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CONSTANT PSEUDO-ANGLE LIGHTLIKE SURFACES

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Abstract. The oriented angles between lightlike vectors cannot be defined properly compared to the timelike vectors in the Minkowski spacetime. Therefore, we use the pseudo-angles between any non-lightlike or lightlike vectors to develop the theory of lightlike surfaces having constant angle with a fixed non-lightlike direction. We investigate some geometric properties on these surfaces such as being a tangent developable. Besides, we construct the constant angle lightlike ruled surfaces by means of the null helices. We give several examples to illustrate the obtained surfaces.

1. Introduction

In the differential geometry and physics, especially in the theory of general relativity, lightlike hypersurfaces play an important role because they are considered as models for different horizon types of black holes. A black hole is a region of space-time containing a huge mass compacted into an extremely small volume. The gravity inside the black hole is so strong that even light with its remarkable speed cannot escape (see [1]). After the Einstein's theory of gravitation was first published in 1915, numerous research papers were devoted to the mathematical and physical theory of black holes. For subsequent information about black holes and the applications of lightlike hypersurfaces, see [3, 7, 11, 12, 23, 24].

A constant angle surface is a surface which has tangent planes making a constant angle with a fixed constant vector field at every point in the Euclidean meaning (for more detail, see [8, 9, 20]). These surfaces are considered as a generalization of the concept of helix. They represent good models to describe some phenomena in physics of interfaces in liquids crystals and of layered fluids (see [6]). Lopez and Munteanu extend the theory of constant angle surfaces to the three dimensional Minkowski spacetime [18]. However, due to the variety of causal characters

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of a vector in Minkowski space, there is not a natural concept of angle between two arbitrary vectors so it is only possible to define the angle between timelike vectors. Therefore, they state that a constant angle surface in Minkowski space is actually a spacelike surface whose unit normal vector makes a constant hyperbolic angle with a fixed timelike vector at every point. In that case, it is possible to define the angle since any unit normal vector field of a spacelike immersion is timelike at each point. However, when we come to the concept of lightlike surfaces, following question arises: Is it possible to define the constant angle lightlike surfaces in Minkowski spacetime? To answer this question we use the concept of pseudo-angles between lightlike (null) vectors and the others. Helzer introduced an oriented pseudo-angle between any two null or non-null unit vectors in [13]. Pseudo-angles provide a generalization of the oriented hyperbolic angles between the unit non-null vectors [4]. That is to say, an oriented hyperbolic angle between non-null unit vectors in Minkowski plane is equivalent to the oriented pseudo-angle between those vectors. In [21], the author introduce pseudo-perpendicular vectors in Minkowski plane. In the mentioned work, it is shown that any unit non-null or null vector can be associated exactly eight vectors which are pseudo-perpendicular to it. So it is given geometric interpretations of the oriented pseudo-angles in terms of the hyperbolic arcs by using the pseudo-perpendicular vectors. Pseudo-angles have applications in several fields, such as in computing Polyakov extrinsic energy of Polyakov string solutions [3] or in Backlund transformations [22].

Ruled surfaces are generated by the continuous movement of a straight line in the space and they are one of the most important topics in differential geometry. Also, ruled surfaces play an important role in the study of rational design problems in spatial mechanisms since they represent the trajectories of the oriented lines embedded in a moving rigid body in spatial motion. This kind of a surface can be used in many scientific fields as well as in Computer Aided Geometric Design (CAGD). Different from the Euclidean space, there exist several types of the ruled surfaces according to the Lorentzian casual characters of lines and curves lying on the surface in Minkowski space. In [25], Kim and Yoon give classifications of the ruled surfaces in Minkowski 3-space. Also, Ali [2] introduces two types of non-lightlike ruled surfaces in Minkowski 3-space: Those of constant slope parallel to the tangent of a timelike general helix and those parallel to the normal of a timelike slant helix. However, there is still a gap in the theory of lightlike ruled surfaces in Minkowski 3-space.

In this paper, first we introduce the concept of lightlike constant-pseudo angle surfaces in Section 3. We give Theorem 3 and Theorem 5 to classify these surfaces in two types. Moreover, we show that any constant pseudo-angle lightlike surface is actually a ruled surface along a spacelike base curve with lightlike rulings. We give some related corollaries and examples. In Section 4, we define a constant angle lightlike ruled surface by means of the Cartan frame of a null helix, a pseudo-null curve as a slant helix or a Cartan slant helix. We see that, they are ruled surfaces along a non-null base curve with null rulings similar to the surfaces introduced in the previous section. We investigate such ruled surfaces in three cases depending on the type of chosen helix. We also give some related examples to support the theory.

2. Preliminaries

2.1. **Pseudo-angles in the Minkowski plane.** In this section, we give a brief information on the pseudo-perpendicular vectors in Minkowski plane and introduce the concept of pseudo-angles between lightlike and non-lightlike vectors in terms of the hyperbolic arcs of finite hyperbolic lengths (for detailed information see [13,21]). The Minkowski plane E_1^2 is an affine plane endowed with the standard indefinite scalar product given by

$$g(x, y) = x_1 y_1 - x_2 y_2$$

for any two vectors $x(x_1, x_2)$ and $y = (y_1, y_2)$. A vector $v \neq 0$ has a casual character spacelike, timelike or lightlike (lightlike) iff $g(v, v) > 0$, $g(v, v) < 0$ or $g(v, v) = 0$, respectively. The vector $v = 0$ is spacelike and the norm of a given vector is defined as $\|v\| = \sqrt{|g(v, v)|}$.
$e_2 = (0, 1)$ is a unit timelike vector and an arbitrary vector v in E_1^2 is called future-pointing or past-pointing if $g(v, e_2) < 0$ or $g(v, e_2) > 0$, respectively. Moreover, any two timelike vectors have the same time-orientation when they are both future pointing or past pointing vectors. On the other hand, if $g(x, y) < 0$ for any two lightlike vectors x and y, we say they have the same time-orientation.
Let $O = e_1, e_2$ be the standard orthonormal basis of E_1^2. Then we define a function $\phi_O(u)$ by

$$\phi_O(u) = \begin{Bmatrix} \ln|a + b| & if & a + b \neq 0 \\ -\ln|a - b| & if & a + b = 0 \end{Bmatrix}$$

where $u = ae_1 + be_2$ is a lightlike or non-lightlike unit vector and $a, b \in \mathbb{R}$ [21].

Definition 1. *If u and v are unit non-lightlike or lightlike vectors, then the oriented pseudo-angle $\phi(u, v)$ from u to v is given by,*

$$\phi(u, v) = \phi_O(u, v) = \phi_O(v) - \phi_O(u)$$

We note that the function $\phi_O(u, v)$ only depends on the orientation of the bases O. Also, one can show that the oriented hyperbolic angles between the unit non-lightlike vectors in the Minkowski plane are actually equal to the oriented pseudo-angles between them [21].

Definition 2. *Let u and v be the unit non-lightlike or lightlike vectors in Minkowski plane. Then we say they are mutually pseudo-perpendicular vectors, if $\phi(u, v) = 0$ [21].*

Moreover, for any unit non-lightlike or lightlike vector in the Minkowski plane, it can be associated eight vectors pseudo-perpendicular to it (for more information see [21]).

Any oriented pseudo-angle $\phi(a,b)$ can be associated a unique hyperbolic arc of finite hyperbolic length. This hyperbolic arc is determined by the central pseudo-angle enclosed by two unit non-lightlike vectors on the non-lightlike unit circle [13] . The central pseudo-angle of the unit spacelike circle is pseudo-angle formed by two unit timelike future-pointing (or past-pointing) vectors. Analogously, central pseudo-angle of the unit timelike circle is pseudo-angle formed by two unit spacelike vectors.

On the other hand, a measure of an unoriented pseudo-angle $|\phi(a,b)|$ is equal to the hyperbolic length of the hyperbolic arc determined by two unit non-lightlike vectors pseudo perpendicular to a and b, where a and b are the unit non-lightlike or lightlike vectors. The oriented and unoriented pseudo-angles between unit non-lightlike or lightlike vectors are distinguished in six cases depending on the causal characters of the vectors a and b in [21].

2.2. **Lightlike surfaces.** In this section, we refer to the fundamental notions about the theory of lightlike surfaces (for a further information on the lightlike surfaces, see [10, 11]).

Let $\bar{M}$ be a 3 dimensional semi-Riemannian manifold endowed with the metric $\bar{g}$. If M is a lightlike surface in $\bar{M}$, there exists a subspace $T_pM^{\perp}$ at every point such that

$$T_pM^{\perp} = \left\{v_p \in T_p\bar{M} : \bar{g}_p\left(v_p, w_p\right) = 0,\ \forall w_p \in T_pM\right\}.$$

where T_pM is the tangent plane on the surface M. Then the radical distribution is defined by,

$$RadT_pM = T_pM \cap T_pM^{\perp} \neq \{0\},\ \forall p \in M.$$

The rank of $RadTM$ is 1 for the lightlike surface M.

The complement vector bundle to $RadTM$ in TM is $S(TM)$ which is called a *screen distribution.* Clearly, $S(TM)$, is a non-degenerate subspace. Hence, one can write the following decomposition,

$$\begin{aligned} TM &= RadTM \oplus_{ort} S(TM) \\ RadTM &= TM \cap TM^{\perp} \end{aligned}$$

where $TM^{\perp} = \underset{p\in M}{\cap} T_pM^{\perp}$.

Theorem 1. *Let $(M, g, S(TM))$ be a lightlike surface in $\bar{M}$. If U is a coordinate neighborhood of M and $RadTM = Span\{\xi\}$. There exist a smooth vector field N such that*

$$\bar{g}\left(\xi, N\right) = 1 \quad and \quad \bar{g}\left(N, W\right) = 0.$$

where W is a non-lightlike vector field in $S(TM)$.

The subspace $ltr(TM) = Span\{N\}$ is called *lightlike transversal vector bundle.* Also, the following decomposition is satisfied;

$$T\bar{M}\mid_M = TM \oplus tr(TM)$$

where $tr(TM) = ltr(TM) \oplus S(TM^{\perp})$. In this case, $\{\xi, W, N\}$ is a quasi-ortonormal basis of $\bar{M}$ along M. The Weingarten equations are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V \tag{2}$$

where $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$. Also, $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$, $\nabla_X Y$ and $\nabla^t_X V$ are the linear connections on M and $tr(TM)$ respectively. Note that ∇ is a torsion free induced linear connection. Also, $A_V X$ and $h(X,Y)$ are the shape operator and second fundamental form on M, respectively. Locally suppose ξ, N is a pair of vector fields on U in Definition 1. Then we define a symmetric bilinear form B and 1-form τ on M by

$$B(X,Y) = \bar{g}(h(X,Y),\xi) \quad \text{and} \quad \tau(X) = \bar{g}(\nabla^t_X N, \xi) \tag{3}$$

The equations (1) and (2) become

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N \tag{4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N. \tag{5}$$

3. Constant Pseudo-Angle Lightlike Surfaces

Let M be a lightlike surface in E^3_1. The tangent plane of M is spanned by the pseudo orthogonal vector fields $\{e_1, \xi\}$ where ξ belongs to the radical distribution and N be the transversal vector field at every point on M. To describe the constant pseudo-angle lightlike surfaces, we consider a fixed non-lightlike vector field U making a constant pseudo-angle with the vector field N. According to the position of U, we classify such surfaces in two types. In all cases, since U is non-lightlike, there exists a non-lightlike vector ν which is pseudo-perpendicular to N at every point. We consider the pseudo-angle ϕ between the vector fields U and N as defined in the Section 2.

Different from the constant angle surfaces in the Euclidean space, pseudo-angle between the transversal vector field N and the constant direction U can be zero on a lightlike surface. Since U and N are pseudo-perpendicular for $\phi = 0$, U is one of the pre-defined eight vectors given in [21]. We assume that ϕ is a non-zero constant throughout this work.

Type I

Let U lies in the plane of $\{N, \xi\}$. We decompose U as,

$$U = a\xi + bN$$

where a and b are constant functions. By using the logarithmic forms of the inverse hyperbolic functions, we reach to the following form:

$$U = \sinh\phi\xi + \cosh\phi N. \tag{6}$$

We denote $\langle \cdot, \cdot \rangle$ as the Lorentzian metric and $\bar{\nabla}$ as the Levi-Civita connection in E_1^3 when g is the metric and ∇ is the L.C. connection in M. Since U is constant, from (6) we have

$$\sinh\phi \bar{\nabla}_X \xi + \cosh\phi \bar{\nabla}_X N = 0$$

where $X \in T_P M$. Also we know that $B(X, \xi) = 0$ and N is a lightlike vector field, we have

$$\sinh\phi \langle \nabla_X \xi, N \rangle = 0.$$

Then we obtain $\tau(X) = 0$ and this implies $\bar{\nabla}_X N = -A_N X$.

Let $\{v_1, v_2\}$ show the local basis in the tangent plane $T_P M$ and we denote

$$b_{ij} = B(v_i, v_j) = -\langle A v_i, v_j \rangle.$$

We can write the following decompositions by using the Gauss and Weingarten formulas given in (2.4) and (2.5):

$$\bar{\nabla}_{v_i} V_j = \nabla_{v_i} V_j + b_{ij} N \tag{7}$$

$$\bar{\nabla}_{v_i} N = b_{i1} v_1 + b_{i2} v_2 \tag{8}$$

where V_j is a tangent vector field that extends v_j. Now, take the derivative of (6) with respect to e_1 then we have

$$\sinh\phi \bar{\nabla}_{e_1} \xi + \cosh\phi \bar{\nabla}_{e_1} N = 0. \tag{9}$$

By combining (8) and (9), we find

$$\bar{\nabla}_{e_1} \xi = -\coth\phi b_{11} e_1.$$

On the other hand, by taking the derivative of (6) with respect to ξ and combining with (8) we find

$$\sinh\phi \bar{\nabla}_\xi \xi = 0$$

and this implies $\bar{\nabla}_\xi \xi = 0$.

According to the above calculations, we can give the following theorem without proof:

Theorem 2. *Let M be a constant pseudo-angle lightlike surface of Type I. The linear connection on M is given by*

$$\begin{aligned} \nabla_{e_1} e_1 &= \coth\phi b_{11} \xi \\ \nabla_{e_1} \xi &= -\coth\phi b_{11} e_1 \\ \nabla_\xi \xi &= \nabla_\xi e_1 = 0. \end{aligned}$$

From this point on, we choose coordinates u and v such that

$$\frac{\partial}{\partial u} = \beta e_1 \quad \text{and} \quad \frac{\partial}{\partial v} = \beta \xi$$

where $\beta = \beta(u, v)$ is a certain smooth function on the surface. We will construct the parameterization $x(u, v)$ of a lightlike constant pseudo-angle surface of Type I. We assume $x(u, v)$ twice continuously-differentiable and from Theorem 2 we obtain,

$$
\begin{aligned}
x_{vv} &= 0 \\
x_{vu} &= \frac{\beta_v}{\beta} x_u \\
x_{uu} &= -\beta\beta_v x_v + \frac{\beta_u}{\beta} x_u + \beta^2 b_{11} N
\end{aligned}
$$

Since $x_{vu} = x_{uv}$ and $\nabla_{e_1}\xi = -\coth\phi b_{11} e_1$, we find that $\frac{\beta_v}{\beta} = -\coth\phi b_{11}$.

Also we have,

$$
\begin{aligned}
N_u &= \bar{\nabla}_{x_u} N = b_{11} x_u \\
N_v &= \bar{\nabla}_{x_v} N = 0 \\
N_{uv} &= 0
\end{aligned}
$$

Using the fact that $N_{uv} = N_{vu}$ we get

$$(b_{11})_v x_u + b_{11} x_{uv} = 0 \tag{10}$$

Substituting the expression of x_{uv} in the last equation gives $\frac{\partial}{\partial v}(b_{11}\beta) = 0$. Hence there exists a smooth function $\psi(u)$ such that

$$b_{11}\beta = \psi(u). \tag{11}$$

Corollary 1. *Let M be a constant pseudo-angle lightlike surface of Type I. If $b_{11} = 0$, then the surface immersion is affinely equivalent to the graph immersion of a certain function $f : M \to \mathbb{R}$.*

Assume that $b_{11} \neq 0$, then from the equation (10) we have

$$(b_{11})_v - \coth\phi (b_{11})^2 = 0.$$

Hence we obtain

$$b_{11} = \frac{1}{\alpha(u) - v\coth\phi} \tag{12}$$

and from the (11) we get

$$\beta(u, v) = \psi(u)(\alpha(u) - v\coth\phi) \tag{13}$$

We can calculate the second derivatives of $x(u, v)$ by using the last two equations as,

$$
\begin{aligned}
x_{uu} &= (\psi(u))^2 \coth\phi(\alpha(u) - v\coth\phi) x_v + \left(\frac{\psi'(u)}{\psi(u)} + \frac{\alpha'(u)}{\alpha(u) - v\coth\phi}\right) x_u \\
&+ (\psi(u))^2 (\alpha(u) - v\coth\phi) N \\
x_{uv} &= \frac{\coth\phi}{v\coth\phi - \alpha(u)} x_u \\
x_{vv} &= 0
\end{aligned} \tag{14}
$$

Using the expression of U given in (6), we calculate

$$\langle U, x_u\rangle = 0 \quad \text{and} \quad \langle U, x_v\rangle = -\cosh\phi.$$

It implies $\langle x, U\rangle_v = -\cosh\phi$ and so we have

$$\langle x, U\rangle = -v\cosh\phi + \mu$$

where $\mu \in \mathbb{R}$.

1. Without loss of generality, we can choose the non-lightlike vector U as E_1 with an isometry of E_1^3, then the parameterization $x(u, v)$ of the surface M is (up to translations):

$$x(u, v) = (v\cosh\phi, x_1(u, v), x_2(u, v))$$

Since ξ is a lightlike vector, $\langle x_v, x_v\rangle = 0$. So there exists a function $\Phi(u, v)$ such that

$$x_v = (\cosh\phi, \cosh\phi\cos\Phi(u, v), \cosh\phi\sin\Phi(u, v)) \tag{15}$$

From the equations in (14) and (15), we have $\Phi_v = 0$. Hence the function Φ depends on solely the parameter u. We can rewrite the expression of x_v as,

$$x_v = \cosh\phi((0, f(u)) + (1, 0, 0)) \tag{16}$$

where $f(u) = (\cos\Phi(u), \sin\Phi(u))$. If we calculate x_{uv} and integrate with respect to v, we obtain

$$x_u = \cosh\phi(0, vf'(u) + h(u))$$

where $h(u)$ is a smooth function. When we substitute the last equation in (14) and equalise it to the derivative of (16), we find that

$$h(u) = -\tanh\phi\alpha(u)f'(u)$$

We can rewrite the expression of x_u by substituting the above function and take the derivative with respect to u, then we obtain

$$x_{uu} = \cosh\phi(v - \alpha(u)\tanh\phi)(0, f''(u)) - \alpha'(u)\sinh\phi(0, f'(u)) \tag{17}$$

Multiplying the expressions of x_{uu} in (14) and (17) by x_v implies that

$$\Phi'(u) = \frac{\psi(u)}{\sqrt{|\cosh\phi\sinh\phi|}}$$

One can make a change in the variable u to obtain $\Phi'(u) = 1$ and this choice does not affect the second derivatives of $x(u, v)$. Then we substitute $\Phi(u)$ in the last expressions of x_u and x_v and obtain

$$x(u, v) = v\cosh\phi(1, \cos u, \sin u) + \eta(u)$$

by integrating x_v. We calculate the function $\eta(u)$ as

$$\eta(u) = \sinh\phi(\int \alpha(u)\sin u du, -\int \alpha(u)\cos u du, 0)$$

2. Now take the non-lightlike vector U as the vector E_3 in E_1^3. Then the parameterization of the surface M is

$$x(u,v) = (x_1(u,v), x_2(u,v), -v\cosh\phi)$$

Following the similar steps in the case i, we obtain

$$x(u,v) = v\cosh\phi(\cosh u, \sinh u, -1) + \eta(u)$$

where the function $\eta(u)$ reads

$$\eta(u) = -\sinh\phi(\int \alpha(u)\sinh u du, \int \alpha(u)\cosh u du, 0).$$

Now we can give the following theorem as a consequence of the above calculations:

Theorem 3. *Let M be a constant pseudo-angle lightlike surface of Type I which is not totally geodesic in E_1^3. Up to the isometries of the ambient space, there exist local coordinates u and v such that M is given by one of the following two parameterizations:*

1.

$$\begin{aligned} x(u,v) &= \eta(u) + v\cosh\phi(1, \cos u, \sin u) \\ \eta(u) &= \sinh\phi(\int \alpha(u)\sin u du, -\int \alpha(u)\cos u du, 0) \end{aligned}$$

2.

$$\begin{aligned} x(u,v) &= \eta(u) + v\cosh\phi(\cosh u, \sinh u, -1) \\ \eta(u) &= -\sinh\phi(\int \alpha(u)\sinh u du, \int \alpha(u)\cosh u du, 0). \end{aligned}$$

where $\alpha(u)$ is a smooth function on a certain interval I and ϕ is the pseudo-angle between the transversal vector field on M and the fixed direction U.

TypeII

Let the fixed direction U lies in the plane of $\{e_1, N\}$. Then we decompose U as,

$$U = U^T + \cosh\phi N$$

where U^T is the projection of U on the tangent plane of M and

$$e_1 = \frac{U^T}{\|U^T\|}.$$

We can write U as in the following form,

$$U = \cosh\phi e_1 + \sinh\phi N. \tag{18}$$

Since U is constant,

$$\cosh\phi\bar{\nabla}_X e_1 + \sinh\phi\bar{\nabla}_X N = 0$$

where $X \in T_P M$. Then we obtain $\tau(e_1) = -b_{11}\coth\phi$ and $\tau(\xi) = 0$. This implies

$$\bar{\nabla}_{v_i} V_j = \nabla_{v_i} V_j + b_{ij} N \tag{19}$$

$$\bar{\nabla}_{v_i} N = b_{i1} v_1 + b_{i2} v_2 - b_{i1} \coth\phi N \tag{20}$$

where V_j is a tangent vector field that extends v_j. We can calculate the Levi Civita connection on M by taking the derivatives of (18) with respect to e_1 and ξ.

Theorem 4. *Let M be a constant pseudo-angle lightlike surface of Type II. The Levi Civita connection on M is given by*

$$\begin{aligned} \nabla_{e_1} e_1 &= -b_{11}(\tanh\phi e_1 + N) \\ \nabla_{e_1}\xi &= b_{11} e_1 \\ \nabla_\xi \xi &= \nabla_\xi e_1 = 0 \end{aligned}$$

Proof. One can easily reach to the given equations by following straightforward calculations similar to the Theorem 2 in Type I. □

Now we choose coordinates u and v as in Type I. To construct the parameterization $x(u,v)$ of a lightlike constant pseudo-angle surface of Type II, we calculate the second derivatives by using Theorem 3 as follows:

$$\begin{aligned} x_{vv} &= 0 \\ x_{vu} &= \frac{\beta_v}{\beta} x_u \\ x_{uu} &= (-\beta_v \tanh\phi + \frac{\beta_u}{\beta}) x_u \end{aligned} \tag{21}$$

We find that $\frac{\beta_v}{\beta} = b_{11}$ and so $(b_{11})_v + b_{11}^2 = 0$. Choose $b_{11} \neq 0$, then we have

$$b_{11} = \frac{1}{v + \alpha(u)}$$

where $\alpha(u)$ is a smooth function. On the other hand using the derivatives given in (20) we calculate $\frac{\partial}{\partial v}(b_{11}\beta) = 0$. Hence we have $b_{11}\beta = \psi(u)$ where $\psi(u)$ is a smooth function. Then we obtain

$$\beta(u,v) = \psi(u)(v + \alpha(u)).$$

One can easily calculate that

$$\langle x_u, U\rangle = \beta\cosh\phi \quad \text{and} \quad \langle x_v, U\rangle = \sinh\phi$$

Integrating second one of the above equations with respect to v, we get

$$\langle x, U\rangle = v\sinh\phi + \eta(u)$$

and we obtain $\eta(u) = \cosh\phi \int \beta(u,v)du + c$ by taking derivative and integrating with respect to u.

Now we take the spacelike fixed direction U as the vector E_3 in E_1^3. Using (18), we conclude that the parameterization of M is in the form:

$$x(u,v) = (x_1(u,v), x_2(u,v), v\sinh\phi + \cosh\phi \int \beta(u,v)du)$$

up to translations. Since $\langle x_u, x_u\rangle = \beta^2$, there exists a function $\Phi(u,v)$ such that

$$x_u = (\beta \sinh\phi \cosh\Phi, \beta \sinh\phi \sinh\Phi, \beta\cosh\phi) \tag{22}$$

Then we calculate

$$x_{uv} = (\beta_v \sinh\phi\cosh\Phi + \beta\Phi_v \sinh\phi\sinh\Phi, \beta_v \sinh\phi\sinh\Phi + \beta\Phi_v \sinh\phi\cosh\Phi, \beta_v\cosh\phi)$$

We use the equality of the second derivatives of x and integrate the above equation with respect to u to obtain,

$$x_v = (\sinh\phi \int \psi(u)\cosh\Phi du, \sinh\phi\int\psi(u)\sinh\Phi du, \cosh\phi\int\psi(u)du) \tag{23}$$

When we equalise the two expressions of x_{uv} given in (21) and the above equation, we find that $\Phi_v = 0$, hence the fuction Φ only depends on the variable u.

On the other hand we find that

$$\frac{d\Phi}{du} = -\psi(u)\tanh\phi\coth\Phi \tag{24}$$

by following similar steps as in Type I. If we solve the equation (24), we obtain

$$\begin{aligned} \cosh\Phi &= e^{-\tanh\phi\int\psi(u)du} \\ \sinh\Phi &= (e^{-2\tanh\phi\int\psi(u)du} - 1)^{\frac{1}{2}} \end{aligned} \tag{25}$$

Now we integrate the equation (23) with respect to v and we have

$$x(u,v) = (v\sinh\phi\int\psi(u)\cosh\Phi du, v\sinh\phi\int\psi(u)\sinh\Phi du, v\cosh\phi\int\psi(u)du) + \mu(u) \tag{26}$$

Then we take derivative of (26) with respect to u and equalise to the expression of x_u given in (22) to find $\mu(u)$. We have

$$\mu(u) = (\sinh\phi\int\psi(u)\alpha(u)\cosh\Phi du, \sinh\phi\int\psi(u)\alpha(u)\sinh\Phi du, \cosh\phi\int\psi(u)\alpha(u)du)$$

By the help of the above calculations, we give following theorem without proof:

Theorem 5. *Let M be a constant pseudo-angle lightlike surface of Type II which is not totally geodesic in E_1^3. Up to the isometries of the ambient space, there exist local coordinates u and v such that M is given by the following parameterization:*

$$x(u,v) = \mu(u) + v(\sinh\phi\int\psi(u)\cosh\Phi du, \sinh\phi\int\psi(u)\sinh\Phi du, \cosh\phi\int\psi(u)du)$$

$$\mu(u) = (\sinh\phi\int\psi(u)\alpha(u)\cosh\Phi du, \sinh\phi\int\psi(u)\alpha(u)\sinh\Phi du, \cosh\phi\int\psi(u)\alpha(u)du)$$

where $\psi(u)$ and $\alpha(u)$ are smooth functions on a certain interval I, ϕ is the pseudo-angle between the transversal vector field on M and the fixed direction U and

$$\begin{aligned} \cosh\Phi &= e^{-\tanh\phi\int\psi(u)du} \\ \sinh\Phi &= (e^{-2\tanh\phi\int\psi(u)du} - 1)^{\frac{1}{2}}. \end{aligned}$$

Proposition 1. *Let the fixed direction U lies in the plane of $\{e_1, \xi\}$. Then the surface immersion is affinely equivalent to the graph immersion of a certain function $f : M \to \mathbb{R}$.*

Proof. We express U as

$$U = \cosh\phi e_1 + \sinh\phi\xi \tag{27}$$

Since U is constant, we calculate

$$\cosh\phi(\nabla_{e_1}e_1 + b_{11}N) + \sinh\phi\nabla_{e_1}\xi = 0$$

Then we obtain $\nabla_{e_1}e_1 = -b_{11}N$ and this implies $b_{11} = 0$. □

Corollary 2. *Any constant pseudo-angle lightlike surface is a ruled surface along a spacelike base curve with lightlike rulings.*

Theorem 6. *A constant pseudo-angle lightlike surface is totally umbilical.*

Proof. Let $x(u, v)$ be a constant pseudo-angle lightlike surface of Type I or Type II. Since $x_u = \beta e_1$ and $x_v = \xi$ we obtain $B(x_u, x_v) = 0 = g(x_u, x_v)$ and $B(x_v, x_v) = 0 = g(x_v, x_v)$. Also, using the equation

$$B(x_u, x_u) = \langle\bar{\nabla}_{x_u}N, x_u\rangle \tag{28}$$

we have

$$B(x_u, x_u) = \beta^2 b_{11} = b_{11}g(x_u, x_u).$$

□

Corollary 3. *If M is a constant pseudo-angle lightlike surface, then the lightlike sectional curvature is negative.*

Theorem 7. *The constant pseudo-angle lightlike surface of Type I is a lightlike developable.*

Proof. Let M be a lightlike surface of Type I which has one of the two parameterizations given in Theorem 3.3. Then we obtain the following partial differentials of the parameterization given in (1) as:

$$\begin{aligned}
X_u &= \eta\prime(u) + v\cosh\phi(0, -\sin u, \cos u) \\
X_v &= \cosh\phi(1, \cos u, \sin u) \\
\eta\prime(u) &= \sinh\phi(0, \alpha(u)\sin u, -\alpha(u)\cos u)
\end{aligned}$$

Since $||X_u \times X_v|| = 0$, the surface is a lightlike developable. For the parameterization given by (2), the proof is similar. □

Theorem 8. *The constant pseudo-angle lightlike surface of Type I cannot be a tangent surface.*

Proof. Assume that M is a tangent surface. If M is one of the surfaces given in Corollary 1, then we find that $\cosh\phi = 0$. However, this is a contradiction. □

Theorem 9. *The constant pseudo-angle lightlike surface of Type I is one of the following surfaces:*

i. A part of a lightlike plane
ii. A part of the lightcone

iii. A mix of the above surfaces

Proof. Proof is clear from the Theorem 5.1 in [14]. □

Theorem 10. *The constant pseudo-angle lightlike surface of Type II cannot be a lightlike developable.*

Proof. Let $x(u,v)$ be a lightlike surface of Type II. Assume that it is a lightlike developable. The expressions of x_u and x_v are given in (22) and (23), respectively. By a straightforward calculation we can state that the vector $V = x_u \times x_v$ is lightlike. This property implies that the function Φ is constant. However, we see that it is only possible when $\beta = 0$ by using the equation (24) and this is a contradiction. □

Theorem 11. *Let the constant pseudo-angle lightlike surface of Type II be a tangent surface. Then the function $\psi(u)$ is in the form:*

$$\psi(u) = \frac{e^u}{\tanh\phi \int \alpha(u) e^u du}$$

where $\alpha(u)$ is a smooth function and ϕ is the pseudo-angle between the transversal vector and a fixed direction.

Proof. Let M be a surface given in Theorem 4 If it is a tangent surface, tangent of the base curve must be equal to the rulings. Hence, we have

$$\psi(u)\alpha(u)\cosh\Phi = \int \psi(u)\cosh\Phi du.$$

If we take derivative of the above equation with respect to u, we obtain

$$\alpha(u) = 1 + \frac{\tanh\phi}{\psi(u)} \int (\psi(u))^2 \alpha(u) du$$

On the other hand, we get $\psi(u)\alpha(u) = \int \psi(u) du$ when we equalise the third components of the tangent vector of $\mu(u)$ and the ruling. Hence it must be

$$\psi(u)\alpha(u) = \psi(u) + \tanh\phi \int (\psi(u))^2 \alpha(u) du. \tag{29}$$

Taking derivative of (29) gives a Riccati differential equation and the solution is

$$\psi(u) = \frac{e^u}{\tanh\phi \int \alpha(u) e^u du}.$$

□

We give following examples to illustrate the introduced surfaces by taking different choises of the functions $\psi(u)$ and $\alpha(u)$.

Example 1. *Let $x(u,v)$ be the parameterization of a constant pseudo-angle lightlike surface of Type I as in Theorem 3 (2). Take the pseudo-angle as $\phi = 5$ and the function $\alpha(u) = u$. Then the surface is obtained as*

$$x(u,v) = \begin{bmatrix} 74.21v\cosh(u) - 74.2u\cosh(u) + 74.2\sinh(u) \\ 74.21v\sinh(u) - 74.2u\sinh(u) + 74.2\cosh(u) \\ -74.21v \end{bmatrix}$$

and it can be seen in the Figure 1(a).

Example 2. *Now take $x(u,v)$ as the parameterization of a constant pseudo-angle lightlike surface of Type I as in Theorem 3 (1). We choose the pseudo-angle as $\phi = 5$ and the function $\alpha(u) = e^u$. Then the surface parameterization is*

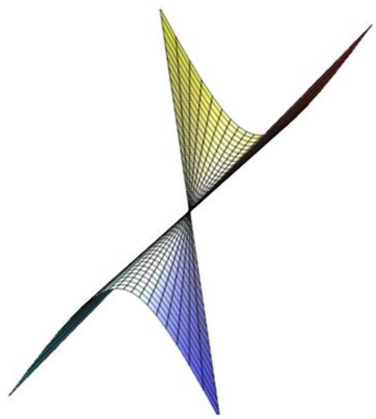
$$x(u,v) = \begin{bmatrix} 74.21v - 37.1\cos(u)\mathrm{e}^u + 37.1\sin(u)\mathrm{e}^u \\ 74.21v\cos(u) - 37.1\cos(u)\mathrm{e}^u - 37.1\sin(u)\mathrm{e}^u \\ 74.21v\sin(u) \end{bmatrix}$$

It can be seen in the Figure 1(b).

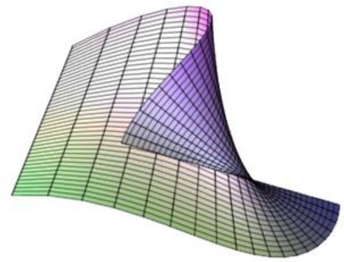
Example 3. *Let $x(u,v)$ be the parameterization of a constant pseudo-angle lightlike surface of Type II as in Theorem 3.4. Take the pseudo-angle as $\phi = 0.5$, $\alpha(u) = 0.001$ and $\psi(u) = -0.5u^3$. We obtained the following parameterization:*

$$x(u,v) = \begin{bmatrix} \left(-0.00008685u^3 - 0.24080v\right)\mathrm{e}^{0.07702}u^3 \\ \left(-0.00008685u^3 - 0.24080v\right)\sqrt{\mathrm{e}^{0.1540u^3} - 1} + \arctan\left(\sqrt{\mathrm{e}^{0.1540u^3} - 1}\right) \\ -0.1879u^3(v + 0.001) \end{bmatrix}$$

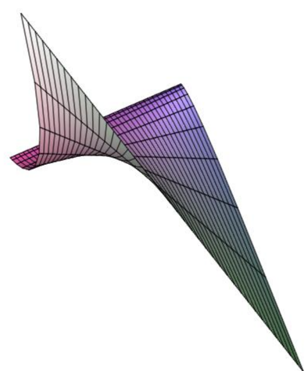
The surface is illustrated in Figure 1(c).



(a) Type I (2) for $\phi = 5$ and $\alpha(u) = u$



(b) Type I (1) for $\phi = 5$ and $\alpha(u) = e^u$



(c) Type II for $\phi = 0.5$, $\alpha(u) = 0.001$ and $\psi(u) = -0.5u^3$

FIGURE 1. Constant pseudo-angle lightlike surfaces of Type I and Type II

4. Constant Angle Lightlike Ruled Surfaces

We investigate the parameterization of a constant angle lightlike ruled surface by means of the Cartan frame on a null helix, a pseudo-null curve as a slant helix or a Cartan slant helix (for further information on these helices see [15], [19], [2], [17], [16], [5]). We classify such ruled surfaces in three cases depending on type of the corresponding helices.

Case 1

Let $\gamma(s)$ be a unit speed null helix equipped with the Cartan frame $\{T, N, B\}$ where the first and second curvatures are $k_1 \neq 0$ and $k_2 = constant$. Here we note that if $k_2 = 0$ then it is a null cubic and the slope axis is a null vector. The slope axis is a non-null vector lies in the rectifying plane if $k_2 \neq 0$.
Now, define a ruled surface as

$$\Psi(s, v) = \alpha(s) + vX(s). \tag{30}$$

Here $\alpha(s)$ and $X(s)$ are expressed by

$$\begin{aligned} \alpha'(s) &= aT + bN + cB \\ X(s) &= x_1T + x_2N + x_3B \end{aligned} \tag{31}$$

where a, b, c, x_1, x_2 and x_3 are smooth functions. If the surface in (30) is lightlike, there exists a lightlike transversal vector field U such that it can be written in the following form by a straightforward calculation:

$$U = U_1T + U_2N + U_3B$$

where

$$U_1 = u_{11} + vu_{12} \quad U_2 = u_{21} + vu_{22} \quad U_3 = u_{31} + vu_{32}. \tag{32}$$

For (30) to be a constant angle surface, we take the lightlike transversal vector U as parallel to the tangent of the curve $\gamma(s)$. Hence $(u_{11}, u_{12}) \neq (0, 0)$. Since there exist spacelike and null vectors in the basis of the tangent plane of $\Psi(s, v)$, we investigate two possibilities:

i. Choose Ψ_v as a null vector, then we have

$$\langle U, \Psi_v \rangle = 1 \quad \text{and} \quad \langle U, \Psi_u \rangle = 0. \tag{33}$$

We can calculate $\langle U, \Psi_v \rangle = x_3U_1 + x_2U_2 + x_1U_3$. Since $U_2 = U_3 = 0$ we have $U_1 = \frac{1}{x_3}$ and this implies $x_3 \neq 0$.
On the other hand, we calculate

$$X'(s) = KT + LN + MB$$

where

$$K = x_1' - x_2k_2 \quad L = x_1 + x_2' + x_3k_2 \quad M = x_3' - x_2. \tag{34}$$

Then we have $c + vM = 0$ which implies $c = 0$ and $x_3' = x_2$. Also, we obtain

$$x_2^2 = -2x_1x_3 \tag{35}$$

$$ax_3 + bx_2 = 0 \tag{36}$$

by using the equations $\langle X, X\rangle = 0$ and $\langle \Psi_s, X\rangle = 0$, respectively. From the equations (35) and (36), we have following ODE;

$$x_3' = \pm\frac{a}{b}x_3. \tag{37}$$

Solving the equation (37) gives

$$\begin{aligned} x_3 &= e^{\pm\int \frac{a}{b}ds} \\ x_2 &= \pm\frac{a}{b}e^{\pm\int \frac{a}{b}ds} \\ x_1 &= -\frac{a^2}{2b^2}e^{\pm\int \frac{a}{b}ds} \end{aligned}$$

where $b \neq 0$.

ii. Now we choose Ψ_s as a null vector. Since

$$\langle U, \Psi_s\rangle = 1 \quad \text{and} \quad \langle U, \Psi_v\rangle = 0$$

we obtain $x_3 = 0$ by using the right hand side of above equation. We also have

$$cu_{11} + v(cu_{12} - x_2u_{11} - vx_2u_{12}) = 1. \tag{38}$$

The equation (38) implies $u_{12} = x_2 = 0$, $u_{11} \neq 0$ and $c \neq 0$. Besides, we can calculate

$$\langle \Psi_v, \Psi_v\rangle = 2x_1x_3 + x_2^2 = 0 \tag{39}$$

and this is a contradiction.

Acording to the above notations, we obtain the expression of a lightlike ruled surface of constant slope as

$$\Psi(s, v) = \int (aT + bN)ds + ve^{\pm\int \frac{a}{b}ds}\left(-\frac{a^2}{2b^2}T + \frac{a}{b}N + B\right) \tag{40}$$

where $b \neq 0$. Note that the surface in (40) is a ruled surface along a spacelike base curve with lightlike rulings. Then we can give the following theorem:

Corollary 4. *Velocity vector of the base curve of a constant angle lightlike ruled surface defined by* (40)*, lies in the osculating plane of a null helix at every point.*

Case 2

Assume that $\gamma(s)$ is a unit speed pseudo-null curve (slant helix) equipped with the Cartan frame $\{T, N, B\}$. If $k_2 = 0$ then any constant vector in E_1^3 can be the slope axis. If $k_2 \neq 0$, the slope axis can be a null or spacelike vector lies in the osculating plane of the curve.

Let the ruled surface defined in (30) with the expressions in (31) be a lightlike surface. We take the transversal vector U expressed in (32) as parallel to the normal vector of $\gamma(s)$. Then we have

$$X' = KT + LN + MB$$

where

$$K = x_1' - x_3 \quad L = x_1 + x_2' \quad M = (x_2 - x_3)k_2 + x_3'.$$

i. We choose Ψ_v as a null vector. Following similar steps as in Case 1, we obtain $x_3 \neq 0$, $u_2 \neq 0$ and $c = 0$. We also have

$$x_2 = x_3 - \frac{x_3'}{k_2} \tag{41}$$

$$-x_2 x_3 = \frac{b^2}{a^2} x_3^2 \tag{42}$$

Substituting (41) in (42), we obtain following ODE:

$$x_3' - k_2(\frac{b^2}{2a^2} + 1)x_3 \tag{43}$$

We find x_1, x_2 and x_3 as;

$$\begin{aligned} x_3 &= e^{\int k_2(\frac{b^2}{2a^2}+1)ds} \\ x_2 &= -\frac{b^2}{2a^2} e^{\int k_2(\frac{b^2}{2a^2}+1)ds} \\ x_1 &= -\frac{b}{a} e^{\int k_2(\frac{b^2}{2a^2}+1)ds} \end{aligned} \tag{44}$$

where $a \neq 0$.

ii. Let Ψ_s be a null vector. Using $\langle U, \Psi_v \rangle = 0$ and $\langle U, \Psi_s \rangle = 1$, we obtain

$$x_2 = 0, u_{22} = 0, c \neq 0 \quad \text{and} \quad u_{21} \neq 0. \tag{45}$$

On the other hand, we find $x_1 = \pm 1$ by calculating $\langle \Psi_v, \Psi_v \rangle = 1$. Hence, one can easily obtain the function c as zero from the equation $\langle \Psi_s, \Psi_s \rangle = 0$. However, this is a contradiction.

Acording to the above notations, we can express a lightlike ruledsurface of constant slope as

$$\Psi(s, v) = \int (aT + bN)ds + v e^{\int k_2(\frac{b^2}{2a^2}+1)ds}(-\frac{b}{a}T - \frac{b^2}{2a^2}N + B) \tag{46}$$

where $a \neq 0$. As in Case 1, the surface in (46) is also a ruled surface along a spacelike base curve with lightlike rulings. So we give the following theorem:

Corollary 5. *Velocity vector of the base curve of a constant angle lightlike ruled surface defined by* (46)*, lies in the osculating plane of a pseudo null curve at every point.*

Case 3

Now, let $\gamma(s)$ be a Cartan slant helix with the attached Cartan frame $\{T, N, B\}$ where $k_2 \neq 0$.

i. Choose Ψ_v as a null vector.

Following similar procedure as in the previous two sections, we find $x_2 \neq 0$, $c \neq 0$, $a \neq 0$ and $b = 0$. Without loss of generality, we also take $ac > 0$. Then the following ODE can be obtained by straightforward calculations:

$$x_3' = \zeta x_3$$

where

$$\zeta = \frac{-(a' + k_2\sqrt{2ac})c + (c' + \sqrt{2ac})a}{2ac}.$$

Using the above equation, we have

$$\begin{aligned} x_3 &= e^{\int \zeta ds} \\ x_2 &= \pm\sqrt{\frac{2a}{c}}e^{\int \zeta ds} \\ x_1 &= -\frac{a}{c}e^{\int \zeta ds} \end{aligned} \qquad (47)$$

ii. If we take Ψ_s as a null vector, we find $x_2 = 0$, $u_{22} = 0$, $b \neq 0$, $u_{21} \neq 0$ and $ac = 0$. Using the inner products of the vectors U, Ψ_s and Ψ_v, we obtain $x_1 = -k_2x_3$. Besides, we also have $\frac{x_1}{x_3} = -\frac{a}{c}$. However, this implies $k_2 = 0$ or indefinite. It is a contradiction.

Acording to the above notations, we can define a constant angle lightlike ruled surface as:

$$\Psi(s,v) = \int (aT + cB)ds + ve^{\int \zeta ds}\left(-\frac{a}{c}T \pm \sqrt{\frac{2a}{c}}N + B\right) \qquad (48)$$

where $c \neq 0$ and $a \neq 0$. Similar to the previous cases, the surface in (43) is also a ruled surface along a non-null base curve with lightlike rulings.

Corollary 6. *Velocity vector of the base curve of a constant angle lightlike ruled surface defined by* (48)*, lies in the rectifying plane of a Cartan slant helix at every point.*

Corollary 7. *A constant angle lightlike ruled surface is constructed by null rulings along a non-null base curve.*

According to the above information mentioned in the three cases, we give following theorem without proof.

Theorem 12. *Let $\gamma(s)$ be a space curve in E_1^3. A constant angle lightlike ruled surface can be defined by one of the equations* (40), (46) *or* (48) *where γ is a null helix, pseudo-null curve or Cartan slant helix, respectively.*

We give some examples to illustrate the theory.

Example 4. *Let γ_1 be a null helix given by*

$$\gamma_1(s) = (s, \sin s, -\cos s).$$

Then the Cartan frame on γ_1 is

$$\begin{aligned} T &= (1, \cos s, \sin s) \\ N &= (0, -\sin s, \cos s) \\ B &= (-\frac{1}{2}, \frac{1}{2}\cos s, \frac{1}{2}\sin s). \end{aligned}$$

Then we obtain the surface given in Figure 2 (a) by choosing the functions $a = 0$ and $b = 1$. Also the base curve $\alpha(s)$ can be seen in the Figure 3 (a).

Example 5. *Let γ_2 be a pseudo null curve given by*

$$\gamma_2(s) = (\frac{s^3}{12}, \frac{s^3+12s}{12\sqrt{2}}, \frac{s^3-12s}{12\sqrt{2}}).$$

Then the Cartan frame on γ_2 is

$$\begin{aligned} T &= (\frac{s^2}{4}, \frac{s^2+4}{4\sqrt{2}}, \frac{s^2-4}{4\sqrt{2}}) \\ N &= (\frac{s}{2}, \frac{s}{2\sqrt{2}}, \frac{s}{2\sqrt{2}}) \\ B &= (-\frac{s^3}{16} - \frac{1}{s}, \frac{s}{2\sqrt{2}} + \frac{1}{\sqrt{2}s} - \frac{s^3}{16\sqrt{2}}, -\frac{s}{2\sqrt{2}} + \frac{1}{\sqrt{2}s} - \frac{s^3}{16\sqrt{2}}) \end{aligned}$$

where $k_2 = \frac{1}{s}$. We obtain the surface given in Figure 2 (b) by choosing the functions $a = 1$ and $b = 1$ and the base curve $\alpha(s)$ can be seen in the Figure 3 (b).

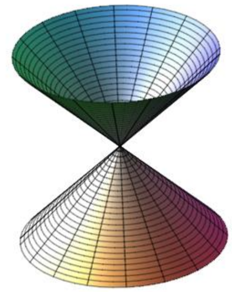
Example 6. *Let γ_3 be a pseudo null curve given by*

$$\gamma_3(s) = (-\frac{s^2}{2}, -\frac{s^2\sqrt{2}(\cos(\ln(s)) + 3\sin(\ln(s)))}{10}, -\frac{s^2\sqrt{2}(\sin(\ln(s)) - 3\cos(\ln(s)))}{10}).$$

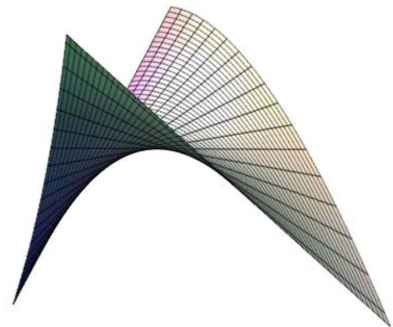
Then the Cartan frame on γ_3 is

$$\begin{aligned} T &= (-s, -\frac{s\sqrt{2}(\cos(\ln(s)) + \sin(\ln(s)))}{2}, \frac{s\sqrt{2}(-\sin(\ln(s)) + \cos(\ln(s)))}{2}) \\ N &= (-1, -\sqrt{2}\cos(\ln(s)), -\sqrt{2}\sin(\ln(s))) \\ B &= (\frac{1}{s}, \frac{\sqrt{2}\left(2\cos(\ln(s))^3 + 2\cos(\ln(s))^2\sin(\ln(s)) - 3\cos(\ln(s)) + \sin(\ln(s))\right)}{4s\sin(\ln(s))\cos(\ln(s)) - 2s}, \\ &\quad \frac{\sqrt{2}\left(2\cos(\ln(s))^2 - 1\right)}{2s(-\sin(\ln(s)) + \cos(\ln(s)))}) \end{aligned}$$

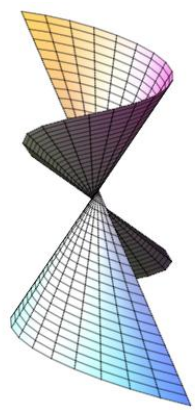
where $k_2 = \frac{1}{s^2}$. We obtain the surface given in Figure 2 (c) by choosing the functions $a = s^2$ and $c = s$ and the base curve $\alpha(s)$ can be seen in the Figure 3 (c).



(a) Case 1 for $a = 0$ and $b = 1$



(b) Case 2 for $a = 1$ and $b = 1$

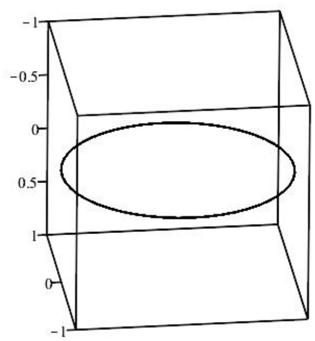


(c) Case 3 for $a = s^2$ and $c = s$

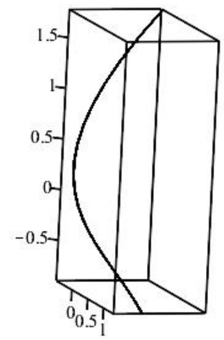
FIGURE 2. Constant angle lightlike ruled surfaces

5. CONCLUSION

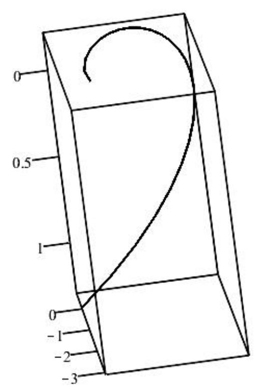
In this paper, we investigate new methods to obtain the parameterizations of lightlike surfaces making constant pseudo-angles with a fixed direction in the Minkowski space. We classify these surfaces by considering the possible casual characters of the fixed direction and show that such surfaces are actually ruled surfaces based on a spacelike curve. Moreover, we give some corrolaries such as; any constant pseudo-angle lightlike surface is totally umbilical and it has negative lightlike sectional curvature, Type I is a lightlike developable and Type II is not. In the given examples one can see the illustrations related to the obtained surfaces



(a) Case 1



(b) Case 2



(c) Case 3

FIGURE 3. Base curve $\alpha(s)$ for (a) $-\pi < s < \pi$, (b) and (c) $-\pi/2 < s < \pi/2$

Type I and Type II.
On the other hand, we obtain corresponding constant angle lightlike ruled surfaces by using the Cartan frame on a null helix, a pseudo-null curve or a Cartan slant helix in section 4. We classify such surfaces according to the casual character of the slope axis. When we assume that the surface itself is lightlike, there exists a lightlike transversal vector field U which is parallel to the tangent vector of the initial curve. We state that a constant angle lightlike ruled surface is constructed

by null rulings along a non-null base curve. The theory is supported by several examples and illustrations.

Declaration of Competing Interests Author declares that there is no conflict of interest in the current manuscript.

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