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SOLITON AND OTHER FUNCTION SOLUTIONS OF THE POTENTIAL KdV EQUATION WITH JACOBI ELLIPTIC FUNCTION METHOD

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Abstract

Original scientific paper

The current study is concerned with analytical solutions of the nonlinear potential KdV equation to the modelling of tsunami waves. Here, we implemented the Jacobi elliptic function method, which is widely used in equations in the fields of science and engineering. As a result, many new soliton, hyperbolic, and periodic wave solutions are obtained through Mathematica. These results will be extremely useful and applicable in scientific and engineering. In addition, the obtained wave solutions are shown with graphs

Keywords: *Jacobi elliptic function method, soliton, the nonlinear partial differential equations, the potential KdV equation.*

JACOBİ ELİPTİC FONKSİYON METOT İLE POTANSİYEL KdV DENKLEMİNİN SOLİTON VE DİĞER FONKSİYON ÇÖZÜMLERİ

Özet

Orijinal bilimsel makale

Mevcut çalışma, tsunami dalgalarının modellenmesiyle oluşan lineer olmayan potansiyel KdV denkleminin analitik çözümleri ile ilgilidir. Burada, fen ve mühendislik alanlarındaki denklemlerde yaygın olarak kullanılan Jacobi eliptik fonksiyon metodunu uyguladık. Sonuç olarak Mathematica ile birçok yeni soliton, hiperbolik ve periyodik dalga çözümleri elde edilmiştir. Bu sonuçlar, fen ve mühendislikte son derece faydalı ve uygulanabilir olacaktır. Ayrıca, elde edilen dalga çözümleri grafiklerle gösterildi

Anahtar Kelimeler: *Jacobi eliptic fonksiyon metod, soliton, lineer olmayan kısmi diferensiyel denklemler, potansiyel KdV denklemi.*

1 Introduction

The concept of Soliton was first discovered by John Scott Rusell in 1834 [1]. The solitons are nonlinear waves that propagate by preserving their shape and speed and continue to maintain these properties after any interaction moment. The fact that it preserves its shape and speed has led many scientists from all fields to work on solitons. Although this theory is related to many areas of mathematics, it has many applications in the engineering, physical, chemical, and biological sciences. Especially in shallow water waves are a growing area of research in mechanical engineering. The well-known nonlinear KdV equation [2] is used to model solitons. Another model, the potential KdV equation, is the nonlinear equation encountered in modeling tsunami waves [3]. The main topic of many research papers is related to the potential KdV equation and its soliton solutions [4-6].

Many methods that yield soliton solutions are used to solve nonlinear partial equations, such as the extended tanh method [7], the first integral method [8], sine-Gordon expansion method [9], generalized tanh method [10], G'/G method [11], Riccati method [12], Kudryashov method [13,14], the F-expansion method [15] etc.

In this paper, the exact solutions of the potential KdV equation are constructed by the Jacobi elliptic function method. This method has been used in many equations formed by modeling structures in the fields of science and engineering [16,17].

1.1 An Overview of Method

Suppose that we have the following nonlinear evolution equation

$$N(u,\ u_x,\ u_t,\ u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \tag{1}$$

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where N is a polynomial in $u(x,t)$. We use the travelling wave transformation

$$u(x, y) = U(\xi), \quad \xi = x - vt. \tag{2}$$

Then Eq.(1) reduce to an ordinary differential equation

$$\tilde{N}(U, U', U'', U''', \ldots) = 0 \tag{3}$$

We can express the solution of Eq.(3) as below

$$U(\xi) = \sum_{i=0}^{n} \alpha_i \psi^i(\xi) \tag{4}$$

where n is the balance number. We determine the positive integer n in Eq. (4) by balancing the highest order derivatives and the nonlinear terms in Eq.(3). $\psi(\xi)$ is a known function as the following Jacobi elliptic function

$$\psi'(\xi) = \sqrt{\Im\psi^4(\xi) + \Re\psi^2(\xi) + \aleph}. \tag{5}$$

Substituting Eq.(4) along with Eq.(5) into Eq.(3), we get a polynomial to of $\psi(\xi)$. Equating each coefficient of polynomial to zero. We derived a system of algebraic equations which can be solved by the aid of Mathematica program [17,18].

2 Wave Solutions of The Potential KdV Equation

2.1 Potansiyel KdV Denkleminin Dalga Çözümleri

$$u_t + au_x^2 + bu_{xxx} = 0 \tag{6}$$

where a and b are real constants, x and t are independent variables that represent the spatial and temporal variables, respectively [4,5]. $u(x,t)$ is the dependent variable that represents the wave profile. The nonlinear u_x^2 term is the transmission term and u_{xxx} term is dispersion term. Solitons are formed the interaction of these two terms.
If Eq.(2) transformation is used, to Eq.(6) becomes as the following ordinary differential equation

$$-vu' + a(u')^2 + bu''' = 0. \tag{7}$$

In Eq.(7), $u'(\xi) = q(\xi)$ conversion is used, Eq.(7) becomes as

$$-vq + aq^2 + bq'' = 0. \tag{8}$$

After that, for $n = 2$, the solution of Eq.(8) is considered as follows

$$q(\xi) = \alpha_0 + \alpha_1\psi(\xi) + \alpha_2\psi^2(\xi) \tag{9}$$

which α_0, α_1 and α_2 are parameters to be determined. By replacing the nontrivial solution Eq.(9) and Eq.(5), we find

$$-\upsilon\alpha_0 + a\alpha_0^2 + 2\aleph b\alpha_0 = 0$$
$$-\upsilon\alpha_1 + \Re b\alpha_1 + 2a\alpha_0\alpha_1 = 0$$
$$a\alpha_1^2 - \upsilon\alpha_2 + 4\Re b\alpha_2 + 2a\alpha_0\alpha_2 = 0$$
$$2\Im b\alpha_1 + 2a\alpha_1\alpha_2 = 0.$$

Solving above algebraic equation system, we obtain

$$a \neq 0, \aleph b \neq 0, \alpha_0 = \frac{\upsilon - 4\Re b}{2a}, \alpha_1 = 0, \quad \alpha_2 = \frac{\upsilon\alpha_0 + 4\Re b\alpha_0}{4\aleph b}. \tag{10}$$

Case1:

$$\Im = m^2, \quad \Re = -\left(1 + m^2\right), \quad \aleph = 1 \quad and \quad \psi(\xi) = sn\xi,$$

we acquire

$$q(\xi) = \tfrac{1}{8ab}((\upsilon + 4(m^2+1)\beta)(4\beta + (\upsilon - 4(1 + m^2)\beta)sn^2(\xi))). \tag{11}$$

When $m \to 1$, then

$$q(\xi) = \tfrac{1}{8ab}\Big((\upsilon + 8b)\big(4b + (\upsilon - 8b)\tanh^2(\xi)\big)\Big). \tag{12}$$

So, we obtain the kink soliton solution

$$u(x, t) = \tfrac{1}{8ab}((\upsilon + 8b)((x - \upsilon t)(\upsilon - 4b) - (\upsilon - 8b)\tanh(x - \upsilon t))). \tag{13}$$

Case2:

$$\Im = -m^2, \quad \Re = 2m^2 - 1, \quad \aleph = 1 - m^2 \; and \quad \psi(\xi) = cn\xi,$$

we acquire

$$q(\xi) = \frac{1}{8(-1 + m^2)\mathrm{ab}}(-(\upsilon + (4 - 8m^2)b)(-4(-1 + m^2)b + (\upsilon - 4b + 8m^2 b)\mathrm{cn}^2)). \tag{14}$$

When $m \to 0$, then

$$q(\xi) = \tfrac{1}{8ab}\Big((\upsilon + 4b)\big(4b + (\upsilon - 4b)\cos^2(\xi)\big)\Big). \tag{15}$$

We can obtain a new periodic solution of Eq.(6), as follows

$$u(x, t) = \frac{1}{32ab}((\upsilon + 4\beta)(2(x - \upsilon t)(\upsilon + 4b) + (\upsilon - \beta)\sin\big(2(x - \upsilon t)\big))). \tag{16}$$

Case3:

$$\Im = 1, \quad \Re = m^2 - 1, \quad \aleph = m^2 \quad and \quad \psi(\xi) = ns\xi,$$

so

$$q(\xi)=\frac{1}{8m^2ab}((\upsilon+4\beta)(1+m^2)(4m^2\beta+(\upsilon-4\beta(1+m^2)\mathrm{ns}^2(\xi)))). \quad (17)$$

When $m\to 1$, then

$$q(\xi)=\frac{1}{8ab}\left((\upsilon+8b)(4b+(\upsilon-8b)\coth^2(\xi))\right) \quad (18)$$

So, singular solution of Eq.(6), as follows

$$u(x,\ t)=\frac{1}{8ab}((\upsilon-4b)(x-\upsilon t)-(\upsilon-8b)\coth(x-\upsilon t)) \quad (19)$$

Case4:

$\Im=1-m^2$, $\Re=2m^2-1$, $\aleph=-m^2$ and $\psi(\xi)=nc\xi$,

from here,

$$q(\xi)=-\frac{1}{8m^2ab}((\upsilon+(4-8m^2)b)(-4m^2b+(\upsilon-4b+8m^2b)\mathrm{nc}^2(\xi))). \quad (20)$$

When $m\to 1$, then

$$q(\xi)=-\frac{1}{8ab}(\upsilon-4b)(-4b+(\upsilon+4b)\cosh^2\xi). \quad (21)$$

Hence, the hyperbolic function solution for the governing Eq. (6) is:

$$u(x,\ t)=-\frac{1}{32ab}(\upsilon-4b)(2(x-\upsilon t)(\upsilon-4b)+(\upsilon+4b)\sinh\big(2(x-\upsilon t)\big)).. \quad (22)$$

Case5:

$\Im=1$, $\Re=2-m^2$, $\aleph=1-m^2$ and $\psi(\xi)=cs\xi$,

for

$$q(\xi)=-\frac{1}{8ab(m^2-1)}(\upsilon+4b(m^2-2))(-4b(m^2-1)+(\upsilon-4b(m^2-2))cs^2(\xi) \quad (23)$$

$m\to 0$, then

$$q(\xi)=\frac{\upsilon-8b}{2a}+\frac{1}{4b}\left(\frac{\upsilon(\upsilon-8b)}{2a}+\frac{4(\upsilon-8b)b}{a}\right)\cot^2(\xi). \quad (24)$$

Hence, the solution for the governing Eq. (6) is:

$$u(x,\ t)=-\frac{1}{8ab}(\upsilon-8b)((x-\upsilon t)(\upsilon+4b)+(\upsilon+8b)\cot(x-\upsilon t)). \quad (25)$$

Case6:

$\Im=-m^2(1-m^2)$, $\Re=2m^2-1$, $\aleph=1$ and $\psi(\xi)=sd\xi$

in this

$$q(\xi)=-\frac{1}{8ab}((\upsilon+(4-8m^2)b)4b+(\upsilon-4b+8m^2b)\mathrm{sd}^2(\xi)). \quad (26)$$

When $m\to 1$, then

$$q(\xi)=\frac{1}{8ab}(\upsilon-4b)\big(4b+(\upsilon+4b)\sinh^2(\xi)\big). \quad (27)$$

So, the hyperbolic function solution for the governing Eq. (6) is:

$$u(x,t)=\frac{1}{32ab}((\upsilon-4b)\big(-2(x-\upsilon t)\big)(\upsilon-4b)+(\upsilon+4b)\sinh\big(2(x-\upsilon t)\big)). \quad (28)$$

and for $m\to 0$,

$$q(\xi)=\frac{1}{8ab}(\upsilon+4b)\big(4b+(\upsilon-4b)\sin(\xi)\big) \quad (29)$$

from here, the periodic solution of Eq.(6) is obtained as

$$u(x,t)=\frac{1}{32ab}(\upsilon+4b)(2(x-\upsilon t)(\upsilon+4b)-(\upsilon-4b)\sin\big(2(x-\upsilon t)\big)). \quad (30)$$

Case7:

$\Im=\frac{1-m^2}{4}$, $\Re=\frac{1+m^2}{2}$, $\aleph=\frac{1-m^2}{4}$ and $\psi(\xi)=nc\xi\pm sc\xi$

for

$$\mathrm{q}(\xi)=\frac{1}{2ab(m^2-1)}((-\upsilon+(2m^2+2)b)(b-m^2b+b(\upsilon+2(m^2+1)))(^n) \quad (31)$$

While $m\to 0$,

$$\mathrm{q}(\xi)=-\frac{1}{2ab}(\upsilon+2b)(b+(\upsilon+2b)(\sec(\xi)+\tan(\xi))^2). \quad (32)$$

After all, the new periodic solution is obtained as

$$u(x,t)=-\frac{1}{2ab}(-\upsilon+2b)(-(x-\upsilon t)(\upsilon+b)+\frac{4(\upsilon+2b)\sin\left(\frac{x-\upsilon t}{2}\right)}{\cos\left(\frac{x-\upsilon t}{2}\right)-\sin\left(\frac{x-\upsilon t}{2}\right)}). \quad (33)$$

Case8:

$\Im=\frac{1-m^2}{4}$, $\Re=\frac{1+m^2}{2}$, $\aleph=\frac{1-m^2}{4}$ and $\psi(\xi)=\frac{cn\xi}{1\pm sn\xi}$

so

$$\mathrm{q}(\xi)=\frac{1}{2ab(m^2-1)}((-\upsilon+(2m^2+2)b)(b-m^2b+b(\upsilon+2(m^2+1)))\left(\frac{cn\xi}{1\pm sn\xi}\right)^2). \quad (34)$$

While $m \to 0$,

$$q(\xi) = -\frac{1}{2ab}((-\upsilon + 2b)(b + (\upsilon + 2b))\frac{\cos^2\xi}{1\pm\sin^2\xi}). \quad (35)$$

So, the new periodic solution is obtained as

$$u(x,t) = \frac{-1}{2ab\left(\cos\left(\frac{x-\upsilon t}{2}\right) + \sin\left(\frac{x-\upsilon t}{2}\right)\right)} (\upsilon - 2b)((x-\upsilon t)(\upsilon + b)\cos\left(\frac{x-\upsilon t}{2}\right) + (\upsilon((x-\upsilon t)-4) + b((x-\upsilon t)-8))\sin\left(\frac{x-\upsilon t}{2}\right)). \quad (36)$$

Case9:

$\mathfrak{I} = \frac{1}{4}$, $\quad \mathfrak{R} = \frac{1-2m^2}{2}$, $\quad \aleph = \frac{1}{4}$ *and* $\psi(\xi) = \frac{sn\xi}{1\pm cn\xi}$,

for

$$q(\xi) = \frac{1}{2ab}\left((\upsilon + 2b)\left(b + (\upsilon - 2b)\left(\frac{sn\xi}{1\pm cn\xi}\right)^2\right)\right). \quad (37)$$

When $m \to 1$,

$$q(\xi) = \frac{1}{2ab}((\upsilon + 2b)(b + (\upsilon - 2b))\left(\frac{\tanh\xi}{1\pm \text{sech}\xi}\right)^2) \quad (38)$$

and the kink soliton is obtained as

$$u(x,t) = \frac{1}{2ab}(\upsilon + 2b)((x - \upsilon t)(\upsilon - b) - 2(\upsilon - 2b)\tanh\left(\frac{x-\upsilon t}{2}\right)). \quad (39)$$

Case10:

$\mathfrak{I} = \frac{(1-m^2)^2}{4}$, $\mathfrak{R} = \frac{1+m^2}{2}$, $\aleph = \frac{1}{4}$ *and* $\psi(\xi) = \frac{sn\xi}{cn\xi \pm dn\xi}$

and

$$q(\xi) = \frac{1}{2ab}(\upsilon - 2b(m^2+1))(b + (\upsilon + 2(1+m^2)b)\left(\frac{sn\xi}{cn\xi\pm dn\xi}\right)^2). \quad (40)$$

When $m \to 1$,

$$q(\xi) = \frac{1}{2ab}(\upsilon - 4b)\left(b + (\upsilon + 4b)\left(\frac{\tanh\xi}{2\text{sech}\xi}\right)^2\right). \quad (41)$$

and the hyperbolic function solution is obtained as

$$u(x,t) = \frac{1}{32ab}(\upsilon - 4b)(-2(x - \upsilon t)(\upsilon - 4b) + (\upsilon + 4b)\sinh(2(x - \upsilon t))). \quad (42)$$

Case11:

$\mathfrak{I} > 0$, $\mathfrak{R} < 0$, $\aleph = \frac{m^2\mathfrak{R}^2}{(1+m^2)^2\mathfrak{I}}$, $\psi(\xi) = \sqrt{\frac{-m^2\mathfrak{R}}{(1+m^2)\mathfrak{I}}}\, sn\left(\sqrt{\frac{-\mathfrak{R}}{(1+m^2)}}\xi\right)$,

so

$$q(\xi) = -\frac{1}{8m^2ab}(^1(\upsilon + 4b)(-\frac{4m^2b}{(1+m^2)^2} + (\upsilon - 4b)\sqrt{\frac{m^2}{(1+m^2)}}\, sn\left(\sqrt{\frac{1}{(1+m^2)}}\xi\right)). \quad (43)$$

When $m \to 1$,

$$q\ \xi = -\frac{1}{2ab}\ \upsilon + 4b\ (-b + \frac{1}{2}\ \upsilon - 4b\ \tanh^2\left(\frac{\xi}{\sqrt{2}}\right)). \quad (44)$$

and the kink soliton solution is obtained as

$$u(x,t) = \frac{1}{4ab}(\upsilon + 4b)(-(x - \upsilon t)(\upsilon - 6b) + \sqrt{2}(\upsilon - 4b)\tanh\left(\frac{x-\upsilon t}{\sqrt{2}}\right)). \quad (45)$$

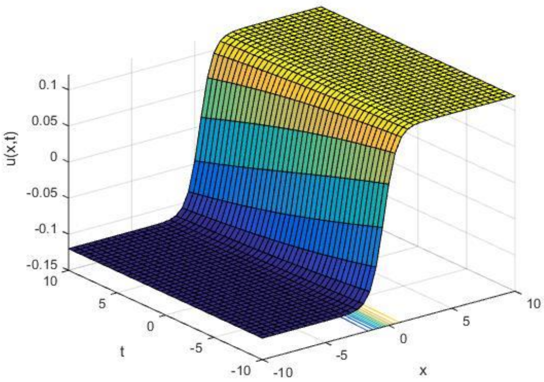

Figure 1. 3D represantation of solution Eq. (13) with $a = 0.5, b = 0.01$.

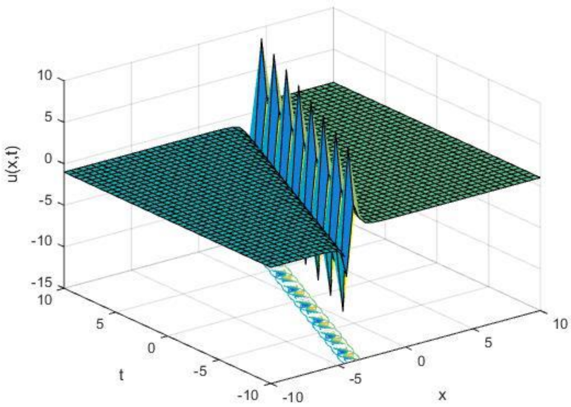

Figure 2. 3D represantation of solution Eq. (19) with $a = 0.5, b = 0.1$.

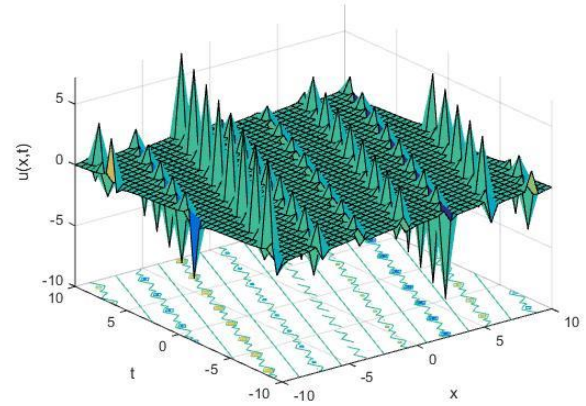

Figure 3. 3D represantation of solution Eq. (25) with $a = 1, b = -0.1$.

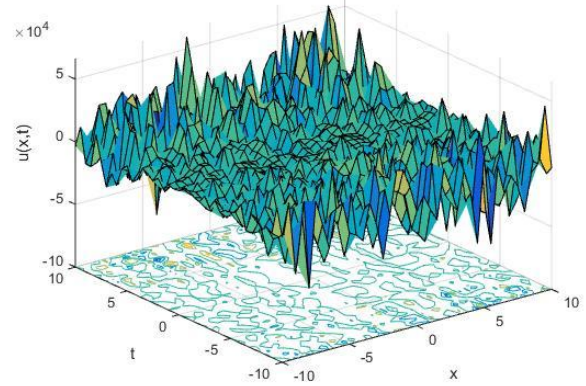

Figure 4. 3D represantation of solution Eq. (36) with $a = 0.1, b = -5$.

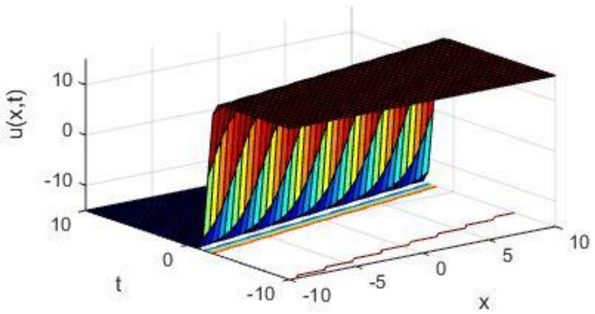

Figure 5. 3D represantation of solution Eq (39) with $a = 1, b = 5$.

3 Results and Discussions

In this section, we have exemplified graphical representation of nonlinear the potential KdV equation. Figs. 1-5 as it illustrate 3D plot of some of our obtained solutions. Fig. 1 represent the profile of kink soliton solution of Eq.(13) for $a=0.5, b=0.01$. Fig. 2 shows the shape of singular soliton solution of Eq.(19) for $a=0.5, b=0.1$. Fig. 3 depicts the periodic solution of Eq. (25) for $a=1, b=-0.1$. Similarly, Fig. 4 shows the periodic solution of Eq. (36) for $a=0.1, b=-5$. And finally, Fig. 5 demonstrated the kink soliton solution of Eq. (39) for $a=1, b=5$.

4 Conclusion

In this work, we presented solitons, hyperbolic, and periodic function wave solutions for the potential KdV, which govern the dynamics of water waves along ocean shores. The Jacobi elliptic function method was developed from new wave solutions to the potential KdV equation researched in science and engineering. In addition, we illustrated solutions in Figures 1-5. After all, obtaining solutions shows that the JEF method is powerful and yields important results. Based on the analysis, it is concluded that the method used is simple and direct.

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Declaration

Ethics committee approval is not required.

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