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ON CENTRAL AUTOMORPHISMS OF FREE METABELIAN LIE ALGEBRAS

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Abstract. Let F_m be the free metabelian Lie algebra of rank m over a field K of characteristic 0. An automorphism φ of F_m is called central if φ commutes with every inner automorphism of F_m. Such automorphisms form the centralizer $\mathrm{C}(\mathrm{Inn}(F_m))$ of inner automorphism group $\mathrm{Inn}(F_m)$ of F_m in $\mathrm{Aut}(F_m)$. We provide an elementary proof to show that $\mathrm{C}(\mathrm{Inn}(F_m)) = \mathrm{Inn}(F_m)$.

1. Introduction

Let F_m be the free metabelian Lie algebra of rank $m \geq 2$ over a field K of characteristic 0 with free generators $x_1, \ldots, x_m$. This is the relatively free algebra of rank m in the variety of Lie algebras $\mathfrak{A}^2$, where $\mathfrak{A}^2$ is the metabelian (solvable of class 2) variety of Lie algebras.

An automorphism σ of a group G is called central if σ commutes with every automorphism in the group of inner automorphisms $\mathrm{Inn}(G)$. For an abelian group G, $\mathrm{Inn}(G)$ is trivial so that the group of central automorphisms $\mathrm{C}(\mathrm{Inn}(G))$ of G is equal to $\mathrm{Aut}(G)$. Thus it is natural to work on non-abelian groups with the extreme situations $\mathrm{C}(\mathrm{Inn}(G)) = \mathrm{Inn}(G)$ and $\mathrm{C}(\mathrm{Inn}(G)) = \mathrm{Aut}(G)$. There has been a lot of work on these problems. For instance G. A. Miller [4] constructed a non-abelian group G of order 64 such that $\mathrm{Aut}(G)$ is abelian and therefore $\mathrm{C}(\mathrm{Inn}(G)) = \mathrm{Aut}(G)$. Another work in this direction was settled by Curran and McChaugan [3] as following:

Theorem 1.1. *Let G be a non-abelian $p-$group. Then* $\mathrm{C}(\mathrm{Inn}(G)) = \mathrm{Inn}(G)$ *if and only if $Z(G) = [G, G]$ and $Z(G)$ is cyclic.*

The goal of our paper is to describe the group of central automorphisms $\mathrm{C}(\mathrm{Inn}(F_m))$ of the Lie algebra F_m and following the result of Curran and McChaugan [3], to show that $\mathrm{C}(\mathrm{Inn}(F_m)) = \mathrm{Inn}(F_m)$.

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A result of Shmel'kin [5] states that the free metabelian Lie algebra F_m can be embedded into the abelian wreath product $A_m \mathrm{wr} B_m$, where A_m and B_m are m-dimensional abelian Lie algebras with bases $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$, respectively. The elements of $A_m \mathrm{wr} B_m$ are of the form $\sum_{i=1}^{m} a_i f_i(t_1, \ldots, t_m) + \sum_{i=1}^{m} \beta_i b_i$, where f_i are polynomials in $K[t_1, \ldots, t_m]$ and $\beta_i \in K$. This allows to make calculations in F'_m with values in $K[t_1, \ldots, t_m]$.

2. Preliminaries

Let F_m be the free metabelian Lie algebra of rank $m \geq 2$ over a field K of characteristic 0 with free generators $x_1, \ldots, x_m$. We use the commutator notation for the Lie multiplication. Our commutators are left normed:

$$[u_1, \ldots, u_{n-1}, u_n] = [[u_1, \ldots, u_{n-1}], u_n], \quad n = 3, 4, \ldots.$$

In particular,

$$F_m^k = \underbrace{[F_m, \ldots, F_m]}_{k \text{ times}}.$$

It is well known, see e.g. [1], that

$$[x_{i_1}, x_{i_2}, x_{i_{\sigma(3)}}, \ldots, x_{i_{\sigma(k)}}] = [x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_k}],$$

where σ is an arbitrary permutation of $3, \ldots, k$ and that F'_m has a basis consisting of all

$$[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_k}], \quad 1 \leq i_j \leq m, \quad i_1 > i_2 \leq i_3 \leq \cdots \leq i_k.$$

For each $v \in F'_m$, the linear operator $\mathrm{ad}v : F_m \to F_m$ defined by

$$u(\mathrm{ad}v) = [u, v], \quad u \in F_m,$$

is a derivation of F_m which is nilpotent because $\mathrm{ad}^2 v = 0$. Hence the linear operator

$$\exp(\mathrm{ad}v) = 1 + \frac{\mathrm{ad}v}{1!} + \frac{\mathrm{ad}^2 v}{2!} + \cdots = 1 + \mathrm{ad}v$$

is well defined and is an automorphism of F_m. The set of all such automorphisms forms a normal subgroup of the group of all automorphisms $\mathrm{Aut}(F_m)$ of F_m. This group is called the inner automorphism group of F_m and is denoted by $\mathrm{Inn}(F_m)$.

An automorphism of F_m is called an IA$-$automorphism if it induces the identity map modulo the commutator ideal of F_m. The set of all such automorphisms forms the normal subgroup of $\mathrm{Aut}(F_m)$ which is denoted by $\mathrm{IA}(F_m)$. The automorphism group $\mathrm{Aut}(L_{m,c})$ is a semidirect product of the normal subgroup $\mathrm{IA}(L_{m,c})$ and the general linear group $\mathrm{GL}_m(K)$. It is clear that $\mathrm{Inn}(L_{m,c}) \subset \mathrm{IA}(F_m)$.

An automorphism of F_m is called central if it commutes with every automorphism in $\mathrm{Inn}(F_m)$. The set of all such automorphisms forms the centralizer of the group $\mathrm{Inn}(F_m)$ in the group $\mathrm{Aut}(F_m)$. We denote this group by $\mathrm{C}(\mathrm{Inn}(F_m))$. Since

$$\exp(\mathrm{ad}u)\exp(\mathrm{ad}v) = \exp(\mathrm{ad}(u+v)), \quad u, v \in F'_m,$$

then $\mathrm{Inn}(F_m)$ is abelian and thus $\mathrm{Inn}(L_{m,c}) \subset \mathrm{C}(\mathrm{Inn}(F_m))$. Let $\varphi \in \mathrm{C}(\mathrm{Inn}(F_m))$. Then $\varphi \exp(\mathrm{ad}u) = \exp(\mathrm{ad}u)\varphi$ for all $u \in F'_m$. Since $\exp(\mathrm{ad}u)$ fixes every element in F'_m we get that

$$\exp(\mathrm{ad}\varphi(u)) = \exp(\mathrm{ad}u).$$

The adjoint representation ad is faithful so we have that $\varphi(u) = u$ for every $u \in F'_m$. Conversely one can easily check that if $\varphi(u) = u$ for every $u \in F'_m$ then $\varphi \in \mathrm{C}(\mathrm{Inn}(F_m))$. Here, we strongly remind that such automorphisms become inner via Proposition 4 as a result of Proposition 3 in [5]. We provide a relatively easy and direct proof of this fact in the section of main results.

Now let us give the necessary information about wreath product. For details and references see e.g. [2]. Let $K[t_1,\dots,t_m]$ be the (commutative) polynomial algebra over K freely generated by the variables $t_1,\dots,t_m$ and let A_m and B_m be the abelian Lie algebras with bases $\{a_1,\dots,a_m\}$ and $\{b_1,\dots,b_m\}$, respectively. Let C_m be the free right $K[t_1,\dots,t_m]$-module with free generators $a_1,\dots,a_m$. We give it the structure of a Lie algebra with trivial multiplication. The abelian wreath product $A_m\mathrm{wr}B_m$ is equal to the semidirect sum $C_m \leftthreetimes B_m$. The elements of $A_m\mathrm{wr}B_m$ are of the form $\sum_{i=1}^m a_i f_i(t_1,\dots,t_m) + \sum_{i=1}^m \beta_i b_i$, where f_i are polynomials in $K[t_1,\dots,t_m]$ and $\beta_i \in K$. The multiplication in $A_m\mathrm{wr}B_m$ is defined by

$$[C_m, C_m] = [B_m, B_m] = 0,$$

$$[a_i f_i(t_1,\dots,t_m), b_j] = a_i f_i(t_1,\dots,t_m)t_j, \quad i,j = 1,\dots,m.$$

Hence $A_m\mathrm{wr}B_m$ is a metabelian Lie algebra and every mapping $\{x_1,\dots,x_m\} \to A_m\mathrm{wr}B_m$ can be extended to a homomorphism $F_m \to A_m\mathrm{wr}B_m$. As a special case of the embedding theorem of Shmel'kin [5], the homomorphism $\varepsilon : F_m \to A_m\mathrm{wr}B_m$ defined by $\varepsilon(x_i) = a_i + b_i$, $i = 1,\dots,m$, is a monomorphism. If

$$f = \sum [x_i, x_j] f_{ij}(\mathrm{ad}x_1,\dots,\mathrm{ad}x_m), \quad f_{ij}(t_1,\dots,t_m) \in K[t_1,\dots,t_m],$$

then

$$\varepsilon(f) = \sum (a_i t_j - a_j t_i) f_{ij}(t_1,\dots,t_m).$$

Let us define $T_j = \{t_j,\dots,t_m\}$ for each $j = 1,\dots,m$.

Lemma 2.1. *Let $t_i \mid (t_j f_j(T_j) + \cdots + t_m f_m(T_m))$ for $i < j$, where $f_j(T_j) \in K[T_j]$, $1 < j \le m$. Then $f_k(T_k) = 0$ for every $j \le k \le m$.*

Proof. Let us define $P_k = t_k f_k(T_k) + \cdots + t_m f_m(T_m)$ for $j \le k \le m$. Now let $t_i \mid P_j$. Assume that $P_j \neq 0$. Then t_i is a factor of $P_j \in K[T_j]$ which is a contradiction. Thus $P_j = 0$ which implies that $t_j \mid P_{j+1}$. Similarly we get that $P_{j+1} = 0$. Repeating this process we get that

$$P_j = \cdots = P_m = 0$$

and thus $f_j(T_j) = f_m(T_m) = 0$. □

3. Main Results

Lemma 3.1. $\mathrm{C}(\mathrm{Inn}(F_m)) \subset \mathrm{IA}(F_m)$.

Proof. Let $\varphi \in \mathrm{C}(\mathrm{Inn}(F_m))$ posses the form

$$\varphi : x_i \to \sum_{k=1}^m c_{ik} x_k + v_i, \quad v_i \in F_m', \quad i = 1,\dots,m.$$

Since φ is a central automorphism then $\varphi([x_1,x_2]) = [x_1,x_2]$ and $\varphi([x_1,x_2,x_j]) = [x_1,x_2,x_j]$ for every $j = 1,\dots,m$. Then we have for each $j = 1,\dots,m$ that

$$\begin{aligned} \varphi([x_1,x_2,x_j]) &= [\varphi([x_1,x_2]), \varphi(x_j)] \\ [x_1,x_2,x_j] &= [x_1,x_2,c_{j1}x_1 + \cdots + c_{jm}x_m + v_j] \\ [x_1,x_2,x_j](1 - c_{jj}) &= \sum_{k\neq j} [x_1,x_2,x_k] c_{jk}. \end{aligned}$$

Using the embedding F_m into the wreath product $A_m \mathrm{wr} B_m$ we get that

$$(a_1t_2 - a_2t_1)t_j(1 - c_{jj}) - (a_1t_2 - a_2t_1)\sum_{k\neq j} t_k c_{jk} = 0$$

Since the generators a_1 and a_2 are free generators, we obtain the equality

$$t_j(1 - c_{jj}) - \sum_{k\neq j} t_k c_{jk} = 0$$

which means that $c_{jj} = 1$ and $c_{jk} = 0$ for every $k \neq j$. □

Now we know that

$$\mathrm{C}(\mathrm{Inn}(F_m)) = \{\varphi \in \mathrm{IA}(F_m) \mid \varphi(u) = u,\, u \in F'_m\}.$$

Lemma 3.2. $\mathrm{C}(\mathrm{Inn}(F_2)) = \mathrm{Inn}(F_2)$.

Proof. Let φ be a central automorphism of the form

$$\begin{aligned}\varphi :& x_1 \to x_1 + [x_2, x_1]f(\mathrm{ad}x_1, \mathrm{ad}x_2)\\ & x_2 \to x_2 + [x_2, x_1]g(\mathrm{ad}x_1, \mathrm{ad}x_2),\end{aligned}$$

where $f = f(t_1, t_2), g = g(t_1, t_2) \in K[t_1, t_2]$. Since φ is central then $\varphi([x_1, x_2]) = [x_1, x_2]$ and

$$\begin{aligned}\varphi([x_1, x_2]) &= [x_1 + [x_2, x_1]f, x_2 + [x_2, x_1]g]\\ [x_1, x_2] &= [x_1, x_2] + [x_2, x_1](\mathrm{ad}x_2 f - \mathrm{ad}x_1 g).\end{aligned}$$

Using the embedding F_2 into the wreath product $A_2 \mathrm{wr} B_2$ we get that

$$(a_2t_1 - a_1t_2)(t_2f - t_1g) = 0$$

Hence we obtain the equality

$$t_2f = t_1g$$

which means that $f(t_1, t_2) = t_1p(t_1, t_2)$ and $g(t_1, t_2) = t_2p(t_1, t_2)$ for a polynomial $p(t_1, t_2)$ in $K[t_1, t_2]$. Thus we get the following

$$\varphi = \exp(\mathrm{ad}u), \quad u = -[x_2, x_1]p(\mathrm{ad}x_1, \mathrm{ad}x_2)$$

which completes the proof. □

Theorem 3.3. *Let φ be a central automorphism of the form*

$$\begin{aligned}\varphi :& x_1 \to x_1 + \sum_{p=2}^{m}[x_p, x_1]f_{p1}^{(1)}(\mathrm{ad}x_2, \ldots, \mathrm{ad}x_m)\\ & x_j \to x_j + \sum_{p>q>1}[x_p, x_q]f_{pq}^{(j)}(\mathrm{ad}x_q, \ldots, \mathrm{ad}x_m), \quad 2 \leq j \leq m,\end{aligned}$$

where $f_{p1}^{(1)}(t_2, \ldots, t_m) \in K[T_2]$ and $f_{pq}^{(j)}(t_q, \ldots, t_m) \in K[T_q]$ for $2 \leq j \leq m$. Then $\varphi = 1$ is the identity map.

Proof. We make induction on the rank $m \geq 2$ of F_m. Let $m = 2$ and φ be of the form

$$\begin{aligned}\varphi :& x_1 \to x_1 + [x_2, x_1]f_{21}^{(1)}(\mathrm{ad}x_2)\\ & x_2 \to x_2\end{aligned}$$

Following Lemma 3.2 we get easily that $f_{21}^{(1)}(\mathrm{ad}x_2) = 0$ and so $\varphi = 1$. Now let $m > 2$ and consider a central automorphism φ of the form

$$\begin{aligned}\varphi : & x_1 \to x_1 + \sum_{p=2}^{m} [x_p, x_1] f_{p1}^{(1)}(\mathrm{ad}x_2, \ldots, \mathrm{ad}x_m) \\ & x_j \to x_j + \sum_{p>q>1} [x_p, x_q] f_{pq}^{(j)}(\mathrm{ad}x_q, \ldots, \mathrm{ad}x_m), \quad 2 \le j \le m,\end{aligned}$$

where $f_{p1}^{(1)}(t_2, \ldots, t_m) \in K[T_2]$ and $f_{pq}^{(j)}(t_q, \ldots, t_m) \in K[T_q]$ for every $2 \le j \le m$. Let (x_1) denote the ideal of the free metabelian Lie algebra F_m by the free generator x_1. Because of the fact that $\varphi((x_1)) \subset (x_1)$ it induces a central automorphism $\widehat{\varphi}$ on $F_m/(x_1) \cong F_{m-1} = K < \bar{x}_2, \ldots, \bar{x}_m >$ such that

$$\widehat{\varphi} : \bar{x}_j \to \bar{x}_j + \sum_{p>q>1} [\bar{x}_p, \bar{x}_q] f_{pq}^{(j)}(\mathrm{ad}\bar{x}_q, \ldots, \mathrm{ad}\bar{x}_m), \quad 2 \le j \le m.$$

By the assumption of induction we get that $f_{pq}^{(j)}(t_q, \ldots, t_m) = 0$ for every $j = 2, \ldots, m$ and $p > q > 1$. Let us rewrite the central automorphism φ:

$$\begin{aligned}\varphi : & x_1 \to x_1 + \sum_{p=2}^{m} [x_p, x_1] f_{p1}^{(1)}(\mathrm{ad}x_2, \ldots, \mathrm{ad}x_m) \\ & x_j \to x_j, \quad j = 2, \ldots, m,\end{aligned}$$

where $f_{p1}^{(1)}(t_2, \ldots, t_m) \in K[T_2]$. Since φ is central then $\varphi([x_1, x_j]) = [x_1, x_j]$ for every $j = 2, \ldots, m$. Using the similar steps in the proof of Lemma 3.2 we get easily that $\varphi = 1$. □

Our next theorem gives the main result.

Theorem 3.4. $\mathrm{C}(\mathrm{Inn}(F_m)) = \mathrm{Inn}(F_m)$.

Proof. It sufficies to show that $\mathrm{C}(\mathrm{Inn}(F_m)) \subset \mathrm{Inn}(F_m)$. We know from Lemma 3.2 that $\mathrm{C}(\mathrm{Inn}(F_2)) = \mathrm{Inn}(F_2)$. Now let $m > 2$ and let a central automorphism φ be of the form

$$\varphi : x_j \to x_j + \sum_{p>q} [x_p, x_q] h_{pq}^{(j)}(\mathrm{ad}x_q, \ldots, \mathrm{ad}x_m), \quad 1 \le j \le m,$$

where $h_{pq}^{(j)}(t_q, \ldots, t_m) \in K[T_q]$ for every $1 \le j \le m$. Let us express $h_{p1}^{(1)}(t_1, \ldots, t_m)$ in the following way:

$$h_{p1}^{(1)}(t_1, \ldots, t_m) = t_1 g_{p1}^{(1)}(t_1, \ldots, t_m) + f_{p1}^{(1)}(t_2, \ldots, t_m), \quad p = 2, \ldots, m,$$

and let us define

$$\psi = \exp(\mathrm{ad}u), \quad u = \sum_{p=2}^{m} [x_p, x_1] g_{p1}^{(1)}(\mathrm{ad}x_1, \ldots, \mathrm{ad}x_m).$$

Then the composition $\phi = \psi\varphi$ is of the form

$$\begin{aligned}\phi : & x_1 \to x_1 + \sum_{p=2}^{m} [x_p, x_1] f_{p1}^{(1)}(\mathrm{ad}x_2, \ldots, \mathrm{ad}x_m) + \sum_{p>q>1}^{m} [x_p, x_q] f_{pq}^{(1)}(\mathrm{ad}x_q, \ldots, \mathrm{ad}x_m) \\ & x_j \to x_j + \sum_{p>q} [x_p, x_q] f_{pq}^{(j)}(\mathrm{ad}x_q, \ldots, \mathrm{ad}x_m), \quad 2 \le j \le m.\end{aligned}$$

Since ϕ is a central automorphism then $\phi([x_1,x_2]) = [x_1,x_2]$. Combining this equality with the embedding of F_m into the wreath product $A_m \mathrm{wr} B_m$ we have the following equation system:

$$t_2(t_{m-1}f^{(1)}_{m(m-1)} + \cdots + t_2 f^{(1)}_{m2} + t_1 f^{(1)}_{m1}) = t_1(t_{m-1}f^{(2)}_{m(m-1)} + \cdots + t_2 f^{(2)}_{m2} + t_1 f^{(2)}_{m1})$$

$$t_2\left(-t_m f^{(1)}_{m(m-1)} + \sum_{k=1}^{m-2} t_k f^{(1)}_{(m-1)k}\right) = t_1\left(-t_m f^{(2)}_{m(m-1)} + \sum_{k=1}^{m-2} t_k f^{(2)}_{(m-1)k}\right)$$

$$\vdots$$

$$t_2\left(-\sum_{k=4}^{m} t_k f^{(1)}_{k3} + t_2 f^{(1)}_{32} + t_1 f^{(1)}_{31}\right) = t_1\left(-\sum_{k=4}^{m} t_k f^{(2)}_{k3} + t_2 f^{(2)}_{32} + t_1 f^{(2)}_{31}\right)$$

$$t_2\left(-\sum_{k=3}^{m} t_k f^{(1)}_{k3} + t_1 f^{(1)}_{21}\right) = t_1\left(-\sum_{k=3}^{m} t_k f^{(1)}_{k2} + t_1 f^{(1)}_{21}\right)$$

where $f^{(1)}_{k1} \in K[T_2]$, $f^{(1)}_{pq} \in K[T_q]$ for every $k = 2,\ldots,m$ and $m \geq p > q > 1$ and $f^{(2)}_{ts} \in K[T_q]$ for every $m \geq t > s \geq 1$. Let us consider the first equality. It is clear that

$$t_1 \mid (t_{m-1}f^{(1)}_{m(m-1)} + \cdots + t_2 f^{(1)}_{m2}).$$

Applying Lemma 2.1 we have that

$$f^{(1)}_{m(m-1)} = \cdots = f^{(1)}_{m2} = 0.$$

Then the equality is transformed into

$$t_2 f^{(1)}_{m1} = t_{m-1}f^{(2)}_{m(m-1)} + \cdots + t_2 f^{(2)}_{m2} + t_1 f^{(2)}_{m1},$$

which means that

$$t_1 \mid (t_2 f^{(1)}_{m1} - t_{m-1}f^{(2)}_{m(m-1)} - \cdots - t_2 f^{(2)}_{m2}) \in K[T_2].$$

Hence $t_2 f^{(1)}_{m1} - t_{m-1}f^{(2)}_{m(m-1)} - \cdots - t_2 f^{(2)}_{m2} = 0$ and $f^{(2)}_{m1} = 0$. Similarly using the other equalities one can get the result of the fact that

$$f^{(1)}_{pq} = 0 \quad \text{and} \quad f^{(2)}_{p1} = 0,$$

for every $m-1 \geq p > q > 1$.

Finally using the action of the central automorphism ϕ on the commutators $[x_1,x_j]$ for $j = 3,\ldots,m$ we get that

$$f^{(3)}_{p1} = \cdots = f^{(m)}_{p1} = 0.$$

Now the central automorphism ϕ satisfies the condition in the Theorem 3.3. Thus $\phi = 1$ and

$$\varphi = \exp(\mathrm{ad}(-u)), \quad u = \sum_{p=2}^{m} [x_p, x_1] g^{(1)}_{p1}(\mathrm{ad}x_1,\ldots,\mathrm{ad}x_m).$$

$\square$

4. Conclusion

In this study an elementary proof of the fact that the group of inner automorphisms of free metabelian Lie algebras coinside with its centralizer. The situation is not the same for free nilpotent Lie algebras, since any inner automorphism of a nilpotent Lie algebra is determined by an arbitrary (including linear sums) element. Inner automorphisms of nilpotent Lie algebras are not commutative in general, hence the group of inner automorphisms is not included in its centralizer.

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