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A NEW TYPE LORENTZIAN ALMOST PARA CONTACT MANIFOLD

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Abstract. The present study initially introduced a new type Lorentzian almost para contact manifold using the generalized symmetric metric connections of type (α, β). Later, some results is given about new type Lorentzian almost para contact manifold.

1. Introduction

A linear connection $\overline{\nabla}$ on a (semi-)Riemannian manifold M is suggested to be a generalized symmetric connection if its torsion tensor T is presented as follows:

$$T(X,Y) = \alpha\{u(Y)X - u(X)Y\} + \beta\{u(Y)\varphi X - u(X)\varphi Y\}, \tag{1.1}$$

for any vector fields X and Y on M, where α and β are constant functions on M [2, 3]. φ can be viewed as a tensor of type $(1,1)$ and u is regarded as a 1-form connected with the vector field which has a non-vanishing smooth non-null unit. Moreover, the connection $\overline{\nabla}$ is called to be metric connection if $\overline{\nabla} g = 0$, with g metric tensor [1].

A linear metric connection satisfying the equation (1.1) is called generalized symmetric metric connections of type (α, β). We remark that generalized symmetric metric connections of type $(1,0)$ is semi-symmetric connection and generalized symmetric metric connections of type $(0,1)$ is quarter-symmetric connection, respectively [4, 5].

In the present paper, we define a new type Lorentzian almost para contact manifold using the generalized symmetric metric connections of type (α, β)

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2. Lorentzian almost para contact manifold with parallelized generalized symmetric metric connection

Let M be a differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and Lorentzian metric g, which satisfies

$$\eta(\xi) = -1, \quad \phi^2(X) = X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$\phi\xi = 0,\ \eta(\phi X) = 0. \tag{2.3}$$

for all vector fields X, Y on M, where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Such manifold (M, ξ, η, g) is called Lorentzian almost para contact manifold. If we mark $\Phi(X,Y) = g(\phi X, Y)$ for all vector fields X, Y on M, then the tensor field Φ is a symmetric $(0,2)$ tensor field [6, 7, 8].

Now, we will give the main characterization theorem.

Theorem 2.1. *For an Lorentzian almost para contact manifold, the generalized symmetric metric connection $\overline{\nabla}$ of type (α,β) is given by*

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)X - g(X,Y)\xi\} + \beta\{\eta(Y)\phi X - g(\phi X, Y)\xi\}. \tag{2.4}$$

Proof. There is the following relationship between a linear connection $\overline{\nabla}$ and Levi-Civita connection ∇

$$\overline{\nabla}_X Y = \nabla_X Y + H(X,Y),$$

for all vector field X and Y. The following is obtained so that $\overline{\nabla}$ is a generalized symmetric metric connection of ∇, in which H is viewed as a tensor of type $(1,2)$;

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T^{'}(X,Y) + T^{'}(Y,X)], \tag{2.5}$$

where T is viewed as the torsion tensor of $\overline{\nabla}$ and

$$g(T^{'}(X,Y), W) = g(T(W,X), Y). \tag{2.6}$$

Thanks to (1.1) and (2.6), we obtain the following;

$$T^{'}(X,Y) = \alpha\{\eta(X)Y - g(X,Y)\xi\} + \beta\{\eta(X)\phi Y - g(\phi X, Y)\xi\}. \tag{2.7}$$

Using (1.1), (2.5) and (2.7) we obtain

$$H(X,Y) = \alpha\{\eta(Y)X - g(X,Y)\xi\} + \beta\{\eta(Y)\phi X - g(\phi X, Y)\xi\}.$$

This proves to our assertion. □

Using (2.4), we have

$$\overline{\nabla}_X \xi = \nabla_X \xi - \alpha\{X + \eta(X)\xi\} - \beta\phi X. \tag{2.8}$$

If the unit timelike vector field ξ is parallel with respect to generalized symmetric metric connection, that is,

$$\overline{\nabla}_X \xi = 0 \Leftrightarrow \nabla_X \xi = \alpha\{X + \eta(X)\xi\} + \beta\phi X, \tag{2.9}$$

then $\overline{\nabla}$ is called generalized symmetric metric ξ connection.

With the help of equations (2.1), (2.2) and (2.4), we obtain

$$(\overline{\nabla}_X \eta)Y = (\nabla_X \eta)Y - \alpha\{\eta(X)\eta(Y) + g(X,Y)\} - \beta g(\phi X, Y) \tag{2.10}$$

and

$$(\overline{\nabla}_X \phi)Y = (\nabla_X \phi)Y - \alpha\{g(X,\phi Y)\xi + \eta(Y)\phi X\} - \beta\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.11}$$

for any vector field X, Y on M.

Now, we suppose that $(\overline{\nabla}_X\eta)Y = 0$ and $(\overline{\nabla}_X\phi)Y = 0$. Then the equations (2.10) and (2.11) will be as follows:

$$(\nabla_X\eta)Y = \alpha\{\eta(X)\eta(Y) + g(X,Y)\} + \beta g(\phi X, Y) \tag{2.12}$$

and

$$(\nabla_X\phi)Y = \alpha\{g(X,\phi Y)\xi + \eta(Y)\phi X\} + \beta\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}. \tag{2.13}$$

A linear $\overline{\nabla}$ satifying equations (2.12) and (2.13) is called $\eta-$ parallel generalized symmetric metric and $\phi-$ parallel generalized symmetric metric connection, respectively.

Definition 2.2. Let M be a Lorentzian almost para contact manifold. If M provide the equations (2.9), (2.12) and (2.13), then M is called Lorentzian almost para contact manifold with parallelized generalized symmetric metric connection, this means that the connection $\overline{\nabla}$ is generalized symmetric metric ξ connection, $\eta-$ parallel generalized symmetric metric connection and $\phi-$ parallel generalized symmetric metric connection.

In this new type manifold, using (2.9), (2.12) and (2.13) we have the following identities.

Proposition 1. In a Lorentzian almost para contact manifold with parallelized generalized symmetric metric connection, curvature tensor R and Ricci tensor S has the following relations

$$R(X,Y)\xi = (\alpha^2+\beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y), \tag{2.14}$$

$$\begin{aligned} \eta(R(X,Y)Z) &= -(\alpha^2+\beta^2)(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)) \\ &+ -2\alpha\beta(\eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)), \end{aligned} \tag{2.15}$$

$$R(\xi,X)Y = (\alpha^2+\beta^2)(g(X,Y)\xi - \eta(Y)X) + 2\alpha\beta(g(\phi Y,X)\xi - \eta(Y)\phi X),$$

$$S(Y,\xi) = \Big\{(\alpha^2+\beta^2)(-n-1) + 2\alpha\beta(trace\phi)\Big\}\eta(Y), \tag{2.16}$$

for any $X,Y,Z \in \chi(M)$.

In a $3-$ dimensional manifold, the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= S(Y,Z)X - g(X,Z)QY + g(Y,Z)QX - S(X,Z)Y \\ &\quad -\frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{2.17}$$

where Q is Ricci operator.

Theorem 2.3. *In a $3-$ dimensional Lorentzian almost para contact manifold with parallelized generalized symmetric metric connection, scalar and Ricci curvature is given by the following expressions*

$$r = \frac{-8}{3}\big\{2(\alpha^2+\beta^2) - \alpha\beta\phi\big\}, \tag{2.18}$$

$$\begin{aligned} S(Y,Z) &= \frac{2}{3}\Big\{\beta^2+\alpha^2-\alpha\beta\phi\Big\}\{g(Y,Z)+\eta(Y)\eta(Z)\} \\ &+ 8\{\beta^2+\alpha^2-4\alpha\beta\phi\}\eta(Y)\eta(Z) \\ &+ (-2\alpha\beta)g(\phi Y,Z). \end{aligned} \tag{2.19}$$

Proof. If we write ξ instead of X in (2.17), we have

$$\begin{aligned}\eta(R(\xi,Y)Z) &= -S(Y,Z)-\eta(Z)S(Y,\xi)+g(Y,Z)S(\xi,\xi)-S(\xi,Z)\eta(Y)\\ &+ \frac{r}{2}\{g(Y,Z)+\eta(Y)\eta(Z)\}. \end{aligned} \tag{2.20}$$

Using Proposition1 in (2.20), we obtain

$$\begin{aligned}S(Y,Z) &= (-\beta^2-\alpha^2+\frac{r}{2}+\Lambda)(g(Y,Z)+\eta(Y)\eta(Z))+(-2\alpha\beta)g(\phi Y,Z)\\ &+ 2\Lambda\eta(Y)\eta(Z),\end{aligned} \tag{2.21}$$

where $\Lambda = 4(\beta^2+\alpha^2)-2\alpha\beta\phi$. Taking an orthanormal frame field in the equation (2.21) over X and Y, we have the scalar curvature.

Using (2.21) in (2.18), the expression of the Ricci tensor is obtained. □

We know that

$$(\nabla_X S)(Y,Z) = XS(Y,Z)-S(\nabla_X Y,Z)-S(Y,\nabla_X Z).$$

Using this with (2.21), we have

$$(\nabla_X S)(Y,Z) = \mu\big\{(\nabla_X\eta)Y\eta(Z)+(\nabla_X\eta)Z\eta(Y)\big\}, \tag{2.22}$$

where $\mu = \frac{1}{3}\big\{\beta^2+\alpha^2-8\alpha\beta\phi\big\}$. If we use the equation (2.13) in (2.22), we obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \mu\Big\{2\alpha\eta(X)\eta(Y)\eta(Z)-\beta g(\phi X,Y)\eta(Z)-\beta g(\phi X,Z)\eta(Y)\\ &\quad +g(X,Y)Z+g(X,Z)Y\Big\}.\end{aligned} \tag{2.23}$$

The equation (2.23) gives the following result

Proposition 2. In a $3-$ dimensional Lorentzian almost para contact manifold with parallelized generalized symmetric metric connection, the manifold is Ricci symmetric if and only if $\beta^2+\alpha^2-8\alpha\beta\phi = 0$.

3. Conclusion

In this study, we introduced new type Lorentzian almost para contact manifold using generalized symmetric metric connection. This concept is similar Lorentzian trans sasakian structure. Thus, all the studies done in this structure can also be examined for the new structure we introduced.

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