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TIMELIKE V-MANNHEIM CURVES IN MINKOWSKI 3-SPACE $\mathbb{E}_1^3$

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Abstract

In this paper, timelike V-Mannheim curve, a new type Mannheim curve in Minkowski 3-Space $\mathbb{E}_1^3$ is characterized. Based on the timelike V-Mannheim curve, the properties of the timelike T, N, and B Mannheim curves are obtained.

Keywords: V-Mannheim Curve, Timelike V-Mannheim Curve, Minkowski 3-Space.

Özet

Bu çalışmada, yeni bir tip Mannheim eğrisi olan zamansı V-Mannheim eğri Minkowski 3-uzayında karakterize edildi. Zamansı V-Mannheim eğriden hareketle zamansı T, N ve B Mannheim eğriler elde edildi.

Anahtar Kelimeler: V-Mannheim Eğri, Zamansı V-Mannheim Eğri, Minkowski 3-uzayı.

1. Introduction

The theory of curves is one of the important study areas of classical differential geometry. In the theory of curves, special curves such as Mannheim curves, Bertrand curves, helix curves and involute-evolute curves have been studied. The characterizations of these curves in Euclidean and Minkowski spaces have been studied.

Mannheim curves which are one of the special curves studied in the theory of curves, were first found by Mannheim in 1878 and their definition were given as follows: The necessary and sufficient condition for any curve to be a Mannheim curve is to provide the equation $\kappa = \lambda(\kappa^2 + \tau^2)$ for the non-zero constant λ.

Some fundamental theorems about Mannheim curves are proved with the help of Riccatti equations in 3D Euclidean space (Blum, 1966). Let α and β be two curves in 3-dimensional Euclidean space. If the principal normal of the curve α and the binormal of the curve β are linearly dependent, the α curve is called the Mannheim curve, the β curve is called the Mannheim partner curve of the α curve (Liu and Wang, 2007). Mannheim curve pairs were defined in 4-dimensional Euclidean space (Matsuda and Yorozu, 2009).

Mannheim curves are also developed in 3-dimensional Minkowski space. The characterization of Mannheim offsets of spacelike and timelike ruled surfaces was obtained by Önder and Uğurlu in 2009. The characterizations of null Mannheim curves were obtained (Ergüt and Öztekin, 2011). There is no null Mannheim curve in Minkowski space (İlarslan et al., 2012). Additionally, a new type of Mannheim curve called the V-Mannheim curve was defined (Camcı,2021).

In Section 2, we present some of the definitions and properties where we used in the following sections.In Section 3, we describe timelike V -Mannheim curves in Minkowski 3-space $\mathbb{E}_1^3$ and give a characterization of a timelike V -Mannheim curve. Finally, we discuss the need for further research. This study is a part of the first author's master's thesis.

2.Preliminaries

Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit-speed curve with arc-length parameters "s". If we denote Serret-Frenet apparatus with $\{T, N, B, \kappa, \tau\}$, then we can define a curve $\beta: I \longrightarrow \mathbb{R}^3$ as

$$\beta(s) = \int V(s)ds + \lambda(s)N(s) \tag{1}$$

where $\lambda: I \to \mathbb{R}$ is a differentiable function, V is a unit vector field with

$$V(s) = u(s)T(s) + v(s)N(s) + w(s)B(s) \tag{2}$$

Definition 2.1. Let $\{\bar{\bar{T}}, \bar{\bar{N}}, \bar{\bar{B}}, \bar{\kappa}, \bar{\tau}\}$ be Serret-Frenet apparatus of the curve β defined in (1). If $N, \bar{\bar{B}}$ are linearly dependent, then (α, β) is V-Mannheim curve mate and α is called V-Mannheim curve. If $V = T$, then (α, β) cupple is called a classical Mannheim curve mate (Camcı, 2021).

Theorem 2.1: The curve α is a V-Mannheim curve if and only if it is satisfies

$$u(s)\kappa(s) - w(s)\tau(s) = \lambda(s)(\kappa^2(s) + \tau^2(s))$$

where

$$\lambda(s) = -\int v(s)ds$$

(Camcı, 2021).

Corollary 2.1. If $u(s) = 1$, $v(s) = w(s) = 0$, we have Mannheim (T-Mannheim) curve. From Theorem 2.1. λ is constant and $\kappa = \lambda(s)(\kappa^2(s) + \tau^2(s))$ (Camcı, 2021).

Corollary 2.2. If $v(s) = 1, u(s) = w(s) = 0$, we have B-Mannheim curve. From Theorem 2.1. λ is constant and $\kappa(s) = -\lambda(s)(\kappa^2(s) + \tau^2(s))$ (Camcı, 2021).

Theorem 2.2: In 3-dimensional Euclidean space, let K be a regular curve with a coordinate neighbourhood (I, α), Frenet apparatus $\{T, N, B, \kappa, \tau\}$ and arc length parameter s. Necessary and sufficient conditions for the curve K to be a V-Mannheim curve

$$2u(s)\kappa(s) = \frac{1}{\lambda}\left[u^2(s) + u(s)\sqrt{u^2(s) + w^2(s)}\cos(2\theta(s) + \theta_0(s))\right]$$

$$2w(s)\tau(s) = \frac{1}{\lambda(s)}\left[-w^2(s) + w(s)\sqrt{u^2(s) + w^2(s)}\sin(2\theta(s) + \theta_0(s))\right]$$

where $\cos\theta_0(s) = \frac{u(s)}{\sqrt{u^2(s)+w^2(s)}}$ and $\sin\theta_0(s) = \frac{w(s)}{\sqrt{u^2(s)+w^2(s)}}$ (Camcı, 2021).

Corollary 2.3. If $u(s) = 1$, $v(s) = w(s) = 0$, we have Mannehim (T-Mannheim) curve. By using Theorem 2.1 and Theorem 2.2. ,the curvature and torsion of the curve α are obtained as:

$$\kappa(s) = R(\cos\theta(s))^2$$

$$\tau(s) = R\cos\theta\,(s)\sin\theta(s)$$

where $R = \frac{1}{\lambda}$ is a constant (Camcı, 2021).

Corollary 2.4. If $v(s) = 1, u(s) = w(s) = 0$, we have B-Mannheim curve. By using Theorem 2.1 and Theorem 2.2. ,the curvature and torsion of the curve α are obtained as:

$$\tau(s) = -R\sin^2\theta(s)$$

$$\kappa(s) = R\cos\theta(s)\sin\theta\,(s)$$

where $R = \frac{1}{\lambda}$ is a constant (Camcı, 2021).

Theorem 2.3. The curve β is V-Mannheim mate of the curve α if only and if

$$\frac{d\bar{\bar{\tau}}}{d\bar{\bar{s}}} = \frac{v\bar{\bar{\tau}}\sqrt{1+\lambda^2\bar{\bar{\tau}}^2}}{\lambda\sqrt{1-v^2}} + \frac{\bar{\bar{\kappa}}}{\lambda}(1+\lambda^2\bar{\bar{\tau}}^2)$$

is satisfied (Camcı, 2021).

Next, recall that Minkowski 3-space$\mathbb{E}_1^3$ is Euclidean 3-space $\mathbb{E}^3$ equipped with the metric

$$g := -{dx_1}^2 + {dx_2}^2 + {dx_3}^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$ [13]. In this space, a vector has one of three casual characters according to this metric. If $g(u,u) > 0$ or $u = 0$, then u is a spacelike vector,

if $g(u,u) < 0$, then u is a timelike vector, and if $g(u,u) < 0$ and $u \neq 0$, then u is a null (lightlike) vector. Moreover, an arbitrary curve α in Minkowski 3-space $\mathbb{E}_1^3$ is called according to the causal chracter of its the velocity vector $\alpha'(s)$. A curve α is called spacelike, timelike, or null, if $\alpha'(s)$ is spacelike, timelike or null, respectively. For a timelike curve with Serret-Frenet apparatus $\{T, N, B, \kappa, \tau\}$, the following formulas hold:

$$T' = \kappa N$$

$$N' = \kappa T + \tau B$$

$$B' = -\tau N$$

$g(T,T) = -1$, $g(B,B) = 1$, $g(N,N) = 1$

$g(T,N) = 0$, $g(B,N) = 0$, $g(T,B) = 0$

$T \times N = \mathrm{B}$, $B \times T = N$, $N \times B = -\mathrm{T}$

3. Timelike V-Mannheim Curve

In this section, we define timelike V -Mannheim curves in Minkowski 3-Space $\mathbb{E}_1^3$ and investigate some of their basic properties. In addition, we give a characterization for this type of curves.

Definition 3.1. Let $\alpha: I \to \mathbb{E}_1^3$, $\alpha = \alpha(\mathrm{s})$ be a unit-speed timelike curve with Frenet apparatus {T, N, B, κ, τ} and $\beta: I \to \mathbb{E}_1^3$, $\beta = \beta(\mathrm{s})$ be a regular curve with Frenet apparatus $\{\bar{\bar{T}}, \bar{\bar{N}}, \bar{\bar{B}}, \bar{\bar{\kappa}}, \bar{\bar{\tau}}\}$. We can define a curve β as

$$\beta(s) = \int V(s)ds + \lambda(s)N(s)$$

where $\alpha: I \to \mathbb{R}_1^3$ is a differentiable function and V is a unit vector field with,

$$V(s) = u(s)T(s) + v(s)N(s) + w(s)B(s)$$

If the vector fields $N, \bar{\bar{B}}$ are linearly dependent, then the curve pair (α, β) is called a timelike V - Mannheim curve mate and γ is called a timelike V -Mannheim curve. Moreover, especially, if V = T (N or B), then (α, β) is a timelike T (N or B)-Mannheim curve mate.

Theorem 3.1. Let unit-speed and timelike curves α and β in 3-dimensional Minkowski space. Let the Serret-Frenet elements of these curves be {T, N, B, κ, τ} and $\{\bar{\bar{T}}, \bar{\bar{N}}, \bar{\bar{B}}, \bar{\bar{\kappa}}, \bar{\bar{\tau}}\}$ respectively and the timelike curve β be the V-Mannheim curve pair of the timelike curve α. Necessary and sufficient conditions for an α curve to be a timelike V-Mannheim curve

$$u(s)\kappa(s) - w(s)\tau(s) = \lambda(s)(\tau^2(s) - \kappa^2(s)) \tag{3.1}$$

$$\lambda(s) = -\int v(s)\, ds \tag{3.2}$$

Proof. Since the timelike curve β is the V-Mannheim curve pair of the timelike curve α, from equation (3.1), we have

$$\beta(s) = \int_0^s V(u)du + \lambda(s)N(s) \tag{3.3}$$

Derivating of the equation (3.3) we get

$$\frac{d\bar{\bar{s}}}{ds}\bar{\bar{T}} = (u(s) + \lambda(s)\kappa(s))T(s) + (v(s) + \lambda'(s))N(s) + (w(s) + \lambda(s)\tau(s))B(s) \tag{3.4}$$

with the definition of Mannheim curve, we get

$$\lambda(s) = -\int v(s)ds.$$

from the equation (3.4), we have

$$\bar{\bar{T}}(\bar{\bar{s}}) = \frac{ds}{d\bar{\bar{s}}}(u(s) + \lambda(s)\,\kappa(s))T(s) + \frac{ds}{d\bar{\bar{s}}}(w(s) + \lambda(s)\tau(s))B(s) \tag{3.5}$$

From the equation (3.5), we get

$$\frac{ds}{d\bar{\bar{s}}}\left(u(s)\ +\lambda(s)\,\kappa(s)\,\right)=\cosh\theta\,(s) \tag{3.6}$$

$$\frac{ds}{d\bar{\bar{s}}}\left(w(s)\ +\lambda(s)\,\tau(s)\,\right)=\sinh\theta\,(s) \tag{3.7}$$

Hence we get the tangent vector of the timelike curve β,

$$\bar{\bar{T}}=\cosh\theta(s)\ T(s)\ +\sinh\theta(s)\,B\,(s) \tag{3.8}$$

by derivating of the equation (3.8), we get

$$\frac{d\bar{\bar{s}}}{ds}\bar{\bar{\kappa}}(\bar{\bar{s}})\bar{\bar{N}}(\bar{\bar{s}})=\left(\frac{d\theta}{ds}\sinh\theta(s)\right)T(s)\ +(\kappa(s)\cosh\theta(s)\ -\tau(s)\ \sinh\theta(s)\,)N(s)\ +\left(\frac{d\theta}{ds}\cosh\theta\,(s)\right)B(s) \tag{3.9}$$

By using the definition of Mannheim curve, we get

$$u(s)\kappa(s)-w(s)\tau(s)=\lambda(s)(\tau^2(s)-\kappa^2(s))$$

On the other hand derivating equation (3.1), we get

$$\bar{\bar{T}}(\bar{\bar{s}})=\cosh\theta(s)\ T(s)\ +\sinh\theta(s)\,B\,(s) \tag{3.10}$$

From the equations (3.9) and (3.10), we get

$$\bar{\bar{N}}(\bar{\bar{s}})=\sinh\theta(s)\,T(s)+\cosh\theta(s)\,B(s) \tag{3.11}$$

$$\bar{\bar{B}}(\bar{\bar{s}})=\left((\kappa(s)\sinh\theta(s)-\tau(s)\cosh\theta(s))\frac{ds}{d\bar{\bar{s}}}\right)N(s) \tag{3.12}$$

From the equation (3.12), if $N,\bar{\bar{B}}$ are linearly dependent then it completes the proof.

Corollary 3.1. In equation (3.2), if $u(s) = 0$, $v(s) = 0$, $w(s) = 1$ are selected, then the curve α becomes the timelike T-Mannheim curve and $\lambda(s)$ is constant. Also, from Theorem 3.1. the curvature of the curve α is following,

$$\kappa(s) = \lambda(s)(\tau^2(s) - \kappa^2(s))$$

Corollary 3.2. In equation (3.2), if $u(s) = 1$, $v(s) = 0$, $w(s) = 0$ is selected, then the curve α becomes the timelike T-Mannheim curve and $\lambda(s)$ is constant. So from Theorem 3.1., we get,

$$\tau(s) = \lambda((\mathrm{s})\kappa^2(\mathrm{s}) - \tau^2(\mathrm{s}))$$

Example 3.1. Let the $\alpha(s) = \left(2\sqrt{3}\sinh s, \sqrt{11}s, 2\sqrt{3}\cosh s\right)$, should be timelike a curve in Minkowski 3-space. Frenet vectors and curvatures of α are as follows:

$$T(s) = \left(2\sqrt{3}\cosh s, \sqrt{11}, 2\sqrt{3}\sinh s\right)$$

$$N(s) = (\sinh s, 0, \cosh s)$$

$$B(s) = \left(\sqrt{11}\cosh s, -2\sqrt{3}, \sqrt{11}\sinh s\right)$$

$$\kappa(s) = 2\sqrt{3}$$

$$\tau(s) = 12\sqrt{11}$$

Let the curve β be the timelike V-Mannheim curve mate of the curve α. In equation (3.2), if $u(s) = 0, v(s) = 0, w(s) = 1$ are chosen, then (α, β) pair becomes B-Mannheim curve mate. In this case, from equation (3.1) we get

$$\lambda(s) = -\frac{\sqrt{11}}{131}$$

From (3.3), we have

$$\beta(s) = \int B(s)ds + \lambda(s)N(s)$$

Hence we obtain the curve β as follows:

$$\beta(s) = \left(\frac{130\sqrt{11}}{131}\sinh(s)\,, -2\sqrt{3}\,s, \frac{130\sqrt{11}}{131}\cosh(s)\right)$$

The graph of the timelike B-Mannheim curve pair with (α, β) is shown in Figure 1.

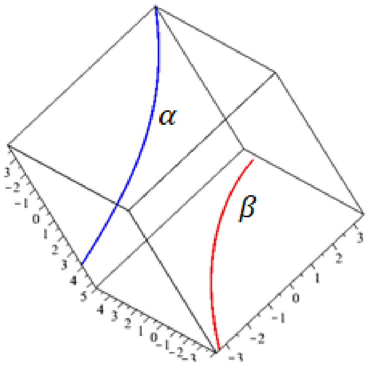

Figure 1. (α, β) timelike B-Mannheim curve mate

Theorem 3.2. Let α be the Serret-Frenet vectors $\{T, N, B\}$, the curvatures κ and τ of the timelike V-Mannheim curve in 3-dimensional Minkowski space. In this case, the timelike curve α is the V-Mannheim curve if and only if

$$2u(s)\kappa(s) = \frac{1}{\lambda}\left[-u^2(s)\lambda(s) + u(s)\sqrt{u^2(s) - w^2(s)}\cosh(2\theta(s) - \theta_0(s))\right] \tag{3.13}$$

$$2w(s)\tau(s) = \frac{1}{\lambda(s)}\left[-w^2(s) + w(s)\sqrt{u^2(s) - w^2(s)}\sinh(2\theta(s) - \theta_0(s))\right] \tag{3.14}$$

where $\cosh\theta_0(s) = \frac{u(s)}{\sqrt{u^2(s) - w^2(s)}}$ and $\sinh\theta_0(s) = \frac{w(s)}{\sqrt{u^2(s) - w^2(s)}}$.

Proof. Since α is a timelike V-Mannheim curve, if $\kappa(s)$ and $\tau(s)$ are subtracted from (3.3), we get

$$\kappa(s) = \sqrt{\frac{u(s)\kappa(s) - w(s)\tau(s)}{\lambda(s)}}\sinh\theta(s) \tag{3.15}$$

$$\tau(s) = \sqrt{\frac{u(s)\kappa(s) - w(s)\tau(s)}{\lambda(s)}}\cosh\theta(s) \tag{3.16}$$

If the expressions (3.15) and (3.16) are substituted in the equation (3.3) and rearranged, we get

$$2u(s)\kappa(s) = \frac{1}{\lambda(s)}\left[-u^2(s) + u(s)\sqrt{u^2 - w^2}\left(\frac{u(s)}{\sqrt{u^2 - w^2}}\cosh 2\theta(s) - \frac{w}{\sqrt{u^2 - w^2}}\sinh 2\theta(s)\right)\right] \tag{3.17}$$

$$2w(s)\tau(s) = \frac{1}{\lambda(s)}\left[-w^2(s) - w(s)\sqrt{u^2(s) - w^2(s)}(\sinh\theta_0(s)\cosh 2\theta(s) + \cosh\theta_0(s)\sinh 2\theta(s))\right] \tag{3.18}$$

Here if we say $\cosh\theta_0(s) = \frac{u(s)}{\sqrt{u^2(s)-w^2(s)}}$ and $\sinh\theta_0(s) = \frac{w(s)}{\sqrt{u^2(s)-w^2(s)}}$ we get

$$2u(s)\kappa(s) = \frac{1}{\lambda(s)}\left[-u^2(s) + u(s)\sqrt{u^2(s) - w^2(s)}\cosh(2\theta(s) - \theta_0(s))\right]$$

$$2w(s)\tau(s) = \frac{1}{\lambda(s)}\left[-w^2(s) + w(s)\sqrt{u^2 - w^2}\sinh(2\theta(s) - \theta_0(s))\right]$$

Corollary 3.3. In equation (3.2), if $u(s) = 0$, $v(s) = 0$, $w(s) = 1$ are chosen, then the curve α becomes the timelike T-Mannheim curve and $\lambda(s)$ is constant. Then the curvature and torsion of the curve α are obtained from Theorem 3.1. and Theorem 3.2. as,

$$\kappa(s) = Rsinh^2\theta(s)$$

$$\tau(s) = R\sinh\theta(s)\cosh\theta(s)$$

where $R = \frac{1}{\lambda(s)}$.

Corollary 3.4. If $u(s) = 0$, $v(s) = 0$, $w(s) = 1$ are selected in the equation (3.2) then the curve α becomes the timelike B-Mannheim curve and $\lambda(s)$ is constant.Then the curvature and the torsion of the curve α are computed from Theorem 3.1. and Theorem 3.2. as the following,

$$\kappa(s) = R\sinh\theta(s)\cosh\theta(s)$$

$$\tau(s) = -Rcosh^2\theta(s)$$

where $R = \frac{1}{\lambda(s)}$.

Theorem 3.3. Let $\{T, N, B, \kappa, \tau\}$ and $\{\bar{\bar{T}}, \bar{\bar{N}}, \bar{\bar{B}}, \bar{\bar{\kappa}}, \bar{\bar{\tau}}\}$ be the Serret-Frenet elemets of the unit speed timelike curves α and β in 3-dimensional Minkowski space, respectively. β is timelike V-Mannheim curve mate of the curve α if and only if

$$\frac{d\bar{\bar{\tau}}}{d\bar{\bar{s}}} = \frac{v(s)\bar{\bar{\tau}}(\bar{\bar{s}})\sqrt{1-\lambda^2(s)\bar{\bar{\tau}}^2(\bar{\bar{s}})}}{\lambda(s)\sqrt{1-v^2(s)}} + \frac{\bar{\bar{\kappa}}(\bar{\bar{s}})}{\lambda(s)}(1-\lambda^2(s)\bar{\bar{\tau}}^2(\bar{\bar{s}})) \tag{3.19}$$

Proof. Let the curve β be the timelike V-Mannheim curve pair of the curve α. In this case, from the equation (3.3) we get,

$$\int_0^s V(u)du = \beta(s) - \lambda(s)\bar{\bar{B}}(\bar{\bar{s}}) \tag{3.20}$$

If the derivative of the equation (3.20) is taken according to the $\bar{\bar{s}}$ arc-length parameter thereby

$$\frac{ds}{d\bar{\bar{s}}}\big(u(s)T(s) + v(s)N(s) + w(s)B(s)\big) = \bar{\bar{T}}(\bar{\bar{s}}) - \frac{d\lambda}{d\bar{\bar{s}}}\bar{\bar{B}}(\bar{\bar{s}}) + \lambda\bar{\bar{\tau}}(\bar{\bar{s}})\bar{\bar{N}}(\bar{\bar{s}}) \tag{3.21}$$

From the definition of Mannheim curve, we get

$$\frac{ds}{d\bar{\bar{s}}}u(s)T(s) + \frac{ds}{d\bar{\bar{s}}}w(s)B(s) = (\cosh\theta + \lambda\bar{\bar{\tau}}\sinh\theta)T(s) + (\sinh\theta + \lambda\bar{\bar{\tau}}\cosh\theta)B(s) \tag{3.22}$$

where

$$\frac{ds}{d\bar{\bar{s}}}u(s) = \cosh\theta(s) + \lambda(s)\bar{\bar{\tau}}(\bar{\bar{s}})\sinh\theta\,(s) \tag{3.23}$$

$$\frac{ds}{d\bar{\bar{s}}}w(s) = \sinh\theta\,(s) + \lambda(s)\bar{\bar{\tau}}(\bar{\bar{s}}\,)\cosh\theta(s) \tag{3.24}$$

The torsion of the curve β computed from (3.23) and (3.24) as

$$\bar{\bar{\tau}}(\bar{\bar{s}}) = \frac{1}{\lambda(s)} \frac{w(s)\cosh\theta(s) - u(s)\sinh\theta(s)}{\mathrm{u}(s)\cosh\theta(s) - w(s)\sinh\theta(s)} \tag{3.25}$$

where

$$(\mathrm{u}(s)\cosh\theta(s) - w(s)\sinh\theta(s))^2 - (w(s)\cosh\theta(s) - u(s)\sinh\theta(s))^2 = u^2(s) - w^2(s) = 1 - v^2(s) \tag{3.26}$$

Let us define a differentiable function as $\varphi: I \to \mathbb{R}\ (s \to \varphi(s))$ where we can use in the equation (3.26). From the equation (3.26), we get

$$\sqrt{1 - v^2(s)}\cosh\varphi(s) = \mathrm{u}(s)\cosh\theta(s) - w(s)\sinh\theta\,(s) \tag{3.27}$$

$$\sqrt{1 - v^2(s)}\sinh\varphi(s) = w(s)\cosh\theta(s) - u(s)\sinh\theta(s) \tag{3.28}$$

Substituting the equations (3.27) and (3.28) in the equation (3.25) we get,

$$\bar{\bar{\tau}}(\bar{\bar{s}}) = \frac{1}{\lambda(s)}\tanh\varphi(s). \tag{3.29}$$

If the derivative of the equation (3.29) taken according to the $\bar{\bar{s}}$ arc parameter we have,

$$\frac{d\bar{\bar{\tau}}}{d\bar{\bar{s}}} = \frac{ds}{d\bar{\bar{s}}}\left(\frac{d\lambda}{ds}\tanh\varphi(s) + \frac{1}{\lambda(s)}\frac{d}{ds}(\tanh\varphi(s))\right) \tag{3.30}$$

If the equation (3.25) arranged again, we get the desired expression,

$$\frac{d\bar{\bar{\tau}}}{d\bar{\bar{s}}} = \frac{v(s)\bar{\bar{\tau}}(\bar{\bar{s}})\sqrt{1 - \lambda^2(s)\bar{\bar{\tau}}^2(\bar{\bar{s}})}}{\lambda(s)\sqrt{1 - v^2(s)}} - \frac{\bar{\bar{\kappa}}(\bar{\bar{s}})}{\lambda(s)}(1 - \lambda^2(s)\bar{\bar{\tau}}^2(\bar{\bar{s}}))$$

Conversely, if the derivative of (3.20) taken according to the $\bar{\bar{s}}$ arc-length parameter, the equation (3.21) obtained. If the equations (3.8) and (3.11) substituted in the equation then we have

$$\frac{ds}{d\bar{\bar{s}}} = \sqrt{\frac{1-\lambda^2(s)\bar{\bar{\tau}}^2(\bar{\bar{s}})}{1-v^2(s)}} \tag{3.31}$$

$$\frac{ds}{d\bar{\bar{s}}}(u(s)T(s)+w(s)B(s)) = \bar{\bar{T}}(\bar{\bar{s}})+\lambda(s)\bar{\bar{\tau}}(\bar{\bar{s}})\bar{\bar{N}}(\bar{\bar{s}}) \tag{3.32}$$

If the derivative of (3.32) taken according to the $\bar{\bar{s}}$ arc parameter, we get

$$\frac{d\bar{\bar{s}}}{ds}\frac{d^2s}{d\bar{\bar{s}}^2}\left(\bar{\bar{T}}+\lambda\bar{\bar{\tau}}\bar{\bar{N}}\right)+(\frac{ds}{d\bar{\bar{s}}})^2(u\kappa-w\tau)N = \lambda\bar{\bar{\tau}}\bar{\bar{\kappa}}\bar{\bar{T}}+\left[\frac{d(\lambda)}{d\bar{\bar{s}}}\bar{\bar{\tau}}+\frac{d(\bar{\bar{\tau}})}{d\bar{s}}\lambda\right]\bar{\bar{N}}+\lambda\bar{\bar{\tau}}^2\bar{\bar{B}} \tag{3.33}$$

where

$$\frac{d\bar{\bar{s}}}{ds}\frac{d^2s}{d\bar{\bar{s}}^2} = \lambda(s)\bar{\bar{\tau}}(\bar{\bar{s}})\bar{\bar{\kappa}}(\bar{\bar{s}}) \tag{3.34}$$

$$\bar{\bar{\kappa}}(\bar{\bar{s}})+\frac{d(\lambda\bar{\bar{\tau}})}{d\bar{\bar{s}}} = \frac{d\bar{\bar{s}}}{ds}\frac{d^2s}{d\bar{\bar{s}}^2}(\lambda\bar{\bar{\tau}}) = \lambda^2(s)\bar{\bar{\tau}}^2(\bar{\bar{s}})\bar{\bar{\kappa}}(\bar{\bar{s}}). \tag{3.35}$$

It can be seen easily from the equations (3.33), (3.34), (3.35) that $\{N,\bar{\bar{B}}\}$ are linearly dependent. From here the proof is completed.

4. Conclusion

This study considers the V-Mannheim curves in 3-dimensional Euclidean space put forward by Camcı (2021). We characterized these curves in 3-dimensional Euclidean space are in Minkowski 3-space. First, we took the pairs of these curves as time curves. Afterwards the temporal V-Mannheim curve definition and characterization are given. Then, based on the temporal V-Mannheim curve definition, temporal T-Mannheim, N-Mannheim and B-Mannheim curve definitions are made and enriched with examples.

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Conflicts of Interest

The authors declare no conflict of interest.

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