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Research Article

Local Analysis of a Competitive Problem with Toxicants

Nihal Özdoğan[1*]

[1*] Bursa Technical University, Faculty of Science and Engineering, Departmant of Mathematics, Bursa, Turkey, (ORCID: 0000-0002-7551-1636), nihal.ozdogan@btu.edu.tr

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Abstract

This study aims to explain the dynamics of a competitive problem affected by toxicants. The effect of toxicants on ecological systems is an interesting topic for mathematical modelling. Discretization of the nonlinear problem is inevitable for right approximation of its solutions due to the difficulty of finding analytical solutions. In this work, a continuous time two species competitive problem was transformed into a discrete time problem. Because, it is very important to create a discrete model that will protect the properties of the original continuous model and the dynamics will be independent of step size. Also, in this study, the dynamic behaviour of a competitive system under the influence of toxicants were investigated. Lastly, the stability properties of each fixed point of the corresponding discrete problem have been examined using some theoretical results.

Keywords: Stability, Toxicants, Fixed point, Competitive system.

Zehirli Maddelerle Rekabetçi Bir Problemin Lokal Analizi

Öz

Bu çalışma, zehirli maddelerden etkilenen rekabetçi bir problemin dinamiklerini açıklamayı amaçlamaktadır. Zehirli maddelerin ekolojik sistemler üzerindeki etkisi matematiksel modelleme için ilginç bir konudur. Analitik çözümleri bulmak zor olduğundan, problemin çözümlerine doğru yaklaşım için lineer olmayan problemin ayrıklaştırılması kaçınılmazdır. Bu çalışmada sürekli haldeki iki tür içeren rekabetçi problem ayrık haldeki probleme dönüştürülmüştür. Çünkü orijinal sürekli modelin özelliklerini koruyacak ayrık bir model oluşturmak çok önemlidir ve modelin dinamikleri adım boyutundan bağımsız olacaktır. Ayrıca bu çalışmada zehirli maddelerin etkisi altındaki rekabetçi bir sistemin dinamik davranışı araştırılmıştır. Son olarak, ilgili ayrık problemin her bir denge noktasının kararlılık özellikleri bazı teorik sonuçlar kullanılarak incelenmiştir.

Anahtar Kelimeler: Kararlılık, Zehirli maddeler, Rekabetçi sistem.

[*] Corresponding Author: nihal.ozdogan@btu.edu.tr

1. Introduction

The effect of toxic substances on ecological systems is an interesting topic for mathematical modelling. The uncontrolled contribution of toxic substances to the nature causes the extinction of many species and many others are at the threshold of extinction. Freedman and Shukla [1], Chattopadhyay [2], Samanta [3], Biswas and Bairagi [4], Das et al. [5] and others worked the effects of toxicants on different ecosystems. Recently, nonlinear ordinary differential equations are often used to illustrate the interaction between two species. Due to the diffuculties of analytical solutions of these problems, discretization has become inevitable in order to reach approximate solutions.

Numerical methods based on finite difference approximations can be used to predict competitive population dynamics. But disadvantage of these methods is that their accuracy and stability depend on the step size [6]. However, these methods do not guarantee positive solutions for positive initial conditions. On the other hand, using nonstandard finite difference (NSFD) method in the proposed discrete system, this problem can be eliminated. Therefore, it is very important to create a discrete model that will protect the properties of the original continuous model. Dimitrov and Kojouharov [7,8,9], Shokri et al. [10], Banda et al. [11], Sajjad et al. [12] and many others have used nonstandard techniques developed by Mickens [13].

In this paper, we analyze the following competitive problem that is affected by toxic substances in [14]:

$$\frac{dx}{dt} = x(K_1 - \alpha_1 x - \beta_{12} y - \gamma_1 xy)$$

$$\frac{dy}{dt} = y(K_2 - \alpha_2 y - \beta_{21} x - \gamma_2 xy) \qquad (1)$$

where $x(t)$ and $y(t)$ denote the population densities at time t. K_1, K_2, α_1, α_2, β_{12}, β_{21}, γ_1 and γ_2 are positive constants. K_i is the intrinsic growth rate of species i, α_i indicates the competition coefficient of species i, β_{ij} measures the impact of species j upon the growth rate of species i $(i \neq j;\ i,j = 1,2)$ and γ_i represents the toxicant coefficients.

2. The Discrezation of a Competitive Problem

First of all, we must install the discrete version of the continuous problem with NSFD method. The NSFD method is based on two basic factors [15, 16]:

(i) $\frac{dx}{dt} = \frac{x_{k+1} - x_k}{\varphi(h)}$

where $\varphi(h) = h + O(h^2)$.

(ii) nonlinear and linear terms may need a nonlocal presentation.

Building a numerical scheme for equation (1), we discretize the time variable at $t_n = nh$, $t(\geq 0)$ where $h(>0)$ is the time step size. To represent the stability analysis of discrete time model, we can employ the following discretizations:

$$x \to x_k, \quad y \to y_k, \quad x^2 \to x_k x_{k+1}, \quad xy \to x_{k+1} y_k,$$

and

$$yx \to y_{k+1} x_{k+1}, \quad xy \to x_{k+1} y_k, \quad y^2 \to y_k y_{k+1}.$$

Thus, positive solutions can be found. By using NSFD scheme, we can obtain the discrete system:

$$\frac{x_{k+1} - x_k}{\varphi(h)} = K_1 x_k - \alpha_1 x_k x_{k+1} - \beta_{12} x_{k+1} y_k - \gamma_1 x_k x_{k+1} y_k$$

$$\frac{y_{k+1} - y_k}{\varphi(h)} = K_2 y_k - \alpha_2 y_k y_{k+1} - \beta_{21} x_{k+1} y_{k+1} - \gamma_2 x_{k+1} y_k y_{k+1}, \qquad (2)$$

where $\varphi(h)$ depends on the step size h and it is called denominator function. Let us suppose $h_1 = \varphi(h)$. Then system (2) becomes

$$x_{k+1} = \frac{x_k(1 + h_1 K_1)}{[1 + h_1 \alpha_1 x_k + h_1 \beta_{12} y_k + h_1 \gamma_1 x_k y_k]},$$

$$y_{k+1} = \frac{y_k(1 + h_1 K_2)}{[1 + h_1 \alpha_2 y_k + h_1 \beta_{21} x_{k+1} + h_1 \gamma_2 x_{k+1} y_k]}. \qquad (3)$$

Since all parameters are positive and the initial values are positive, the solution of the discrete system (3) remains positive.

3. Equilibrium Points and Local Stability Analysis

The competitive problem (1) has four equilibrium points: $E_1^* = (0,0)$, $E_2^* = \left(\frac{K_1}{\alpha_1},\ 0\right)$, $E_3^* = \left(0,\ \frac{K_2}{\alpha_2}\right)$ and $E_4^* = (x^*, y^*)$.

If we take $x_{k+1} = x_k = x$ and $y_{k+1} = y_k = y$ at system (3), the fixed points are satisfying the following equations:

$$x_{k+1} = \frac{x_k(1+h_1K_1)}{[1+h_1\alpha_1 x_k + h_1\beta_{12} y_k + h_1\gamma_1 x_k y_k]} = x_k,$$

$$y_{k+1} = \frac{y_k(1+h_1K_2)}{[1+h_1\alpha_2 y_k + h_1\beta_{21}x_{k+1} + h_1\gamma_2 x_{k+1}y_k]} = y_k ,$$

$$K_1 = \alpha_1 x_k + \beta_{12} y_k + \gamma_1 x_k y_k , \quad (4)$$

$$K_2 = \alpha_2 y_k + \beta_{21} x_{k+1} + \gamma_2 x_{k+1} y_k . \quad (5)$$

In this case the fixed point (x^*, y^*) is satisfying:

$K_1 = \alpha_1 x^* + \beta_{12} y^* + \gamma_1 x^* y^*$,

and

$K_2 = \alpha_2 y^* + \beta_{21} x^* + \gamma_2 x^* y^*$.

Jacobian matrix of system (3) evaluated at an arbitrary fixed point $(x_{k,}, y_k)$ is found by

$$J(x_{k,}, y_k) = \begin{pmatrix} \frac{\partial x_{k+1}}{\partial x_k} & \frac{\partial x_{k+1}}{\partial y_k} \\ \frac{\partial y_{k+1}}{\partial x_k} & \frac{\partial y_{k+1}}{\partial y_k} \end{pmatrix}$$

where

$$J_{11}(x_k, y_k) = \frac{(1+h_1K_1)}{[1+h_1\alpha_1 x_k + h_1\beta_{12} y_k + h_1\gamma_1 x_k y_k]} - \frac{(1+h_1K_1)x_k(h_1\alpha_1 + h_1\gamma_1 y_k)}{[1+h_1\alpha_1 x_k + h_1\beta_{12} y_k + h_1\gamma_1 x_k y_k]^2},$$

$$J_{12}(x_k, y_k) = \frac{-x_k(1+h_1K_1)(h_1\beta_{12} + h_1\gamma_1 x_k)}{[1+h_1\alpha_1 x_k + h_1\beta_{12} y_k + h_1\gamma_1 x_k y_k]^2},$$

$$J_{21}(x_k, y_k) = \frac{-y_k(1+h_1K_2)(h_1\beta_{21} + h_1\gamma_2 y_k)J_{11}}{[1+h_1\alpha_2 y_k + h_1\beta_{21} x_k + h_1\gamma_2 x_k y_k]^2},$$

$$J_{22}(x_k, y_k) = \frac{(1+h_1K_2)}{[1+h_1\alpha_2 y_k + h_1\beta_{21} x_k + h_1\gamma_2 x_k y_k]} - \frac{(1+h_1K_2)(h_1\alpha_2 + h_1\gamma_2 x_k + h_1(\beta_{21} + \gamma_2 y_k)) J_{12}}{[1+h_1\alpha_2 y_k + h_1\beta_{21} x_k + h_1\gamma_2 x_k y_k]^2}.$$

Assume that λ_1 and λ_2 are the eigenvalues of the Jacobian matrix.

Lemma 3.1 The fixed point (x_k, y_k) of system (3) is defined stable (sink) if $|\lambda_1| < 1$, $|\lambda_2| < 1$ and unstable (source) if $|\lambda_1| > 1$, $|\lambda_2| > 1$. It is defined unstable (saddle) if $|\lambda_1| > 1$, $|\lambda_2| < 1$ or $|\lambda_1| < 1$, $|\lambda_2| > 1$ and non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$ [7].

Theorem 3.1 The fixed point E_1^* is a source.

Proof. At the fixed point E_1^* , Jacobian matrix is

$$J(E_1^*) = \begin{pmatrix} J_{11}(E_1^*) & J_{12}(E_1^*) \\ J_{21}(E_1^*) & J_{22}(E_1^*) \end{pmatrix},$$

where

$J_{11}(E_1^*) = 1 + h_1K_1$,

$J_{12}(E_1^*) = 0$,

$J_{21}(E_1^*) = 0$,

$J_{22}(E_1^*) = 1 + h_1K_2$.

The eigenvalues are $\lambda_1 = 1 + h_1K_1$ and $\lambda_2 = 1 + h_1K_2$. Since the eigenvalues are positive and they are greater than one, the fixed point E_1^* is a source.

If we take $y_k = y_{k+1} = 0$ and $x_k = x_{k+1} = 0$ at equation (4), we can obtain x_k and y_k as follows:

$x_k = \frac{K_1}{\alpha_1}$, $y_k = \frac{K_2}{\alpha_2}$.

Theorem 3.2 The fixed point $E_2^* = \left(\frac{K_1}{\alpha_1}, 0\right)$ is stable (sink) if $\frac{\alpha_1}{\beta_{21}} < \frac{K_1}{K_2}$ and it isn't a source. The fixed point $E_2^* = \left(\frac{K_1}{\alpha_1}, 0\right)$ is a saddle if $\frac{\alpha_1}{\beta_{21}} > \frac{K_1}{K_2}$ and it is unstable. The fixed point $E_2^* = \left(\frac{K_1}{\alpha_1}, 0\right)$ is non-hyperbolic if $\frac{\alpha_1}{\beta_{21}} = \frac{K_1}{K_2}$ [4].

Proof. At the fixed point E_2^* , Jacobian matrix is

$$J(E_2^*) = \begin{pmatrix} J_{11}(E_2^*) & J_{12}(E_2^*) \\ J_{21}(E_2^*) & J_{22}(E_2^*) \end{pmatrix},$$

where

$J_{11}(E_2^*) = 1 - \frac{h_1K_1}{(1+h_1K_1)}$,

$J_{12}(E_2^*) = \frac{-h_1K_1(\beta_{12}\alpha_1 + \gamma_1K_1)}{(1+h_1K_1)\alpha_1{}^2}$,

$J_{21}(E_2^*) = 0$,

$J_{22}(E_2^*) = \frac{1+h_1K_2}{1+\frac{h_1K_1\beta_{21}}{\alpha_1}}$.

The eigenvalues are $\lambda_1 = 1 - \frac{h_1K_1}{(1+h_1K_1)}$, $\lambda_2 = \frac{1+h_1K_2}{1+\frac{h_1K_1\beta_{21}}{\alpha_1}}$.

Since $|\lambda_1|$ is less than one for any $h_1 > 0$, E_2^* isn't a source. If $\frac{\alpha_1}{\beta_{21}} < \frac{K_1}{K_2}$ (for any $h_1 > 0$), $|\lambda_2| < 1$ and E_2^* becomes stable. Also, if $\frac{\alpha_1}{\beta_{21}} > \frac{K_1}{K_2}$ (for any $h_1 > 0$), $|\lambda_2| > 1$. Thus E_2^* is a saddle. On the other hand, if $\frac{\alpha_1}{\beta_{21}} = \frac{K_1}{K_2}$, $|\lambda_2| = 1$ and E_2^* is non-hyperbolic.

Theorem 3.3 The fixed point $E_3^* = \left(0, \frac{K_2}{\alpha_2}\right)$ is stable (sink) if $\frac{K_1}{K_2} < \frac{\beta_{12}}{\alpha_2}$ and it isn't a source. The fixed point $E_3^* = \left(0, \frac{K_2}{\alpha_2}\right)$ is a saddle if $\frac{K_1}{K_2} > \frac{\beta_{12}}{\alpha_2}$ and it is unstable. The fixed point $E_3^* = \left(0, \frac{K_2}{\alpha_2}\right)$ is non-hyperbolic if $\frac{K_1}{K_2} = \frac{\beta_{12}}{\alpha_2}$ [4].

Proof. At the fixed point E_3^*, Jacobian matrix is

$$J(E_3^*) = \begin{pmatrix} J_{11}(E_3^*) & J_{12}(E_3^*) \\ J_{21}(E_3^*) & J_{22}(E_3^*) \end{pmatrix},$$

where

$$J_{11}(E_3^*) = \frac{1+h_1K_1}{1+\frac{h_1K_2\beta_{12}}{\alpha_2}},$$

$$J_{12}(E_3^*) = 0,$$

$$J_{21}(E_3^*) = \frac{-\frac{K_2\left(\left(h_1\beta_{21}+h_1\gamma_2\frac{K_2}{\alpha_2}\right)J_{11}\right)}{\alpha_2}}{(1+h_1K_2)},$$

$$J_{22}(E_3^*) = 1 - \frac{h_1K_2}{(1+h_1K_2)}.$$

The eigenvalues are $\lambda_1 = \frac{1+h_1K_1}{1+\frac{h_1K_2\beta_{12}}{\alpha_2}}$, $\lambda_2 = 1 - \frac{h_1K_2}{(1+h_1K_2)}$.

Since $|\lambda_2|$ is less than one for any $h_1 > 0$, E_3^* isn't a source. If $\frac{K_1}{K_2} < \frac{\beta_{12}}{\alpha_2}$ (for any $h_1 > 0$), $|\lambda_1| < 1$ and E_3^* becomes stable. Also, if $\frac{K_1}{K_2} > \frac{\beta_{12}}{\alpha_2}$ (for any $h_1 > 0$), $|\lambda_1| > 1$. Thus E_3^* is a saddle. On the other hand, if $\frac{K_1}{K_2} = \frac{\beta_{12}}{\alpha_2}$, $|\lambda_1| = 1$ and E_3^* is non-hyperbolic.

Lemma 3.2 Let λ_1 and λ_2 are the eigenvalues of the Jacobian,

$$J(x^*, y^*) = \begin{pmatrix} J_{11}(x^*, y^*) & J_{12}(x^*, y^*) \\ J_{21}(x^*, y^*) & J_{22}(x^*, y^*) \end{pmatrix}.$$

Then $|\lambda_i| < 1$ $(i = 1, 2)$ if the following situations are hold:

(1) $0 < \det\left(J(x^*, y^*)\right) < 1$,

(2) $1 + \det\left(J(x^*, y^*)\right) + \mathrm{tr}\left(J(x^*, y^*)\right) > 0$,

(3) $1 + \det\left(J(x^*, y^*)\right) - \mathrm{tr}\left(J(x^*, y^*)\right) > 0$ [17,18].

Theorem 3.4 The fixed point $E_4^* = (x^*, y^*)$ is stable if the conditions of Lemma 2 satisfy.

Proof. Using the equations $K_1 = \alpha_1x^* + \beta_{12}y^* + \gamma_1x^*y^*$, $\alpha_1x^* + \gamma_1x^*y^* = K_1 - \beta_{12}y^*$ and $K_2 = \alpha_2y^* + \beta_{21}x^* + \gamma_2x^*y^*$, $K_2 - \beta_{21}x^* = \alpha_2y^* + \gamma_2x^*y^*$

the Jacobian $J(x^*, y^*)$ at the fixed point E_4^* can be finded by:

$$J_{11}(x^*, y^*) = 1 - \frac{h_1K_1}{1+h_1K_1}\frac{K_1 - \beta_{12}y^*}{K_1},$$

$$J_{12}(x^*, y^*) = -\frac{h_1K_1}{1+h_1K_1}\frac{x^*(\beta_{12}+\gamma_1x^*)}{K_1},$$

$$J_{21}(x^*, y^*) = -\frac{h_1K_2}{1+h_1K_2}\frac{y^*(\beta_{21}+\gamma_2y^*)J_{11}}{K_2},$$

and

$$J_{22}(x^*, y^*) = 1 - \frac{h_1K_2}{1+h_1K_2}\frac{K_2 - \beta_{21}x^*}{K_2} - \frac{y^*h_1(\beta_{21}+\gamma_2y^*)\,J_{12}}{1+h_1K_2}.$$

It is clearly seen that, $J_{12}(x^*, y^*)$, $J_{21}(x^*, y^*)$ are negative and $0 < J_{11}(x^*, y^*) < 1$, $J_{22}(x^*, y^*) > 0$.

Thus,

$$\begin{aligned}&\det\left(J(x^*, y^*)\right)\\ &= J_{11}\left\{1 - \frac{h_1K_2}{1+h_1K_2}\frac{K_2 - \beta_{21}x^*}{K_2}\right\}\\ &- J_{11}\left\{\frac{y^*h_1(\beta_{21}+\gamma_2y^*)}{1+h_1K_2}\right\}\left\{-\frac{h_1K_1}{1+h_1K_1}\frac{x^*(\beta_{12}+\gamma_1x^*)}{K_1}\right\}\\ &- \left\{\frac{h_1K_1}{1+h_1K_1}\frac{x^*(\beta_{12}+\gamma_1x^*)}{K_1}\right\}\left\{\frac{h_1K_2}{1+h_1K_2}\frac{y^*(\beta_{21}+\gamma_2y^*)J_{11}}{K_2}\right\}\end{aligned}$$

$$0 < \det\left(J(x^*, y^*)\right) = J_{11}\left\{1 - \frac{h_1K_2}{1+h_1K_2}\frac{K_2 - \beta_{21}x^*}{K_2}\right\} < 1$$

This shows us that the first condition of Lemma 2 is satisfied. Since $J_{11}(x^*, y^*) > 0$ and $J_{22}(x^*, y^*) > 0$, $\mathrm{tr}\left(J(x^*, y^*)\right) > 0$. As $\det\left(J(x^*, y^*)\right) > 0$ and $\mathrm{tr}\left(J(x^*, y^*)\right) > 0$,

$1 + \det\left(J(x^*, y^*)\right) + \mathrm{tr}\left(J(x^*, y^*)\right) > 0$. Thus, the second condition is satisfied.

The calculations below show us that

$$\begin{aligned}&1 + \det\left(J(x^*, y^*)\right)\\ &- \mathrm{tr}\left(J(x^*, y^*)\right)\\ &= 1 + 1 - \frac{h_1K_2}{1+h_1K_2}\frac{K_2 - \beta_{21}x^*}{K_2} - \frac{h_1K_1}{1+h_1K_1}\frac{K_1 - \beta_{12}y^*}{K_1}\\ &+ \frac{h_1K_1}{1+h_1K_1}\frac{h_1K_2}{1+h_1K_2}\frac{K_1 - \beta_{12}y^*}{K_1}\frac{K_2 - \beta_{21}x^*}{K_2} - 1\\ &+ \frac{h_1K_1}{1+h_1K_1}\frac{K_1 - \beta_{12}y^*}{K_1} - 1 + \frac{h_1K_2}{1+h_1K_2}\frac{K_2 - \beta_{21}x^*}{K_2}\\ &- \frac{y^*h_1(\beta_{21}+\gamma_2y^*)}{1+h_1K_2}\frac{h_1K_1}{1+h_1K_1}\frac{x^*(\beta_{12}+\gamma_1x^*)}{K_1},\end{aligned}$$

$$1+\det\big(J(x^*,y^*)\big)-\operatorname{tr}\big(J(x^*,y^*)\big)=\frac{h_1K_1}{1+h_1K_1}\frac{h_1K_2}{1+h_1K_2}\frac{K_1-\beta_{12}y^*}{K_1}\frac{K_2-\beta_{21}x^*}{K_2}-\frac{y^*h_1(\beta_{21}+\gamma_2y^*)}{1+h_1K_2}\frac{h_1K_1}{1+h_1K_1}\frac{x^*(\beta_{12}+\gamma_1x^*)}{K_1}$$

Using the equations $K_1-\beta_{12}y^*=\alpha_1x^*+\gamma_1x^*y^*$ and $K_2-\beta_{21}x^*=\alpha_2y^*+\gamma_2x^*y^*$

$$1+\det\big(J(x^*,y^*)\big)-\operatorname{tr}\big(J(x^*,y^*)\big)=\frac{{h_1}^2x^*y^*}{(1+h_1K_1)(1+h_1K_2)}\{(\alpha_1\alpha_2-\beta_{21}\beta_{12})+x^*(\alpha_1\gamma_2-\beta_{21}\gamma_1)+y^*(\alpha_2\gamma_1-\beta_{12}+\gamma_2)\}.$$

It can be easily shown that $1+\det\big(J(x^*,y^*)\big)-\operatorname{tr}\big(J(x^*,y^*)\big)>0$. So the proof is complete.

4. Conclusion

In this study, it was aimed to observe the dynamic behaviours of the competitive problem with toxicants. Stability analysis of fixed points of the discretized problem were made with the help of some important theoretical evidence. We believe that these analysis will be useful to researchers for the theory of competitive problems.

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