, J Geom Phys., **142**(2019), 111–120.
[11] Galloway, G.J., *Lecture notes on spacetime geometry*, Beijing Int. Math. Research Center, (2007), 1–55.
[12] Gezer, A., Cengiz, N,Salimov, A., *On integrability of golden Riemannian structure*, Turk J Math., **37**(2013), 693–703.
[13] Hretcanu, C.E., Crasmareanu, M.C., *On some invariant submanifolds in a Riemannian manifold with golden structure*, Analele Stiintifice Ale Universitatii Al.I. Cuza Iasi (N.S.), **53**(2007), 199–211.
[14] Hretcanu, C.E., Crasmareanu, M.C., *Metallic structure on Riemannian manifolds*, Rev Un Mat Argen., **54**(2013), 15–27.
[15] Hretcanu, C.E., Blaga, A.M., *Submanifolds in metallic semi-Riemannian manifolds*, Differ Geom Dynm Syst., **20**(2018), 83–97.
[16] Kalia, S., The generalizations of the golden ratio, their powers, continued fractions and convergents, http://math.mit.edu/research/highschool/primes/papers.php.
[17] Önen Poyraz, N., Yaşar, E., *Lightlike hypersurfaces of a golden semi-Riemannian manifold*, Mediter J Math., **14**(2017), 1–20.
[18] Spinadel, V.W., *The metallic means family and forbidden symmetries*, Int Math J., **2**(2002), 279–288.
[19] Şahin, B., *Almost poly-Norden manifolds*, Int J Maps Math., **1**(2018), 68–79.
[20] Yüksel Perktaş, S., Erdoğan, F.E., *On generalized CR-lightlike submanifolds*, Palestine J Math., **8**(2019), 200–208.
[21] Yüksel Perktaş, S., *Submanifolds of almost poly-Norden Riemannian manifolds*, Turk J Math., **44**(2020), 31–49.