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POINTWISE INNER AUTOMORPHISMS OF RELATIVELY FREE LIE ALGEBRAS

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Abstract. Let K be a field of characteristic zero, and $L_{m,c}$ be the free metabelian nilpotent Lie algebra of class c of rank m over K. We call an automorphism ϕ pointwise inner, if there exists an inner automorphism ξ_i for each generator x_i, $i = 1, \ldots, m$, such that $\phi(x_i) = \xi_i(x_i)$. In this study, we exemine the group $PI(L_{m,c})$ of pointwise inner automorphisms of the Lie algebra $L_{m,c}$, and we provide a set of generators for this group.

1. Introduction

Let G be a free group. An automorphism ϕ of a group G is pointwise inner if $\phi(t)$ is conjugate to t for any $t \in G$. The set of all pointwise inner automorphisms of G forms a group denoted by $Aut_{pwi}(G)$. An automorphism ϕ of G said to be normal if $\phi(H) = H$ for each normal subgroup H and the group of all normal automorphisms is denoted by $Aut_n(G)$. Grossmann showed in [3] that in free group, each pointwise inner automorphism is inner. Endimioni in [1] proved that this property remains true in the free nilpotent group of finite rank. Clearly we have $Inn(G) \leq Aut_{pwi}(G) \leq Aut(G)$. It is clear that $Inn(G)$ is a normal subgroup of $Aut_{pwi}(G)$ as a result of being a normal subgroup of the group of all automorphisms of G. Lubotzsky [4] showed that $Inn(G) = Aut_n(G)$, when G is non-cyclic free group. In the case of a free nilpotent group Endimioni [1] (Theorem 2) have showed that this situation is the different.

In this paper we assume K is a field of charecteristic zero. Let F_m be the free Lie algebra of rank m over the field K with free generators $x_1, \ldots, x_m$. Let $F'_m = [F_m, F_m]$, $F''_m = [F'_m, F'_m]$ be the first and second terms of derived series of F_m. The terms of lower central series of F_m are as follows: $F^1_m = F_m$, $F^2_m = [F^1_m, F^1_m]$,..., $F^{c+1}_m = [F^c_m, F_m]$. Let us denote by $L_m = F_m/F''_m$ the free metabelian Lie algebra and $L_{m,c} = F_m/(F''_m + F^{c+1}_m)$ be the free metabelian nilpotent Lie algebra of nilpotency class c. Let $PI(L_{m,c})$ be the pointwise inner automorphism group of the Lie algebra $L_{m,c}$. The aim of this paper is to describe a generating set of the group $PI(L_{m,c})$. The automorphism group $Aut(L_{m,c})$ is a semidirect

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product of the normal subgroup $IA(L_{m,c})$ and the general linear group $GL_m(K)$, where $IA(L_{m,c})$ is the subgroup of automorphisms which induce the identity map modulo the commutator ideal $L'_{m,c}$. That is, $Aut(L_{m,c}) = GL_m(K) \rtimes IA(L_{m,c})$. Fındık has investigated in [2] the relations between the normal automorphism group $Aut_n(L_{m,c})$ and normal subgroup $IA(L_{m,c})$. Temizyürek and Aydın in [5] have defined the pointwise inner automorphisms of free nilpotent Lie algebras and proved that under the some special conditions, each pointwise inner automorphisms of the free nilpotent Lie algebra is inner.

2. Preliminaries

Definition 2.1. Lie algebras are important examples of non-associative, non-commutative algebras satisfying the identities:

$$[x, x] = 0,$$

and

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The first identity is equivalent to the identity

$$[x, y] = -[y, x]$$

when the base field is not of characteristic 2, and the latter identity is called Jacobi identity.

Note that the free metabelian nilpotent Lie algebra $L_{m,c}$ defined in the previous section satisfies the following identities:

$$[[x, y], [z, t]] = 0, \qquad \text{and} \qquad [y_1, y_2, \dots, y_{c+1}] = 0 \qquad (*)$$

for all $x, y, z, t, y_1, y_2, \dots, y_{c+1} \in L_{m,c}$.

We use the left normed commutator notation for the Lie multiplication:

$$[u_1, \dots, u_{n-1}, u_n] = [[u_1, \dots, u_{n-1}], u_n], \quad n = 3, 4, \dots.$$

For each $v \in L_{m,c}$, the linear operator $\mathrm{ad}v : L_{m,c} \to L_{m,c}$ defined by

$$\mathrm{ad}v(u) = [u, v], \quad u \in L_{m,c},$$

is a derivation of $L_{m,c}$ which is nilpotent and $\mathrm{ad}^c v = (\mathrm{ad}v)^c = 0$ because $L^{c+1}_{m,c} = 0$.

Hence the linear operator

$$\begin{aligned}\exp(\mathrm{ad}v) &= 1 + \frac{\mathrm{ad}v}{1!} + \frac{\mathrm{ad}^2 v}{2!} + \frac{\mathrm{ad}^3 v}{3!} + \frac{\mathrm{ad}^4 v}{4!} + \cdots \\ &= 1 + \frac{\mathrm{ad}v}{1!} + \frac{\mathrm{ad}^2 v}{2!} + \cdots + \frac{\mathrm{ad}^{c-1} v}{(c-1)!}\end{aligned}$$

is well defined and is an automorphism of $L_{m,c}$.

Definition 2.2. The set of all automorphisms are of the form exp(adv), $v \in L_{m,c}$, is called the **inner** automorphism group of $L_{m,c}$ and is denoted by $\mathrm{Inn}(L_{m,c})$.

$\mathrm{Inn}(L_{m,c})$ is a normal subgroup of the group of all automorphisms $\mathrm{Aut}(L_{m,c})$ of $L_{m,c}$.

Definition 2.3. The factor group $\mathrm{Aut}(L_{m,c})/\mathrm{Inn}(L_{m,c})$ is called the **outer** automorphism group of $L_{m,c}$ and is denoted by $\mathrm{Out}(L_{m,c})$.

Definition 2.4. An automorphism ϕ of a group G is called **normal** if

$$\phi(N) = N$$

for every normal subgroup N of G.

We define normal automorphisms of the free metabelian Lie algebra similarly:

Definition 2.5. An automorphism ϕ of $L_{m,c}$ called **normal** if

$$\phi(I) = I$$

for every ideal I of $L_{m,c}$.

Normal automorphisms of $L_{m,c}$ forms a normal subgroup of $\mathrm{Aut}(L_{m,c})$. We shall denote it by $\mathrm{N}(L_{m,c})$. Hence naturally we define another automorphism class:

Definition 2.6. $\Gamma\mathrm{N}(L_{m,c}) = \mathrm{Aut}(L_{m,c})/\mathrm{N}(L_{m,c})$ is called the group of **normally outer** automorphisms of $L_{m,c}$.

Every inner automorphism fixes the ideals of $L_{m,c}$, so that $\mathrm{Inn}(L_{m,c})$ is a normal subgroup of $\mathrm{N}(L_{m,c})$. This arises the following:

Definition 2.7. $\mathrm{OutN}(L_{m,c}) = \mathrm{N}(L_{m,c})/\mathrm{Inn}(L_{m,c})$ is called the group of **outer normal** automorphisms of $L_{m,c}$.

Lemma 2.8. [2] (i) *If $m \geq 3$ and $c = 2$, then* $\mathrm{N}(L_{m,2}) \subset \mathrm{IA}(L_{m,2})$.
(ii) *If $m \geq 3$ and $c = 3$, then* $\mathrm{N}(L_{m,3}) \subset \mathrm{IA}(L_{m,3})$.
(iii) *If $m \geq 2$ and $c \geq 4$, then* $\mathrm{N}(L_{m,c}) \subset \mathrm{IA}(L_{m,c})$.

We consider only the group of normal automorphisms in $\mathrm{IA}(L_{m,c})$ and denote it by $\mathrm{IN}(L_{m,c})$.

Theorem 2.9. [2] (i) $\mathrm{IN}(L_{m,2})$ *is abelian;*
(ii) $\mathrm{IN}(L_{m,3})$ *is nilpotent of class 3;*
(iii) $\mathrm{IN}(L_{m,c})$ *is metabelian when $c \geq 4$.*

Theorem 2.10. [2] *A normal automorphism ψ of the algebra $L_{m,c}$ is of the form*

$$\psi : x_i \to x_i + \sum_{j=1}^{m} [x_i, x_j] f_j, \quad i = 1, \ldots, m,$$

where $f_j \in K[\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m]$.

Now we illustrate some explicit examples for low ranks and nilpotency classes as follows.

An example of a normal (inner) auto for $m = c = 3$

$$\begin{aligned} \phi(x_1) &= x_1 \\ \phi(x_2) &= x_2 + [x_2, x_1] + [x_2, x_1, x_1]/2 \\ \phi(x_3) &= x_3 + [x_3, x_1] + [x_3, x_2, x_2]/2 \end{aligned}$$

A simple example of a normally outer auto for $m = c = 3$

$$\begin{aligned} \phi(x_1) &= x_1 \\ \phi(x_2) &= x_2 + [x_2, x_1] + [x_2, x_1, x_1] \\ \phi(x_3) &= x_3 + [x_3, x_1] + [x_3, x_2, x_2] \end{aligned}$$

A simple example of an outer normal auto for $m = c = 3$

$$\begin{aligned}\phi(x_1) &= x_1\\ \phi(x_2) &= x_2 + [x_2, x_1] + [x_2, x_1, x_1]\\ \phi(x_3) &= x_3 + [x_3, x_1] + [x_3, x_1, x_1]\end{aligned}$$

Theorem 2.11. [2] *The free Lie algebra of finite rank does not have inner and normal automorphisms.*

Our goal is to describe the group of pointwise inner automorphisms $\mathrm{PInn}(L_{m,c})$ of the algebra $L_{m,c}$.

Approach:

- The automorphism group $\mathrm{Aut}(L_{m,c})$ is a semidirect product of the normal subgroup $\mathrm{IA}(L_{m,c})$ of the automorphisms which induce the identity map modulo the commutator ideal of $L_{m,c}$ and the general linear group $\mathrm{GL}_m(K)$.
- Since $\mathrm{PInn}(L_{m,c}) \subset \mathrm{IA}(L_{m,c})$, it is sufficient to work in only $\mathrm{IA}(L_{m,c})$.

3. Main Results

In this section, we define a set of generators for the group $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms of the free metabelian nilpotent Lie algebra $L_{m,c}$ of rank m. Every automorphism φ in $\mathrm{PInn}(L_{m,c})$ is of the form $\varphi = (u_1, \dots, u_m)$; i.e., $\varphi(x_i) = e^{\mathrm{ad}(u_i)}(x_i)$, for some $u_i \in L_{m,c}$, $i = 1, \dots, m$. Let us define the set

$$I_i = \{\varphi_u = (0, \dots, 0, u, 0, \dots, 0) \mid u \in L_{m,c}\}, \quad i = 1, \dots, m,$$

consisting of m-tuples where each coordinate except for i-th position is necessarily filled by zero.

Theorem 3.1. *The set I_i is a group for every $i = 1, \dots, m$.*

Proof. Let $i = 1$, and let $\varphi_u = (u, 0, \dots, 0), \varphi_v = (v, 0, \dots, 0) \in I_1$, where $u, v \in L_{m,c}$. Then $\varphi_u \phi_v(x_i) = x_i$, $i = 2, \dots, m$, and

$$\begin{aligned}\varphi_u \phi_v(x_1) =& \varphi_u \left(e^{\mathrm{ad}(v)}(x_1)\right)\\ =& \varphi_u \left(x_1 + [x_1, v] \sum_{k=0}^{c-2} \frac{\mathrm{ad}_{\bar{v}}^k}{(k+1)!}\right)\\ =& \varphi_u(x_1) + [\varphi_u(x_1), \varphi_u(v)] \sum_{k=0}^{c-2} \frac{\mathrm{ad}_{\varphi_u(\bar{v})}^k}{(k+1)!}\\ =& e^{\mathrm{ad}(\varphi_u(v))}(\phi_u(x_1)) = e^{\mathrm{ad}(\varphi_u(v))} e^{\mathrm{ad}(u)}(x_1).\end{aligned}$$

On the other hand, since inner automorphisms form a group, then there exists an element $w \in L_{m,c}$ such that $e^{\mathrm{ad}(\varphi_u(v))} e^{\mathrm{ad}(u)} = e^{\mathrm{ad}(w)}$, that is

$$\varphi_u \phi_v = (w, 0, \dots, 0) = \varphi_w.$$

Now let us show that φ_u has an inverse in I_1. Consider the map φ_{-u}. Clearly there exists a linear combination of smaller length of elements in $\varphi_u(x_1) - x_1$ comparing with the element $(\varphi_{-u}\varphi_u)(x_1) - x_1$. In other words, the least length of monomials in the element $(\varphi_{-u}\varphi_u)(x_1) - x_1$ is longer than that of $\varphi_u(x_1) - x_1$. Hence, after finite number of this process we reach to the identity map, due to nilpotency. Therefore I_1 is a group. One may prove similarly that I_i is a group for $i = 2, \ldots, m$. □

Theorem 3.2. *The set* $\mathrm{PInn}(L_{m,c})$ *of pointwise inner automorphisms of the free metabelian nilpotent Lie algebra* $L_{m,c}$ *forms a group generated by the set* $I_1 \cup \cdots \cup I_m$.

Proof. Consider an arbitrary element $\varphi = (u_1, \ldots, u_m)$ for some $u_i \in L_{m,c}$, $i = 1, \ldots, m$. One may easily observe that $(\varphi_{u_1}^{-1}\varphi) = (0, u_2, \ldots, u_m)$. Similarly we have that

$$\varphi_{u_m}^{-1} \cdots \varphi_{u_1}^{-1}\varphi = 1$$

and that $\varphi = \varphi_{u_m} \cdots \varphi_{u_1}$ which means that $\varphi \in < I_1 \cup \cdots \cup I_m >$. □

4. Conclusion

In this study, it was shown that the set of piontwise inner automorphisms form a subgroup of the group of all automorphisms of the free metabelian Lie algebra of rank $m \geq 2$ in the nilpotency class $c \geq 2$ over a field of characteristic zero. In addition, the main result provides a generating set for this subgroup.

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