Journal of Universal Mathematics
Vol.5 No.2 pp.81-87 (2022)
ISSN-2618-5660
DOI: 10.33773/jum.1142805

A TILING INTERPRETATION FOR (p, q)-FIBONACCI AND (p, q)-LUCAS NUMBERS

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Abstract. In this paper, we introduce a tiling approach to (p, q)-Fibonacci and (p, q)-Lucas numbers that generalize the well-known Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, k -Fibonacci and k-Lucas numbers. We show that nth (p, q)-Fibonacci number is interpreted as the number of ways to tile a $1 \times n$ board with cells labeled $1,\ 2,\ \ldots,\ n$ using colored 1×1 squares and colored 1×2 dominoes, where there are p different colors for squares and q different colors for dominoes. Then nth (p, q)-Lucas number is interpreted as the number of ways to tile a circular $1 \times n$ board with colored squares and colored dominoes. We also present some generalized Fibonacci and Lucas identities using this tiling approach.

1. Introduction

The theory and application of Fibonacci and Lucas numbes in modern science have received great interest. The Fibonacci numbers F_n are the terms of the sequence $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ by recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 1,\quad F_1 = 1$ and Lucas numbers L_n are the terms of the sequence $\{2, 1, 3, 4, 7, 11, 18, 29, \ldots\}$ by recurrence relation $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2,\quad L_1 = 1$ for $n \geq 2$ [1]. Fibonacci and Lucas numbers have been curiously studied, and using the recurrence relations of sequences of Fibonacci and Lucas numbers and different initial conditions, new number sequences can be created as Fibonacci type sequences and Lucas type sequences, respectively. For $n \geq 2$, $k \geq 1$, Falcon and Plaza introduced the k-Fibonacci sequences and k-Lucas sequences by recurrence relations $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ with $F_{k,0} = 0$, $F_{k,1} = 1$ and $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$ with $L_{k,0} = 2$, $L_{k,1} = k$ [2, 3]. Several authors presented the Binet-like formulas, matrix representations, summation formulas and the new family of k-Fibonacci sequences, and k-Lucas sequences, and period according to the m modulo of this new family [4, 5, 6].

Date: **Received:** 2022-07-09; **Accepted:** 2022-07-30.
2000 *Mathematics Subject Classification.* 11B37; 11B39; 11B83.
Key words and phrases. Fibonacci Number, Generalized Fibonacci Numbers, Lucas Number, Tilings.

The sequences of (p,q)-Fibonacci and (p,q)-Lucas numbers, which are generalizations of the Fibonacci type and the Lucas type sequences, are defined by second order recurrence relations

$$F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}, F_{p,q,0} = 1,\ F_{p,q,1} = p, \quad n \geq 2 \tag{1.1}$$

and

$$L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2}, L_{p,q,0} = 2,\ L_{p,q,1} = p, \quad n \geq 2 \tag{1.2}$$

respectively. Some authors have studied the fundamental identities of these numbers similar to the well-known properties of Fibonacci numbers, providing various general formulas for these numbers [7, 8, 9]. Koshy introduced one of the most comprehensive sources contains the applications and generalizations of Fibonacci and Lucas sequences [10].

More generalized versions of the Fibonacci type and the Lucas type sequences have been obtained by using various different interpretations, such as the tiling interpretation, known as the number of tiling a board of n length using dominoes of various lengths. For one of them, the nth Fibonacci number F_n is interpreted combinatorially as the number of tiles of a $(1 \times n)$-board with 1×1 squares and 1×2 dominoes, called the n-board, and is showed that the sum of the squares of the first n Fibonacci numbers is verified using the geometric figures [11, 12]. These interpretations have been used to proven tiling proofs of many Fibonacci-type relations [13, 14]. For example, there are 5 different ways to tile a 1×4 board using 1×1 and 1×2 tiles as provided in the figure 1. So $F_4 = 5$.

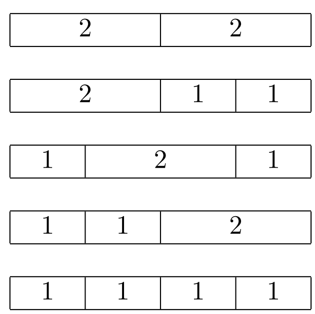

Figure 1. Tiling interpretations of F_5 number.

The aim of this study is to introduce the (p,q)-Fibonacci and the (p,q)-Lucas numbers with the tiling approach. The nth (p,q)-Fibonacci and the nth (p,q)-Lucas number are to express by tiling interpretations which allows one to derive properties of them via tiling proof. It is also to represent sums formulas of odd and even subscript numbers and well-known relations among these numbers.

2. A TILING APPROACH TO (p,q)-FIBONACCI AND (p,q)-LUCAS NUMBERS

In this section, we introduce tiling interpretations for (p,q)-Fibonacci and the (p,q)-Lucas numbers. We explore the nth (p,q)-Fibonacci and the nth (p,q)-Lucas number with the tiling approach. Moreover, some fundamental identities and sums formulas for these numbers are given by tiling interpretations. Let us first consider the nth (p,q)-Fibonacci number which is related to a $1\times n$ board. We present (p,q)-Fibonacci numbers count the number of distinct ways to tile a $1\times n$ board using colored squares and colored dominoes. The first two tiles are given initial conditions such that there are $F_{p,q,1}$ condition for an initial square and $qF_{p,q,0}$ condition for an initial domino.

Theorem 2.1. *For $n\geq 1$, $F_{p,q,n}$ counts the number of ways to tile a $1\times n$ board with colored 1×1 tiles (squares) and colored 1×2 tiles (dominoes), where there are p different colors for squares and q different colors for dominoes.*

Proof. Let $F_{p,q,n}$ represent the number of distinct ways to tile a $1\times n$ board with cells labeled $1,\ 2,\ \ldots,\ n$ using colored 1×1 tiles (squares) and colored 1×2 tiles (dominoes), where there are p different colors for squares and q different colors for dominoes. It is easy to obtain that the initial (p,q)-Fibonacci conditions satisfy, since there is one way to tile a empty board, $F_{p,q,0}=1$ and, since a board of length one can be obtained by one colored square, $F_{p,q,1}=p$. For $n\geq 2$, the first tile in all tilings is either colored 1×1 and colored 1×2 tiles. If the first tile is 1×1 tile, then p different colors can be selected for the square, and we get $F_{p,q,n-1}$ ways to tile $1\times(n-1)$ board. If the first tile is a 1×2 tile, then q different colors can be selected for the domino, and we get $F_{p,q,n-2}$ ways to tile $1\times(n-2)$ board. Thus $F_{p,q,n}=pF_{p,q,n-1}+qF_{p,q,n-2}$, in relation 1.1 is obtained, and this completes the proof. □

Similary, the (p,q)-Lucas numbers can be interpreted as the number of ways to tile a circular $1\times n$ board with squares and dominoes. The first two tiles are given initial conditions such that there are $L_{p,q,1}$ condition for an initial square and $qL_{p,q,0}$ condition for an initial domino.

Theorem 2.2. *For $n\geq 1$, $L_{p,q,n}$ counts the number of ways to tile a circular $1\times n$ board with colored 1×1 tiles (squares) and colored 1×2 tiles (dominoes), where there are p different colors for squares and q different colors for dominoes.*

Proof. Let $L_{p,q,n}$ represent the number of distinct ways to tile a circular $1\times n$ board, called a $1\times n$ circle, with squares and dominoes. If there is no domino covering cells n and 1, we call a $1\times n$ circle in-circuit, and a $1\times n$ circle out-of circuit if there is a domino covering cells n and 1. The empty circle can be either in-circuit or out-of circuit, then $L_{p,q,0}=2$. Since a 1×1 circle can be tiled only by a square $L_{p,q,1}=p$. For $n\geq 2$, according to the first tile, there are two cases. If a square is added to the left of the first tile, $1\times n$ circle is obtained from $1\times(n-1)$ circle, we get $L_{p,q,n-1}$ ways. If a domino is added to the left of the first tile, $1\times n$ circle is obtained from $1\times(n-2)$ circle, which we get $L_{p,q,n-2}$ ways. Then for $n\geq 2$, we have $L_{p,q,n}=pL_{p,q,n-1}+qL_{p,q,n-2}$, in the relation 1.2, and this completes the proof. □

This tiling interpretation allows us to present many identities for (p,q)-Fibonacci and the (p,q)-Lucas numbers. We represent well-known relations among these numbers with the following theorems using the tiling interpretations. Now, we give sum formulas for the (p,q)-Fibonacci sequences and the (p,q)-Lucas sequences.

Theorem 2.3.

$$\textbf{\textit{i.}}\quad q\sum_{i=0}^{n} p^{n-i}F_{p,q,i}+p^{n+2}=F_{p,q,n+2}$$

$$\textbf{\textit{ii.}}\quad q\sum_{i=0}^{n} p^{n-i}L_{p,q,i}+p^{n+2}=L_{p,q,n+2}$$

Proof. ***i.*** From Theorem 2.1, the number of ways to tile a $1\times(n+2)$ board with colored 1×1 tiles and colored 1×2 tiles equal to $F_{p,q,n+2}$. Thus, the right-hand side of this identity is counted. If it is shown that the left-hand side of the identity gives the same count, the proof is completed. Suppose that we have a $1\times(n+2)$ board. The number of the tilings consisting of all squares is p^{n+2}. All other tilings contain at least one domino. Tilings containing a 1×2 tile can be partitioned according to the location of last domino covers cells $i+1$ and $i+2$ for $0\le i\le n$. For $1\le i\le n$, there are $F_{p,q,i}$ ways to tile cells 1 through i, q different colors can be selected for the domino on cells $i+1$ and $i+2$, and p^{n-i} different colors can be selected on cells $i+3$ to $n+2$; consequently there are $qF_{p,q,i}p^{n-i}$ such tilings ways. If $i=0$, there are $qF_{p,q,0}$ ways to select the initial domino and p^n different colors can be selected for the remaining squares, and we get $qF_{p,q,0}p^n$ ways. Summing gives the desired result, we get $q\sum_{i=0}^{n}p^{n-i}F_{p,q,i}+p^{n+2}=F_{p,q,n+2}$. So the proof is completed.

ii. The proof of ii. is similarly obtained using Theorem 2.2.

□

Sum formulas of the first n terms with odd subscripts of the (p,q)-Fibonacci sequences and the (p,q)-Lucas sequences using the interpretation of tiling a board of $2n$ length are given by the following theorems.

Theorem 2.4.

$$\textbf{\textit{i.}}\quad p\sum_{i=1}^{n} q^{n-i}F_{p,q,2i-1}+q^{n}=F_{p,q,2n}$$

$$\textbf{\textit{ii.}}\quad p\sum_{i=1}^{n} q^{n-i}L_{p,q,2i-1}+2q^{n}=L_{p,q,2n}$$

Proof. ***i.*** From Theorem 2.1, the number of ways to tile a $1\times2n$ board with colored 1×1 tiles and colored 1×2 tiles equal to $F_{p,q,2n}$. Thus, the right-hand side of this identity is counted. If it is shown that the left-hand side of the identity gives the same count, the proof is completed. Suppose that we have a $1\times2n$ board. The number of the tilings consisting of all dominoes is q^n. The other tilings containing a 1×1 tile can be partitioned according to the location of last square covers cell $2i$ for $1\le i\le n$. There are $F_{p,q,2i-1}$ ways to tile cells 1 through $2i-1$, and pq^{n-i} different colors can be selected for last square and the remaining $n-i$ dominoes. Summing gives the desired result, we get $p\sum_{i=1}^{n}q^{n-i}F_{p,q,2i-1}+q^n=F_{p,q,2n}$. So the proof is completed.

ii. The proof of ii. is similarly obtained using Theorem 2.2.

□

Sum formulas of the first n terms with even subscripts of the (p,q)-Fibonacci sequences and the (p,q)-Lucas sequences using the interpretation of tiling a board of $2n+1$ length are given by the following theorems.

Theorem 2.5.

$$\textbf{\textit{i.}}\quad p\sum_{i=1}^{n} q^{n-i}F_{p,q,2i}+pq^{n}=F_{p,q,2n+1}$$

$$\textbf{\textit{ii.}}\quad p\sum_{i=1}^{n} q^{n-i}L_{p,q,2i}+pq^{n}=L_{p,q,2n+1}$$

Proof. ***i.*** From Theorem 2.1, the number of ways to tile a $1\times(2n+1)$ board with colored 1×1 tiles and colored 1×2 tiles equal to $F_{p,q,2n+1}$. Thus, the right-hand side of this identity is counted. If it is shown that the left-hand side of the identity gives the same count, the proof is completed. Suppose that we have a $1\times(2n+1)$ board. All tilings have at least one 1×1 tile. So, the number of the tilings consisting of maximum dominoes is pq^n. The other tilings containing a 1×1 tile can be partitioned according to the location of last square covers cell $2i+1$ for $0\leq i\leq\ n$. There are $F_{p,q,2i}$ ways to tile cells 1 through $2i$, and pq^{n-i} different colors can be selected for last square and the remaining $n-i$ dominoes. Separating the $i=0$ case, summing gives the desired result, and we get $p\sum_{i=1}^{n} q^{n-i}F_{p,q,2i}+pq^{n}=F_{p,q,2n+1}$. So the proof is completed.

ii. The proof of ii. is similarly obtained using Theorem 2.2.

□

The relation between the nth the (p,q)-Fibonacci number, $F_{p,q,n}$ and nth (p,q)-Lucas number, $L_{p,q,n}$ is given by the following theorem.

Theorem 2.6. *Let $m,\ n\geq1$ be integers and we have*

$$2F_{p,q,m+n-1}=F_{p,q,m-1}L_{p,q,n}+F_{p,q,n-1}L_{p,q,m}$$

Proof. Begin by tiling the board with a $1\times(m+n-1)$ board. From Theorem 2.1, $F_{p,q,m+n-1}$ counts the number of ways to tile a $1\times(m+n-1)$ board with colored 1×1 tiles and colored 1×2 tiles. Thus, the left-hand side of this identity is counted. If it is shown that the right-hand side of the identity gives the same count, the proof is completed. On the other hand, we tile a $1\times(m+n-1)$ board in two ways:

1. The $1\times(m+n-1)$ tiling has no dominoes covering the nth and $(n+1)$th cells, and there is a line between cells n and $n+1$, which is at nth cell, then the $1\times(m+n-1)$ tiling can be divided into a $1\times n$ tiling and a $1\times(m-1)$ tiling. Now we attach the right side of the nth cell to the left side of the first cell of the $1\times n$ tiling, so we get an in-circuit $1\times n$ circle. Let $L^1_{p,q,n}$ denote the number of ways to tile an in-circuit $1\times n$ circle.
2. The $1\times(m+n-1)$ tiling has a domino covering the nth and $(n+1)$th cells, and there is not a line between cells n and $n+1$, which is at n-th cell, then it is breakable at $(n-1)$th cell. In this case, we create a $1\times(n-1)$ tiling and an out-of circuit $1\times m$ circle. Let $L^2_{p,q,m}$ denote the number of ways to tile an out-of circuit $1\times m$ circle.

Now, considering whether the $1 \times (m+n-1)$ tile has a line in the mth cell, we apply the same approach for the mth cell. So, we get

$$\begin{aligned} 2F_{p,q,m+n-1} =& L^1_{p,q,n}F_{p,q,m-1} + F_{p,q,n-1}L^2_{p,q,m} + L^1_{p,q,m}F_{p,q,n-1} + F_{p,q,m-1}L^2_{p,q,n} \\ =& F_{p,q,m-1}\left(L^1_{p,q,n} + L^2_{p,q,n}\right) + F_{p,q,n-1}(L^1_{p,q,m} + L^2_{p,q,m}) \\ =& F_{p,q,m-1}L_{p,q,n} + F_{p,q,n-1}L_{p,q,m} \end{aligned}$$

where $L^1_{p,q,n} + L^2_{p,q,n} = L_{p,q,n}$ and $L^1_{p,q,m} + L^2_{p,q,m} = L_{p,q,m}$. □

3. Conclusion

In this paper, the (p, q)-Fibonacci numbers that generalize the Fibonacci type numbers, among others, Pell, Jacobsthal, k-Fibonacci numbers, and the (p, q)-Lucas numbers that generalize the Lucas type numbers, among others, Pell-Lucas, Jacobsthal-Lucas, k-Lucas numbers, are expressed by tiling interpretations. We obtain that nth (p, q)-Fibonacci number counts the number of ways to tile a $1 \times n$ board with cells labeled $1,\ 2,\ \ldots,\ n$ using colored 1×1 squares and colored 1×2 dominoes, where there are p different colors for squares and q different colors for dominoes, and nth (p, q)-Lucas number counts the number of ways to tile a circular $1 \times n$ board with colored squares and colored dominoes. We present well-known relations among these numbers. Formulas to find the sum of the terms of the (p, q)-Fibonacci sequences and the (p, q)-Lucas sequences as well as the sum of terms of even and odd subscripts are given by using these interpretations. More general interpretations to express the (p, q)-Fibonacci and (p, q)-Lucas numbers can be studied and the relations between these interpretations and their previously found properties can be examined.

4. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The author(s) declared that has no received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author(s) declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of

Universal Mathematics.

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