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Two Distinct Perspectives on Ratios: Additive and Multiplicative Relationships between Quantities[1]

Oranlar Üzerine İki Farklı Yaklaşım: Nicelikler Arasındaki Toplamsal ve Çarpımsal İlişkiler

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ABSTRACT. The purpose of this study was to investigate how pre-service teachers' formation of additive and multiplicative relationships support and constrain their understandings of ratios and proportional relationships in terms of quantities. Six pre-service teachers were selected purposefully based on their performances in a previous course. An explanatory case study with multiple cases was used to make comparisons within and across cases. A semi-structured interview was conducted with pairs of pre-service teachers. The results revealed that pre-service teachers' heavy reliance on additive or multiplicative relationships critically shaped their reasoning about ratios from the two perspectives. Pre-service teachers who only attended to multiplicative relationships were found to have a robust understanding of proportional relationships. Pre-service teachers, who did attend to additive relationships, even if they used multiplicative relationships, struggled to form appropriate proportional relationships from the two perspectives.
Keywords: Ratio and Proportion, Addition and Multiplication, Pre-service Teachers

ÖZ. Bu çalışmada, öğretmen adaylarının nicelikler arasında oluşturduğu toplamsal ve çarpımsal ilişkilerin onların oranlar ve orantısal ilişkiler konusundaki anlayışlarını nasıl desteklediği ya da sınırlandırdığı araştırılmıştır. Çalışmaya dâhil edilen altı öğretmen adayı bir önceki lisans dersindeki performanslarına göre amaçlı bir şekilde belirlenmiştir. Durum içi ve durumlar arası karşılaştırmalar yapabilmek için çoklu durumla açıklayıcı durum çalışması kullanılmıştır. İkili gruplara ayrılan öğretmen adaylarıyla yarı-yapılandırılmış birer görüşme yapılmıştır. Sonuçlara göre, öğretmen adaylarının nicelikler arasındaki toplamsal ya da çarpımsal ilişkilere odaklanması, onların oranlarla ilgili iki yaklaşımı kullanarak akıl yürütmelerini kritik bir şekilde etkilemiştir. Nicelikler arasında yalnızca çarpımsal ilişkiler kuran öğretmen adaylarının orantısal ilişkilerle ilgili sağlam bir anlayışa sahip olduğu bulunmuştur. Diğer taraftan, nicelikler arasındaki toplamsal ilişkilere odaklanan öğretmen adayları, çarpımsal ilişkiler de kursalar, oranlarla ilgili iki yaklaşımı kullanarak orantısal ilişkiler oluşturmada zorluk yaşamışlardır.
Anahtar Kelimeler: Oran ve Orantı, Toplama ve Çarpma, Öğretmen Adayları

ÖZET

Amaç ve Önem: Öğrencilerin orantısal akıl yürütmelerini araştıran geniş bir alanyazına kıyasla (Karplus, Pulos ve Stage, 1983; Noelting, 1980a, 1980b; detaylı bir inceleme için, Lamon, 2007), öğretmenlerin bu konudaki anlayışlarını sorgulayan göreceli olarak çok az çalışma vardır. Konuyla ilgili var olan bu sınırlı sayıdaki çalışmalarda, orantısal akıl yürütme konusunda öğretmenlerin öğrencilerle benzer zorluklar yaşadıkları belirlenmiştir (Harel ve Behr, 1995; Pitta-Pantazi ve Christou, 2011; Sowder, Philipp, Armstrong ve Scappelle, 1998). Öğrencilerde gözlemlendiği gibi, öğretmenlerin de orantısal ilişki problemlerini çözerken içler-dışlar çarpımı gibi ezber odaklı yöntemlere başvurabilecekleri (Riley, 2010), iki nicelik arasındaki orantısal bir ilişkiyi koordine etmekte zorlanabilecekleri (Orrill ve Brown, 2012), ve verilen bir problemde hangi aritmetik işlemi kullanacaklarını tahmin yoluyla belirleyebilecekleri bulunmuştur (Harel ve Behr, 1995). Öğretmen ve de öğretmen adaylarının orantısal akıl yürütmeleriyle ilgili az sayıdaki çalışmaya ek olarak, nicelikler arasındaki toplamsal ve çarpımsal ilişkilere odaklanmanın, öğretmen adaylarının oranlarla ilgili iki yaklaşım üzerine orantısal akıl yürütmelerini nasıl etkilediğini araştıran hiç bir çalışma bulunmamaktadır. Bu çalışma, alanyazındaki bu eksikliğe katkıda bulunmayı amaçlamaktadır.

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Yöntem: Araştırmaya Amerika Birleşik Devletleri'nin güneydoğusundaki bir üniversiteden altı ilköğretim matematik öğretmenliği son sınıf öğretmen adayı katılmıştır. Çoklu durumlarla açıklayıcı durum analizi yöntemi kullanılmıştır. Bir önceki dönem aynı öğretmen tarafından verilen "Sayılar ve İşlemler" dersini alan ve bu çalışmaya katılmak için gönüllü olan öğretmen adayları arasından, çarpma, bölme ve kesirler konularında yüksek ders ve sınav performansına sahip olan iki öğretmen adayı dersin öğretmeni tarafından Grup 1, ortalama performansa sahip iki öğrenci Grup 2 ve düşük performans gösteren iki öğrenci de Grup 3 olarak belirlenmiştir. Biri dersin öğretmeni olmak üzere iki uzmanın tecrübelerine dayanılarak, görüşmeleri ikili gruplar şeklinde yapmanın öğretmen adaylarının kamera ortamındaki görüşlerini daha rahat ifade etmelerini sağlayacağı düşünülmüştür. Her bir ikili grupla yaklaşık birer saat süren yarı-yapılandırılmış görüşmeler yapılmıştır. Her görüşmede orantısal ilişkileri içeren üç problem verilmiş ve öğrencilerden bu problemleri oranlarla ilgili iki yaklaşımı kullanarak nasıl çözebileceklerini bireysel olarak anlatmaları istenmiştir. Problem soruları, biri dersin öğretmeni olmak üzere, iki uzman tarafından hazırlanmıştır. Her bir ikili grupla görüşmeler, oranlarla ilgili iki yaklaşım konusu derste işlendikten sonra gerçekleştirilmiştir. Bu çalışmanın verileri, öğrencilerin görüşme esnasındaki sözlü ve yazılı ifadelerinden oluşmaktadır.

Bulgular: Açıklayıcı durum analizine göre, Grup 1 öğretmen adayları, Amy ve Paul, orantısal ilişkiler üzerine akıl yürütebilme bakımından en yüksek performans gösteren ikiliydi. Amy ve Paul, oranlarla ilgili iki yaklaşımı, (*çoklu-küme* ve *değişken-parçalar* yaklaşımları) birbirine karıştırmadan kullanabildi. Ayrıca, orantısal ilişkiler üzerine akıl yürütürken, her zaman çarpımsal ilişkilere bağlı kaldı. Grup 2, Chip ve Amber, Grup 1'e göre orantısal ilişkiler konusunda düşük bir performans gösterdi. *Çoklu-küme* ve *değişken-parçalar* yaklaşımlarını uygun akıl yürütme yöntemleriyle değerlendiremedi. Chip ve Amber'in oranlarla ilgili bu iki yaklaşım konusunda yaşadıkları zorluklar, onların nicelikler arasındaki toplamsal ilişkilere odaklanmasıyla paralel göründü. Görüşme boyunca, Chip toplamsal ve çarpımsal ilişkilere odaklanırken, Amber sadece toplamsal ilişkilere yöneldi. En düşük performans gösteren ikili ise Grup 3, Lisa ve Tess'di. Bu ikili, sıklıkla *çoklu-küme* ve *değişken-parçalar* yaklaşımlarını birbirine karıştırarak, orantısal akıl yürütme konusunda zayıf anlayışa sahip olduklarını gösterdiler. Hem Lisa hem de Tess, nicelikler arasında çoğunlukla toplamsal ilişkiler oluşturdu ve bu durum onların nicelikler arasındaki sabit oranları görmelerine engel oldu. Yoğun bir şekilde kullandıkları "her bir" kelimesinin, dikkatlerini kümeler arasındaki toplama ve çıkarma işlemlerine ve nicelikler arasındaki farka, dolayısıyla toplamsal ilişkilere yönelttiği gözlemlendi.

Sonuç ve Öneriler: Bu çalışmanın en temel sonucu, öğretmen adaylarının nicelikler arasındaki toplamsal ya da çarpımsal ilişkilere odaklanmasının, onların oranlarla ilgili iki yaklaşım üzerine akıl yürütmelerinde önemli bir rol oynadığının gösterilmesidir. Diğer bir deyişle, toplamsal ya da çarpımsal ilişkilere odaklanmak, öğretmen adaylarının *çoklu-küme* ve *değişken-parçalar* yaklaşımlarını birbirine karıştırıp karıştırmamalarını belirlemiştir. Öğretmen adaylarının yalnızca çarpımsal ilişkiler yerine hem toplamsal hem de çarpımsal ilişkiler oluşturdukları durumlarda, nicelikler arasındaki orantısal ilişkileri kurmada zorlandıkları tespit edilmiştir. Buradan hareketle, nicelikler arasındaki toplamsal ilişkilere odaklanmanın orantısal akıl yürütmeyi engellediği çıkarımı yapılabilir. Diğer taraftan, nicelikler arasında yalnızca çarpımsal ilişkiler kurulduğunda ise, öğretmen adaylarının oranlarla ilgili her iki yaklaşımı uygun bir şekilde oluşturabildikleri, dolayısıyla da orantısal akıl yürütme konusunda güçlü bir anlayışa sahip oldukları görülmüştür. İlerideki çalışmalar, bu çalışmadaki yalnızca çarpımsal ilişkilere odaklanmanın orantısal akıl yürütmeyi geliştireceği bulgusunun, diğer öğretmen adaylarına genellenebilme durumunu araştırmalıdır. Önemli bir öneri olarak, öğretmen adaylarını yetiştiren matematik öğretmenliği programlarında verilen matematik öğretimine yönelik dersler, orantısal akıl yürütme kazanımını sağlayacak şekilde yeniden düzenlenmelidir. Özellikle, bu dersler çarpma, bölme ve orantısal ilişkiler konularına kapsamlı bir şekilde yer ayırmalıdır. Bu çalışma, çarpımsal ilişkilerin orantısal akıl yürütme anlayışı için zorunlu olduğunu bulmanın yanında, öğretmen adaylarının toplamsal ilişkilere odaklanmasının önlenmesi için de acil bir çağrı niteliğindedir.

INTRODUCTION

Ratios and proportional relationships have a pivotal role in elementary and secondary mathematics education (e.g., Kilpatrick, Swafford, & Findell, 2001; Lamon, 2007, Lesh, Post, & Behr, 1988; National Council of Teachers of Mathematics, 2000) and provide the foundations for diverse topics such as linear functions, slope, geometric similarity, and probability (e.g., Ben-Chaim, Keret, & Ilany, 2012; Lobato & Ellis, 2010; Simon & Blume, 1994). Lesh et al. (1988) gave utmost importance to proportional reasoning as the capstone of elementary mathematics and the cornerstone of high school mathematics. Lamon (2007) regards the concepts of ratios and proportions, together with fractions, as "the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites" (p. 629). Proportional reasoning is expected to develop gradually over time with adequate instruction, and many adults lack proportional reasoning (Lamon, 2007; Lobato & Ellis, 2010; Tourniaire & Pulos, 1985).

It is widely known among researchers that instruction and research on proportional relationships mostly consists of comparison problems and missing-value problems (Lamon, 2007). In comparison problems, the values of quantities a, b, c, and d are placed accordingly, and students are asked to decide the order relation between the ratios a:b and c:d. In missing-value problems, three values of a, b, c, and d are given, and students are asked to obtain the unknown (missing) value. To solve such problems, students are often encouraged to use rote numerical procedures such as cross-multiplication. Despite a traditional focus on cross-multiplication, proportional reasoning requires much more than appropriate rote computations to solve these problems. A robust understanding of proportional relationships includes coordinating two quantities in a way that preserves the invariance relationship between them that can be acquired either through additive relationships or multiplicative relationships.

The distinction between additive and multiplicative relationships is aligned with the distinction between the operations of addition and multiplication. Although some mathematics education researchers and textbooks have characterized multiplication as repeated addition (e.g., Fischbein, Deri, Nello, & Marino, 1985), many studies have confirmed that multiplication is much more than simply doing addition (e.g., Clark & Kamii, 1996; Greer, 1992; Piaget, 1987; Steffe, 1994; Van Dooren, De Bock, & Verschaffel, 2010; Vergnaud, 1983). While the operation of addition consists of situations involving "adding, joining, subtracting, separating, and removing" (Lamon, 2007, p. 650), the multiplication operation covers situations of "shrinking, enlarging, scaling, duplicating, exponentiating, and fair sharing" (Lamon, 2007, p. 650). In an example involving 7 marbles in 4 boxes, the appropriate multiplication operation is 4 x 7 because 4 is the number of groups and 7 is the number of units in each group (see Beckmann & Izsák, 2015, for a review). The first factor of the operation (i.e., 7) will scale the second factor (i.e., 4) by transforming the size of the second factor proportionally, which will result in a multiplicative change rather than an additive one. On the other hand, when considering 4 x 7 as repeated addition, the second factor (i.e., 7) will be replicated (or iterated) the first factor (i.e., 4) times by adding 7+7+7+7 repeatedly, implying additive changes.

Studies of proportional reasoning have not examined the role that teachers' and students' formation of additive and multiplicative relationships play in their ability to use the two perspectives on ratios. While a large body of the research has focused on the proportional reasoning of students (e.g., Karplus, Pulos, & Stage, 1983; Noelting, 1980a, 1980b; see Lamon, 2007 for a review), there exists only a small body of literature on teachers' reasoning of proportionality. In these relatively few studies, teachers were found to perform poorly on proportional relationship tasks and to have difficulties similar to those of students (e.g., Harel & Behr, 1995; Pitta-Pantazi & Christou, 2011; Sowder, Philipp, Armstrong, & Scappelle, 1998). Just as students do, teachers might rely on rote computation procedures, such as cross-multiplication, and apply it inappropriately (e.g., Riley, 2010); might not focus on the invariance relationship between two co-varying quantities (e.g., Lim, 2009); and, might have difficulty coordinating two quantities in a proportional relationship (e.g., Orrill & Brown, 2012).

In this paper, I report on the results from semi-structured interviews during which three pairs of pre-service middle-grades teachers worked on tasks that required forming proportional relationships by using the two perspectives on ratios. The study makes at least two contributions. First, very little research has been conducted on teachers' reasoning with regard to proportional relationships (e.g., Orrill & Brown, 2012), and the results of this study demonstrate that they have difficulties in proportional reasoning based on their use of additive vs. multiplicative relationships in forming ratios. Second, no research has been reported about the effects of additive and multiplicative relationships on reasoning proportionally from the two perspectives on ratios, and the results of this study suggest that pre-service teachers' ability to use the two perspectives on ratios with appropriate reasoning, drawings, and words depends on how much they relied on multiplicative and additive relationships.

The purpose of this study was to investigate how pre-service teachers' formation of additive and multiplicative relationships supported and constrained their understandings of ratios and proportional relationships in terms of quantities. The following research questions are addressed:

1. What are pre-service teachers' facilities using the two perspectives on ratios and proportional relationships?

2. What are the effects of pre-service teachers' uses of additive and multiplicative relationships in forming ratios?

Theoretical Framework

The conceptual structure is framed by Vergnaud's (1983, 1988) *multiplicative conceptual field*, which places ratios and proportional relationships at the center of many interrelated topics such as multiplication, division, fractions, slope, and linear functions. Moreover, the theoretical framework for this study is based on Beckmann and Izsák's (2015) *mathematical analysis* of ratios and proportional relationships that extends previous literature by constructing parallels between multiplication, division and the two perspectives on ratios.

By assuming M, N, and P as known constants, and x and y as unknowns, Beckmann and Izsák (2015) interpret the equation "M • N = P" to mean the number of groups, M, times the number of units in each group, N, equals the number of units in M groups, P:

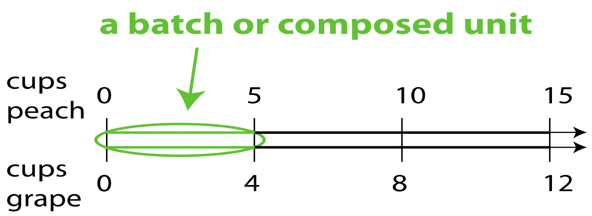
$$M \bullet N = P$$

(# of groups) • (# of units in each/one whole group) = (# of units in M groups)

$M \bullet N = x$ [Equation A] Unknown product, multiplication	$M \bullet x = P$ [Equation B] "How many in each group?" division	$x \bullet N = P$ [Equation C] "How many groups?" division
$x \bullet y = P$ [Equation D] Inversely proportional relationship	$x \bullet N = y$ [Equation E] "Variable number of fixed amounts" proportional relationship	$M \bullet x = y$ [Equation F] "Fixed numbers of variable parts" proportional relationship

Figure 1. *Mathematical Analysis* (From Beckmann & Izsák, 2015, p. 19)

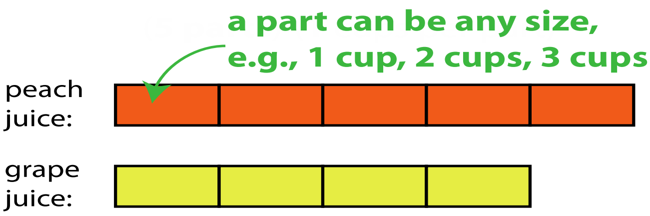
Beckmann and Izsák (2015) defined one of the two perspectives on ratios as the "*variable number of fixed quantities*" (or simply, "*multiple batches*") perspective, which is in the form of $x \bullet N = y$ (Equation E in Figure 1). Figure 2 illustrates the multiple-batches perspective on quantities of peach and grape juice in a mixture with a 5 to 4 ratio. In this perspective, 5 cups of the first quantity (i.e., peach juice) and 4 cups of the second quantity (i.e., grape juice) can be viewed as "1 composed unit" or "1 batch," and the ratio 5 to 4 can consist of two quantities in which their amounts are multiples of those fixed measurements (batches). For example, while 10 cups of peach juice and 8 cups of grape juice form 2 batches, 5/3 cups of peach and 4/3 cups of grape juice demonstrate 1/3 of a batch, and so on. In the multiple-batches perspective, while the number of groups (or batches) varies, the size of each group (or batch) is fixed.



batches	cups peach	cups grape	cups total
1	5	4	9
2	10	8	18
3	15	12	27

Figure 2. *Multiple-batches Perspective*

Moreover, Beckmann and Izsák (2015) introduced the second perspective on ratios as the "*fixed numbers of variable parts*" (or simply, *variable parts*) perspective, which is in the form $M \bullet x = y$ (Equation F in Figure 1). Figure 3 illustrates the variable-parts perspective by using the same quantities with the same ratio 5 to 4 as in Figure 2a. In this perspective, peach and grape juice are in a 5 to 4 ratio because there are 5 parts of peach and 4 parts of grape juice with each part being the same size. For example, while 10 cups of peach juice and 8 cups of grape juice form 5 parts of peach and 4 parts of grape juice with 2 cups in each part, 5/3 cups of peach and 4/3 of grape juice still indicate 5 parts of peach and 4 parts of grape juice but with 1/3 cups in each part. Hence, in contrast to the multiple-batches perspective, the variable-parts perspective forms a fixed number of "parts" for each of the two quantities with the size of each part varying.



cups per part	cups peach	cups grape	cups total
1	5	4	9
2	10	8	18
3	15	12	27

Figure 3. *Variable-parts Perspective*

METHOD

The research design for this study is *an explanatory case study* because "*Case study research* is appropriate to use when trying to attribute causal relationships—and not just wanting to explore or describe a situation." (Yin, 1993, p. 31) In order to gain a more general sense of pre-service teachers' reasoning about ratios and proportional relationships, multiple cases were selected to improve the generalizability and external validity of the study (Gay, Mills, & Airasian, 2008).

Participants and Context

This study was conducted with three pairs of pre-service teachers from the middle-grades teacher education program (Grades 4-8) at one large university in the Southeastern U.S. The program includes coursework in two subject area emphases (from among mathematics, science, language arts, and social studies) and teaching methods related to middle grades' curriculum and students. Pre-service teachers in this program are required to take a first semester calculus course followed by specialized content courses in the Department of Mathematics and methods courses in the College of Education. Before Fall 2012, the pre-service teachers in this study had already taken paired content and methods courses on numbers and operations, where the main focus was on certain concepts of Vergnaud's (1983, 1988) *multiplicative conceptual field* such as multiplication, division, and fractions. By Spring 2012, they had completed one content and one methods course in geometry, and in Fall 2012, at the time of the study, they were enrolled in the algebra course. The recruitment of these teachers was based on their earlier performances in the course on numbers and operations. Both the course on numbers and operations and the algebra course were taught by the same instructor. At the beginning of the algebra course, two volunteer participants were assigned as Group 1, Group 2, and Group 3, respectively based on high, medium, and low performance in the previous course on numbers and operations. While each pair had demonstrated similar performances in the previous course, they had not received any instruction about ratios and proportional relationships before the algebra course.

The textbook for the algebra course was *Mathematics for Elementary Teachers*, 3rd edition (Beckmann, 2011), and the pre-service teachers were taught ratios and proportional relationships throughout the course. The aim of the course was to develop their understanding of multiplication, division, fractions, and ratios and proportions in ways consistent with the Common Core State Standards (Common Core State Standards Initiative, 2010). The instructor often asked the pre-service teachers to explain their solutions to problem situations with quantities in group discussions, on their homework, and on exams, rather than only solving the problems. The course also addressed both the multiple-batches perspective with the use of double number lines (Figure 2), and the variable-parts perspective with the use of strip diagrams (Figure 3) to represent the proportionally-related quantities.

Data Collection

One semi-structured (e.g., Bernard, 1994, Chapter 10) hour-long interview with each pair of pre-service teachers was videotaped. The pre-service teachers were paid $25. The interviews were conducted after 3 weeks of instruction, when the two perspectives on proportional relationships were introduced. The tasks used in this study were constructed as a result of expert judgments (Table 1).

Table 1. *Interview Tasks*

Task 1	A fragrant oil was made by mixing 3 milliliters of lavender oil with 2 milliliters of rose oil. What other amounts of lavender oil and rose oil can be mixed to make a mixture that has exactly the same fragrance?
Task 2	What does it mean to say that lavender oil and rose oil are mixed in a 3 to 2 ratio?
Task 3	If I give you some amounts of lavender oil and rose oil, how could you tell if they are mixed in a 3 to 2 ratio? For example, consider each of these mixtures: 12 milliliters of lavender oil, 8 milliliters of rose oil; 21 milliliters of lavender oil, 12 milliliters of rose oil; 14 milliliters of lavender oil, 8 milliliters of rose oil; 5 milliliters of lavender oil, 3 milliliters of rose oil.

In the interviews, a separate piece of paper was given to each participant for each task. The interviewer read the task, and each participant worked individually and explained his or her reasoning out loud. Because the participants in each group took turns in explaining their thinking to the interviewer before moving to the next task, there was no possibility that the participants influenced each other's thinking. The reason for conducting paired interviews instead of single ones was based on the two experts' experience with such interviews. They agreed that paired interviews would enable the participants to feel more relaxed in explaining their reasoning regarding tasks when being video-recorded in comparison to their feelings in individual interviews. Each interview was video-recorded using two cameras, one for capturing the interviewer and the pair and one for capturing the written work of the pair. Then, the two video files from the two cameras were combined into one video file for a restored view (Hall, 2000) and were transcribed verbatim. Hence, the data in this study consists of one interview for each pair, transcriptions of the interviews, and the written work of the participants.

Analysis of the Data

Each interview was conducted in the same order starting from Task 1 to Task 3. A scenario with lavender oil and rose oil was provided with a 3 to 2 ratio in each task, and pre-service teachers were asked to reason about proportional relationships between quantities by focusing on the two perspectives on ratios. Pre-service teachers were also asked to draw strip diagrams when using the variable-parts perspective, and double number lines when using the multiple-batches perspective. After the data were collected, multiple passes were taken through the data by reviewing the transcripts side-by-side with the videos. The pre-service teachers' words, gestures, and inscriptions were concentrated to gather evidence about their thinking processes. To analyze the transcripts, detailed summaries of each video were written, and an attempt was made to identify the mathematical ideas in the pre-service teachers' thinking. In the first pass, it was realized that pre-service teachers had a hard time reasoning from the two perspectives on ratios. In particular, in cases in which they were focusing on the variable-parts perspective by drawing strip diagrams, their reasoning about quantities and language was mostly related to the multiple-batches perspective. Similarly, in cases in which they were interpreting the multiple-batches perspective with a double number line, they usually seemed to think from the variable-parts perspective with inappropriate word selections. Mixing the two perspectives in such a way instead of keeping them separate indicated some weaknesses in pre-service teachers' reasoning about proportional relationships. As more passes were taken through the data, it became increasingly apparent that there was substantial diversity in the pre-service teachers' formation of additive vs. multiplicative relationships when reasoning from the two perspectives on ratios.

During the discussion of the data in the following sections, there is mostly reference to "*keeping the two perspectives separate*" and "*mixing the two perspectives.*" The purpose of the use of "*keeping the two perspectives separate,*" is to indicate that the student could use the multiple-batches and the variable-parts perspectives, using appropriate reasoning and wording with each perspective. For example, reasoning about the fixed number of sizes and varying number of groups and wording related to replication or iteration of the batches are suitable for the multiple-batches perspective. On the other hand, thinking about the fixed number of groups and varying amount of sizes and wording related to changing the size of each part are appropriate for the variable-parts perspective. Any situations other than these examples are considered "*mixing the two perspectives.*" As an example, when responding to Task 1 involving a mixture of lavender oil and rose oil with a 3 to 2 ratio, replication (or iteration) of parts of a strip diagram as if they are batches as in Figure 4 demonstrates "*mixing the two perspectives*":

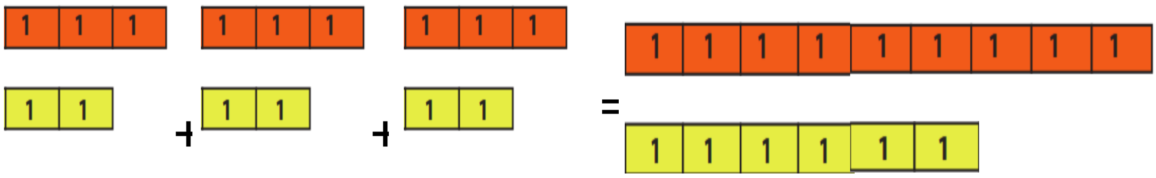


Figure 4. *An Example of Mixing the Two Perspectives*

In terms of the variable-parts perspective, the pre-service teacher in this example should have changed the amount in each part of the strip diagram on the left side of the equation rather than adding more parts to the diagram. In terms of the multiple-batches perspective, the teacher should have drawn a double number line and then replicated the amount in each batch consisting of 3 milliliters of lavender oil and 2 milliliters of rose oil. Such uses of the two perspectives would indicate his/her ability of *"keeping the two perspectives separate."* During the analysis process, "*keeping the two perspectives separate*" suggested a deeper understanding of proportional relationships, while "*mixing the two perspectives*" suggested a weaker understanding of such relationships.

In this study, "multiplicative relationships" were defined as having a sense that when a quantity is multiplied or divided by a number, another quantity must also be multiplied or divided by the same number to maintain the same ratio. Hence, the multiplication and division of batches by the same number, forming multiplicative comparisons within or between measure spaces among the quantities, are indicators of multiplicative relationships. Moreover, "additive relationships" were defined as including any type of addition such as repeated addition or subtraction of batches, simply adding or subtracting quantities in constructing ratios, and the use of the phrase "for every."

FINDINGS

Pair 1: Amy and Paul

Summary: Amy and Paul's work demonstrated "*keeping the two perspectives separate.*" In other words, they used appropriate reasoning, drawings and words with the multiple-batches and the variable-parts perspectives. Moreover, they seemed to focus only on multiplicative relationships when reasoning about proportional relationships across tasks.

As soon as the interviewer finished reading Task 1, Amy demonstrated reasoning with the multiple-batches perspective by generating multiples of the original mixture from a ratio table (Figure 5):

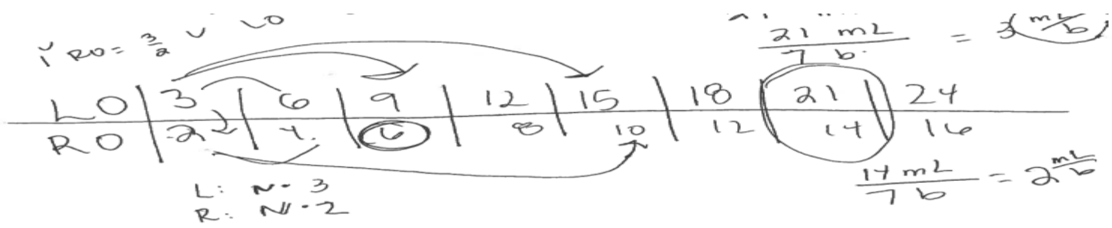


Figure 5. *Amy's Ratio Table Drawing*

Amy: *"Since it is 3 milliliters of lavender oil and 2 milliliters of rose oil that are in sets of 3 and 2, I just find out what other amounts of the oil can be mixed to make the exact same fragrance. Each time we want to make more, we had to multiply these by the same number. So like 3 times 2 and 2 times 2 or 3 times 5 and 2 times 5. To make the same smell, it can be any number and recipes or batches; multiply that number times 3 to find how many milliliters of lavender oil; multiply it by 2 to find out how many milliliters of rose oil."*

Amy: *"Because we kept these in the same ratio, like no matter which of these little sets of numbers you choose, they simplify to 3 and 2. So, if you take this set and divide them each by 7, they simplify to 3 and 2, so they are all the same ratio, so they will all smell the same."*

These data indicated that she considered 3 milliliters of lavender oil and 2 milliliters of rose oil as "1 batch" and calculated larger batches by using multiplicative relationships. For example, her wording as well as her drawing indicated "two batches" when she suggested multiplying 3 milliliters of lavender oil and 2 milliliters of rose oil by 2, and "5 batches" when she proposed multiplication by 5. The strongest evidence for her reliance on multiplicative relationships came a moment later when the interviewer asked whether there was any way to explain the same fragrance:

Amy: *"You could do this where you have like, if this number is 2/3 of this number, so if the amount of rose oil is 2/3 the amount of lavender oil you used, it will smell the same. Or vice versa, if the amount of lavender oil is 3/2 the amount of rose oil, it will smell the same."*

Amy's multiplicative comparison between the two quantities demonstrated an especially thorough understanding of multiplicative relationships: She was also the only pre-service teacher who made explicit multiplicative comparisons between measure spaces during the interview. A moment later, when the interviewer asked Amy to compare her multiple-batches perspective thinking with her variable-parts perspective thinking for the case of 21 milliliters of lavender oil and 14 milliliters of rose oil, she used the variable-parts perspective by drawing a strip diagram (Figure 6):

Figure 6. *Amy's Strip Diagram Drawing*

Amy: *"If you knew that this was 21 and this was 14, then you would kind of simplify to see if 2x has to equal 14, and 3x has to equal 21. And so, then, you would be simplifying to find out the 7 here."*

These data demonstrated that Amy could keep the number of parts of the oil fixed (i.e., 3 parts of lavender oil and 2 parts of rose oil) and alter the amount in each part. While each part represented 1 milliliters of the oil in the original situation, she put appropriately 7 milliliters in each part rather than changing the number of parts. Therefore, Amy was able to use the two perspectives without mixing them, implying her ability to keep these two perspectives separate in addition to her reliance on multiplicative relationships.

Paul, on the other hand, demonstrated multiple-batches reasoning on Task 1 by making multiples of the original mixture shown on a double number line (Figure 7):

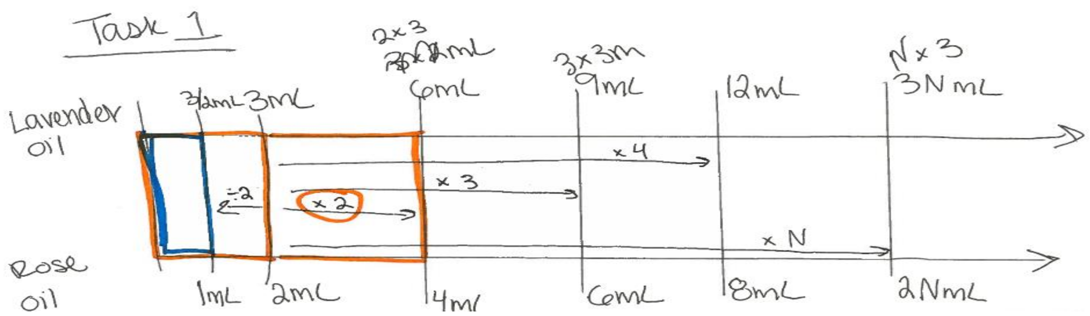


Figure 7. *Paul's Double Number Line Drawing*

Paul: *"Say you want to think like it's a recipe or something like that. This is one recipe for it [pointing to 3 and 2], and then to go to the next one which is like the times 2 part, you know if you get to 6 milliliters you actually have 2 of the recipe, like 2 times. In reality, it is like 3 times 2 milliliters and or 2 times 2 milliliters and you're able to keep going up. And also like you are able to see this is actually literally half of like the original recipe. So, if you want, you can go down."*

These data provided evidence that Paul interpreted 3 milliliters of lavender oil and 2 milliliters of rose oil as "1 recipe" or "1 batch" and could generate larger and smaller batches by using multiplicative relationships. For example, he pointed out that "2 recipes" or "2 batches" are 2 times 3 milliliters of lavender oil and 2 times 2 milliliters of rose oil by inserting a multiplication symbol on his drawing. Thus, he was able to keep the size of "1 batch" fixed and to change the number of batches, implying the multiple-batches perspective. When the interviewer asked if there was any other way to solve Task 1, he used the variable-parts perspective by drawing a strip diagram (Figure 8):

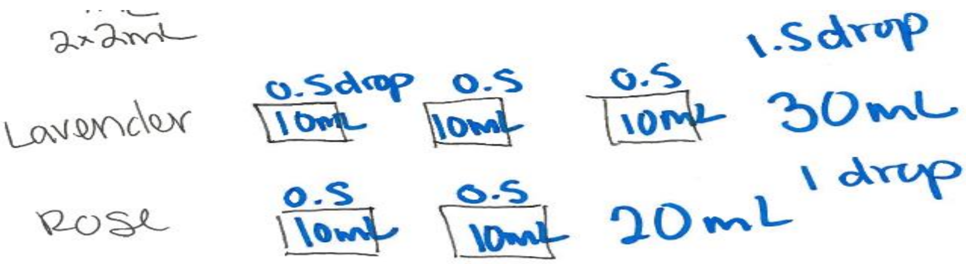


Figure 8. *Paul's Strip Diagram Drawing*

Paul: *"You can do the strip diagram type of thing, too, if you wanted to in which whatever you put in one of these boxes has to go into every single one of the other boxes. If you put 10 milliliters in the first box, that means that you have to put 10 milliliters , 10 milliliters for every single part, in which then you would end up with like 30 total and 20 total. Say you want to put half a drop, you have to put 0.5, 0.5 for every single one of these. And then you get 1.5 drops to 1 drop."*

Paul: *"You can see this [pointing to one part of the strip diagram], 1/10 for instance, you have 3 parts of the 1/10. And then you have here 2/10s [pointing to 2 parts of the rose oil]. You have like 2 parts where it's 1/10 of the whole. You can like separate each of those, you have 1/10 here, 1/10 here, 1/10 here [pointing to each part of the lavender oil] and that gives you 3/10. And the same as right here where you have 1/10, 1/10. That's 2/10."*

These data indicated that Paul could keep the number of parts of lavender and rose oil fixed and change the amount in each part. By keeping the 3 parts of lavender oil and 2 parts of rose oil the same, he explained that the size of each part could take any value, such as 10 milliliters or 0.5 drops, with the condition that each part has to be the same value. Therefore, Paul was able to use the two perspectives without mixing them. This, in turn, implied that Paul was able to keep these two perspectives separate and to use multiplicative relationships in reasoning about proportional relationships, similar to Amy.

Pair 2: Chip and Amber

Summary: Chip and Amber were not able to use the multiple-batches and the variable-parts perspectives with appropriate reasoning, drawings, and words. Instead, they demonstrated *"mixing the two perspectives,"* implying a weak understanding of ratios and proportional relationships. While Chip attended to both multiplicative and additive relationships between proportionally related quantities, Amber relied only on additive relationships when reasoning about these quantities.

When the interviewer presented Task 1, Chip used the multiple-batches perspective and drew a double number line (Figure 9):

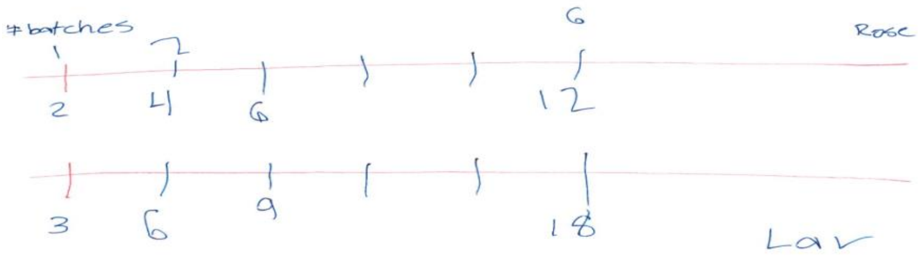


Figure 9. *Chip's Double Number Line Drawing*

Chip: *"We can do like we are in class, like this is one batch. Like the number of batches up here. So that's 1 and that's 2. And then if we get all ... Say this is the sixth batch, then all we're going to do is our initial ratio 2 to 3 times that 6. So, 2 times 6 is twelve, and then the 3 times 6 would be 18 and you can keep going as high as you want to."*

Chip's drawing and explanation clearly meant that he interpreted 3 milliliters of lavender oil and 2 milliliters of rose oil as "1 batch" and demonstrated whole number multiples of 3 and 2 as the number of batches on the double number line (Figure 9). These data also gave evidence for Chip's use of multiplicative relationships such as the multiplication operation between the number of batches and the amount of lavender and rose oil in each batch. However, when the interviewer asked Chip to explain mixing the same fragrance without using the words ratio, fraction or proportion, Chip attended to additive relationships between the quantities through repeated subtraction as follows:

Chip: *"I think a good way to explain this is, for every 3 milliliters of lavender you have 2 milliliters of rose. Even if you have a huge tub of it, if you can separate it out to where 3 lavender goes to 2 rose and just keep separating it out, then eventually get to where there is none left. You don't have any leftovers but you just have a bunch of groups of the 2 to 3 like oil, then that would mean that it would smell the same."*

Task 2 asked what lavender and rose oil mixed in a 3 to 2 ratio meant to Chip and Amber. Chip responded to this question from the variable-parts perspective, while Amber's reasoning and language suggested the multiple-batches perspective. In order to explain the meaning of the 3 to 2 ratio, Chip talked about a strip diagram and said:

Chip: *"As long as there is 3 parts of the lavender to every 2 parts of the rose, then they are in that 3 to 2 ratio. Like I said earlier, 3 to 2 ratio just means for every 3 lavender you have 2 rose. If you have 50 things laying on the table, and if you can break them up, 3 lavender 2 rose here, and then that's a group. So now you have 45 left. Out of those 45 you go 3, 2 again and now you are down to 40. You just keep doing that till you get all the way down to zero. Then, that's what that 3 to 2 ratio means."*

Chip's reference to the replication (or iteration) of the parts of a strip diagram by focusing on "for every" language was related to the multiple-batches perspective reasoning because he was suggesting changing the number of parts instead of fixing them. On the other hand, he mentioned the strip diagram representation of the variable-parts perspective, implying his use of *"mixing the two perspectives."* These data also indicated Chip's use of additive relationships because he discussed subtracting the amount of lavender and rose oil in a repeated way. On the one hand, he was attending to multiplicative relationships by taking whole number multiples of the batches, and on the another hand, he was focusing on additive relationships by subtracting the amount in each batch. Use of “for every” language seemed to regulate his transition from multiplicative relationships to additive ones.

In Task 3, when the interviewer asked Chip and Amber to explain with a drawing whether 12 milliliters of lavender oil and 8 milliliters of rose oil are in a 3 to 2 ratio, Amber combined a strip diagram with multiple-batches reasoning (Figure 10):

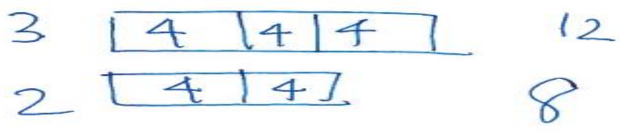


Figure 10. *Amber’s Strip Diagram Drawing*

Amber: *“You’re just creating this 4 times. So you are doing 3 times 4 equals 12 and 2 times 4 equals 8. So you’re still keeping the 3 to 2 ratio. You’re just adding the 3 to 2 four times to get to 12/8ths. You’re just increasing the number of batches of the 3 to 2.”*

These data demonstrated Amber’s use of *“mixing the two perspectives”* and her reliance on additive relationships among proportionally related quantities. In particular, although Amber constructed a strip diagram, which is better suited for the variable-parts perspective, she reasoned from the multiple-batches perspective by considering parts of the strip diagram as “batches” and referring to the change in the number of parts rather than the size in each part. Therefore, her insertion of the batch numbers inside each part of the strip diagram indicated her use of the *“mixing the two perspectives.”* Moreover, her emphasis on the addition of 3 milliliters of lavender and 2 milliliters of rose oil indicated the use of additive relationships. Other evidence for her use of additive relationships came a moment later when the interviewer asked her to explain whether 21 milliliters of lavender oil and 12 milliliters of rose oil are in a 3 to 2 ratio. She used multiple-batches reasoning to explain that the ratio 21 to 12 would smell more lavendery because 21 milliliters of lavender oil and 12 milliliters of rose oil corresponded to 7th and 6th batches, respectively and she explained that the lavender oil was 3 milliliters more than the rose oil, indicating an additive relationship.

Pair 3: Lisa and Tess

Summary: Lisa and Tess were not able to use the multiple-batches and the variable-parts perspectives with appropriate reasoning, drawings, and words. Instead, they often demonstrated *“mixing the two perspectives.”* Both Lisa and Tess relied heavily on additive relationships when reasoning about quantities. This, in turn, caused difficulties when they thought about ratios and proportional relationships.

In Task 1, Lisa and Tess demonstrated the multiple-batches perspective reasoning by making multiples of the original mixture (i.e., “1 batch”) through Lisa’s ratio table (Figure 11):

Batches (#)	1	2	3	4
mL lavender oil	3	6	9	12
mL rose oil	2	4	6	8

Figure 11. *Lisa’s Ratio Table Drawing*

Lisa: *“For every 3 milliliters of lavender oil in the mixture, we have 2 milliliters of rose oil and those are the values that make up one batch of the mixture. So if we want to double the initial batch, then we just multiply each value of the initial ratio by the number of batches.*

So, for two batches, we multiply 3 milliliters by 2 to get 6 milliliters lavender oil. And then 2 milliliters by 2 to get 4 milliliters of rose oil, and then same for three and four batches. So it's kind of like additional copies."

Tess: *"Yeah whatever your number of batches, you're multiplying that by your first batch."*

The drawing and their explanations suggested that Lisa and Tess considered 3 milliliters of lavender and 2 milliliters of rose oil as "1 batch" and discussed obtaining larger batches by multiplying the amount of "1 batch" with the number of desired batches. Hence, their emphases on changing the number of batches but keeping the size of each batch fixed reflected the multiple-batches perspective. In contrast to Tess' focus on multiplicative relationships, Lisa attended to both multiplicative and additive relationships between quantities. Specifically, while Lisa's use of "for every" language to describe the relationship between the amount of lavender and rose oil implied an additive relationship, her multiplication of the original mixture by the batch numbers indicated a multiplicative relationship. As a follow-up question, when the interviewer asked whether there was another way to explain why the two mixtures would smell the same, Tess resorted to additive relationships by focusing on the addition and subtraction between the quantities as follows:

Tess: *"You know that you have 3, 3, 3, 3, 2, 2, 2, 2. If you take those away, they are still the same as the first batch. Like if you take the previous 3's away, that makes sense. Like from here we added 3, we added 3, we added 3 and then we added 2, we added 2, we added 2. So you are doing the same thing every time."*

In Task 2, when the interviewer asked Lisa and Tess how to interpret the statement that the lavender and rose oil were mixed in a 3 to 2 ratio, they explained what a 3 to 2 ratio meant to them:

Tess: *"I think it means that for every 3 milliliters of lavender oil, you have 2 milliliters of rose oil. Because the ratio is 3 to 2, when you add 3 of lavender you have to add 2 of rose oil for them to be the same."*

Lisa: *"I guess here you can go the parts approach where for every 3 parts lavender oil, you have 2 parts rose oil and then any volume quantity could represent."*

These data showed that Lisa was referring to the variable-parts perspective to explain the meaning of a 3 to 2 ratio, but she was suggesting changing the number of parts by replicating (or iterating) the parts of a strip diagram as if the parts were batches. In other words, she was *"mixing the two perspectives"* by using the multiple-batches reasoning with words and a drawing suitable for the variable-parts Perspective. Tess' interpretation of the 3 to 2 ratio, on the other hand, provided further evidence for her use of additive relationships. The interview continued with a follow-up question asking whether there were any visual representations showing what a 3 to 2 ratio meant to them. They thought about specific examples such as 9 and 12 milliliters of lavender oil, and 6 and 8 milliliters of rose oil by making strip diagram drawings (Figure 12, Figure 13):

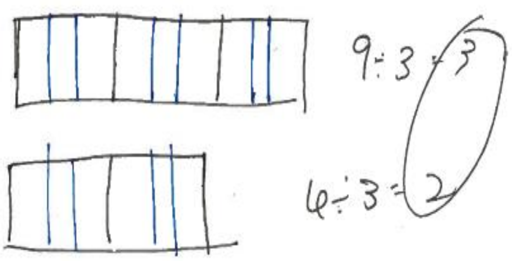


Figure 12. *Lisa's Strip Diagram Drawing*

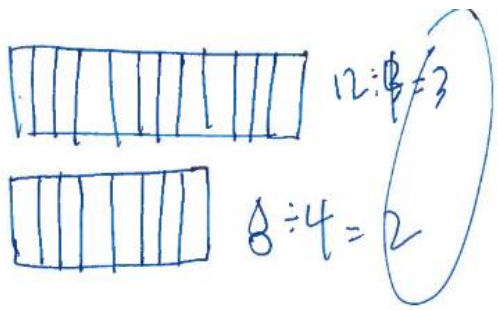


Figure 13. *Tess' Strip Diagram Drawing*

Instead of changing the amount of milliliters in each part (i.e., the size of each part), they formed additional parts by changing the number of parts based on their "for every" language, implying that they were *"mixing the two perspectives."* For example, while Lisa drew 9 parts of lavender and 6 parts of rose oil, Tess made 12 parts of lavender and 8 parts of rose oil instead of keeping the lavender oil as 3 parts and the rose oil as 2 parts. An exchange later, the interviewer asked whether the expression "for every 3 milliliters of lavender oil, there are 2 milliliters of rose oil" was related to their strip diagram drawings. The following was additional evidence for their use of *"mixing the two perspectives:"*

> Tess: *"I think it [the expression] is related to this [pointing to her strip diagram drawing], like it is similar, like how we have been explaining it with the parts, for every 3 parts, there is 2 parts."*
> Lisa: *"'For every' makes me think about a repeated addition. For every 3 means anytime you add 3, you just add 2. If I were to look at this [pointing to her strip diagram drawing] and you told me for every 3 parts, there were 2 parts, I'm trying to think okay when I add 3 here, then I'm going to add 2 here."*

Their explanations about their strip diagram drawings made it clear that Lisa and Tess were *"mixing the two perspectives"* in addition to their reliance on additive relationships. On the one hand, they were attempting to reason from the variable-parts perspective by drawing strip diagrams. However, on the other hand, they were attending to the multiple-batches perspective reasoning and words by focusing on the change in the number of parts. It also became apparent that "for every" language directed their attention towards the use of additive relationships between quantities.

Cross-case Analysis

I have illustrated three cases involving six pre-service teachers ranging from more to less proficient in their use of additive versus multiplicative relationships and two perspectives on ratios. Based on the results in the previous section, a cross-case analysis is presented as a tree diagram (Figure 14):

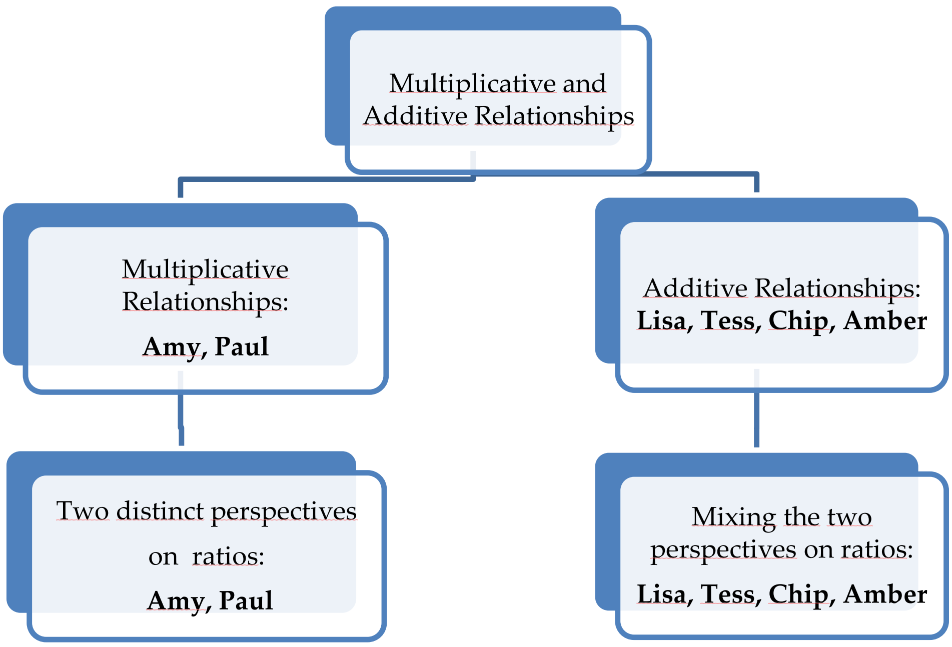


Figure 14. *Additive and Multiplicative Relationships on Ratios*

The results revealed that Pair 1, Amy and Paul, was the most proficient pair in terms of reasoning about proportional relationships among quantities. In particular, they provided evidence of *"keeping the two perspectives separate"* in the form of appropriate reasoning, drawing, and words related to the multiple-batches and the variable-parts perspectives. When reasoning about proportional relationships, they always emphasized multiplicative relationships.

Pair 2, Chip and Amber, was less proficient than Pair 1 in terms of reasoning about proportionally related quantities. Neither was able to use the multiple-batches and the variable-parts perspectives with appropriate reasoning, drawing, and words. Instead, they demonstrated *"mixing the two perspectives."* In situations in which they attempted to use the variable-parts perspective, their reasoning included the multiple-batches perspective, such as putting the batch numbers inside each part of a strip diagram, and vice versa. Such difficulties in reasoning, in turn, seemed to be parallel with their use of additive relationships. While Chip attended to both multiplicative and additive relationships, Amber attended only to additive relationships.

The least proficient pair was Pair 3, Lisa and Tess. They provided noticeably more instances of *"mixing the two perspectives"* than Pair 2 when responding to the tasks. Moreover, they mostly relied on additive relationships instead of multiplicative ones. Such a reliance on additive relationships caused them to ignore the ratios among quantities and the characteristics of the two perspectives on ratios. The uses of "for every" language seemed to direct their attention toward additive relationships such as repeated addition and subtraction of batches, and the difference between the two quantities.

CONCLUSION AND EDUCATIONAL IMPLICATIONS

The findings of this study suggest new lines of research on ratios and proportional relationships. Past research documented students' and teachers' consistent difficulties when reasoning about proportional relationships between quantities, but there has been no study investigating the relationship between students' or teachers' formation of additive and multiplicative relationships and their use of the two perspectives on ratios, which are key concepts in the *multiplicative conceptual field*.

The main result of this study is that pre-service teachers' formation of multiplicative and additive relationships between proportionally related quantities played an important role in their ability to solve tasks by using the two perspectives on ratios. In particular, forming such relationships corresponded with the extent to which pre-service teachers had abilities for *"keeping the two perspectives separate"* and *"mixing the two perspectives."* In cases in which pre-service teachers attended to both multiplicative and additive relationships instead of attending only to multiplicative ones, they struggled to form proportional relationships among quantities, and they demonstrated the use of *"mixing the two perspectives."* In contrast, in cases in which pre-service teachers only attended to multiplicative relationships, they maintained appropriate distinctions between the two perspectives with accurate reasoning, drawings, and words. Therefore, an emphasis on multiplicative relationships seemed to be critical in ensuring a robust understanding of ratios and proportional relationships. Future studies should continue to examine whether underlying facility for forming multiplicative relationships extends to the performances of further future teacher candidates.

The second main result of this study is the heavy reliance on additive relationships such as repeated addition and subtraction of batches, focusing on the difference rather than the multiplicative comparison between proportionally related quantities, and the use of the phrase "for every" caused pre-service teachers to struggle in their reasoning about proportional relationships. Sowder et al. (1998) reported that the concept of ratio is crucial in shifting from additive to multiplicative reasoning because a ratio requires the multiplicative comparison of two quantities. Similarly, the results I have presented suggest that an ability to use both the multiple-batches and the variable-parts perspectives with appropriate reasoning, drawing, and words requires a complete transition from additive to multiplicative relationships.

Another main result of this study is the number of pre-service teachers who attended to additive relationships in comparison to those who attended to multiplicative relationships when reasoning about proportional relationships. Orrill and Brown (2012) found in their study that middle grades teachers mostly relied on addition and subtraction rather than multiplication in tasks with ratios and proportions. In this study, while two of the six pre-service teachers used only multiplicative relationships, the remaining four resorted to additive relationships. Although these numbers cannot be generalized to the pre-service teacher population due to a small convenient sample, they at least suggest that teacher education programs in the U.S. should place significant emphasis on multiplicative relationships in mathematics content courses.

Repeated addition through replication or iteration of the quantities in proportional relationships tasks is sometimes described as "building-up strategies" that consist of forming ratios by extending the original ratio with additive changes (Piaget, Grize, Szeminska, & Bang, 1968, as cited in Lamon, 2007). Lamon (2007) did not consider the "building-up strategies" as proportional reasoning due to their lack of emphasis on the constant ratio between the proportionally related quantities. The results I have presented are consistent with Lamon's (2007) study in the way that "building-up strategies" seemed to constrain pre-service teachers from reasoning about proportional relationships, because they might have directed pre-service teachers' focus toward *"mixing the two perspectives"* and additive relationships. Further studies should continue to examine the specific effects of "building-up strategies" on ratios and proportional relationships.

Finally, an important implication of this study is that mathematics courses for future middle grades teachers should be designed to deliberately support proportional reasoning. In particular, these courses should include all the topics in the *multiplicative conceptual field* such as multiplication, division, and ratios and proportional relationships. This study suggested that a heavy emphasis on multiplicative relationships is critical in *"keeping the two perspectives separate,"* which is a key aspect of a robust understanding of proportional relationships. Similarly, this study demonstrated an urgent need for preventing future teachers from focusing on additive relationships at least when reasoning about proportional relationships tasks.

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